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Representation of Integers of the Form $x^2 + my^2 - z^2$

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Abstract

Let m be a positive integer. A positive integer k is called m-special if every integer n can be expressed as $n = x^2 + my^2 - kz^2$ for some nonzero integers x, y and z. In this article, we show that 1 is m-special if and only if m is not divisible by 4.

1 Introduction

Representations of integers as sums of squares have a long history. Ramanujan [4] proved that there exist 54 quadruples (a, b, c, d) with $1 \leq a \leq b \leq c \leq d$ such that every natural number n is representable in the form $aw^2 + bx^2 + cy^2 + dz^2$. Panaitopol [3] proved that there exists no triple (a, b, c) with $1 \leq a \leq b \leq c$ such that every natural number is representable in the form $ax^2 + by^2 + cz^2$. However, if we allow c in the representation $ax^2 + by^2 + cz^2$ to be negative, then the representation is possible. Nowicki [2] showed that if all natural numbers are representable in the form $x^2 + y^2 - cz^2$, then c is of the form q or $2q$, where either $q = 1$ or q is a product of primes of the form $4m + 1$. Lam [1] proved its sufficiency. In what follows, we are concerned with the problem of representing natural numbers n in the form $n = x^2 + my^2 - z^2$ for a given positive integer m, where $xyz \neq 0$. We provide necessary and sufficient conditions for representing all integers n in the form $x^2 + my^2 - z^2$ where x, y and z are nonzero.

Definition 1. A positive integer k is *m-special* if for every integer n there exist nonzero integers x, y, and z such that $n = x^2 + my^2 - kz^2$.

Nowicki [2] showed that 1 is 1-special, that is, for every integer n there exist nonzero integers x, y, and z such that $n = x^2 + y^2 - z^2$. Nowicki [2] and Lam [1] showed that k is 1-special if and only if k is of the form q or $2q$ where either $q = 1$ or q is a product of primes of the form $4m + 1$.

2 Main Results

Theorem 2. *Let m be a positive integer. If m is divisible by 4, then 1 is not m-special.*

Proof. Let m be a positive integer divisible by 4. Assume that 1 is m-special. Then there exist nonzero integers x, y, z such that $x^2 + my^2 - z^2 = 2$. So we have $x^2 - z^2 \equiv 2 \pmod 4$. Since quadratic residues modulo 4 are 0 and 1, it follows that $x^2 - z^2 \equiv 0, 1, 3 \pmod 4$. This is a contradiction. Hence 1 is not m-special. □

In 2015, Nowicki [2] showed that 1 is 1-special.

Lemma 3. *[2] 1 is 1-special.*

We present an elementary result. The following well-known lemma plays an important role in our main result. The proof is given in [5, Theorem 3.18]:

Lemma 4. *Let m be a rational number. Then m is an integer if and only if m^2 is an integer.*

Theorem 5. *Let $m > 1$ be a positive integer . If m is not divisible by 4, then 1 is m-special.*

Proof. Let $m > 1$ be a positive integer not divisible by 4. We show that for any integers n there exist nonzero integers x, y and z such that $n = x^2 + my^2 - z^2$, i.e., $x^2 - z^2 = (x - z)(x + z) = n - my^2$. In each case, we provide a choice of nonzero integers x, y and z.

Case 1: Suppose $n \equiv 0 \pmod 4$. Thus $n = 4l$ for some integer l. If $l \neq 0$, then we choose

$$y = 4l + 2, \quad x = 1 + l - m(2l + 1)^2, \quad z = -1 + l - m(2l + 1)^2.$$

Obviously, $y \neq 0$. We next show that x and z are nonzero. If $x = 0$ or $z = 0$, then $m(2l + 1)^2 = l + 1$ or $m(2l + 1)^2 = l - 1$ respectively. Since $m > 1$, we have $l > 0$ or $l > 3$, respectively. Both cases imply that m is not an integer, which is a contradiction.

Case 2: Suppose $n \equiv 1 \pmod 4$. Thus $n = 4l + 1$ for some integer l. In this case, we choose

$$y = 2(4l + 1), \quad x = 1 + 2l - 2m(4l + 1), \quad z = -2l - 2m(4l + 1).$$

Again, we have that $y \neq 0$. If $x = 0$, then 1 is divisible by 2, which is a contradiction. If $z = 0$, then $m^2(4l+1)^2 = l^2$. Thus m^2 is not an integer for all $l \neq 0$. Then by Lemma 4, we get that m is not an integer. Thus x and z are nonzero.

Case 3: Suppose $n \equiv 2 \pmod 4$. Thus $n = 4l + 2$ for some integer l.

Case 3.1: Suppose $m \equiv 2 \pmod 4$. Hence $m = 4k+2$ for some non-negative integer k. We choose

$$y = 2l + 1, \quad x = l - k - 2kl + 1, \quad z = -3l - k - 2kl - 1.$$

It is easy to see that $y \neq 0$. If $x = 0$ or $z = 0$, then $k = \frac{l+1}{2l+1}$ or $k = \frac{-(3l+1)}{2l+1} = -1 - \frac{l}{2l+1}$ respectively. Since k is an integer, we have $l = -1$ or 0. Therefore x or z can be zeros only if $(m, n) = (2, -2), (6, 2)$. To handle these cases, we can use the representations

$$\begin{aligned} 2 &= x^2 + 6y^2 - z^2 \quad \text{where} \quad (x, y, z) = (12, 3, 14) \\ -2 &= x^2 + 2y^2 - z^2 \quad \text{where} \quad (x, y, z) = (4, 3, 6). \end{aligned}$$

Case 3.2: Suppose $m \equiv 1 \pmod 2$. Therefore $m = 2k+1$ for some positive integer k. Here we choose

$$y = 2l + 1, \quad x = 1 - k - 2kl, \quad z = -k - 2l - 2kl.$$

Since y is odd, we have that y is nonzero. If $x = 0$, then $l = 0$ and $k = 1$. We obtain $m = 3$ and $n = 2$. For this exceptional case, we use the representation $2 = 12^2 + 3(3)^2 - 13^2$. If $z = 0$, then $k^2(1 + 2l)^2 = (2l)^2$. Thus k^2 is not an integer for $l \neq 0$. Thus again by Lemma 4, k is not an integer for $l \neq 0$.

Case 4: Let $n \equiv 3 \pmod 4$. Thus $n = 4l + 3$ for some integer l. We choose

$$y = 2(4l + 3), \quad x = 2l - 6m - 8ml + 2, \quad z = -2l - 6m - 8ml - 1.$$

It is easy to see that y and z are nonzero. If $x = 0$, then $m(4l+3) = l+1$. Since $m = \frac{l+1}{4l+3} > 1$, we obtain that l is not an integer which is a contradiction.

$\square$

In conclusion, we have proved the following theorem.

Theorem 6. *Let m be a positive integer. Then 1 is a m-special if and only if m is not divisible by 4.*

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