Journal of Integer Sequences, Vol. 24 (2021),
Article 21.5.4

Complete Sequences of Weighted r-Generalized Fibonacci Powers

Passawan Noppakaew and Thanakorn Prinyasart[1]
Department of Mathematics
Faculty of Science
Silpakorn University
Nakhon Pathom 73000
Thailand
noppakaew_p@silpakorn.edu
prinyasart_t@silpakorn.edu, thanakorn_dpst@hotmail.com

Abstract

In this paper, we study the least number $L \in \mathbb{N}$ that makes the L-fold of a weighted r-generalized Fibonacci power complete. We can establish an upper bound and a lower bound for L, depending on the first r terms and the limit of the consecutive ratios. We also give the explicit value of L in some special cases. In the end, we study a particular minimal complete sequence called a generalized distinguished sequence and show that there exists a generalized distinguished sequence of a weighted r-generalized Fibonacci mth power for all large $m \in \mathbb{N}$.

1 Introduction

In 1960, Hoggatt and King [5] defined a *complete* sequence (a_n) as a sequence of positive integers such that for each $m \in \mathbb{N}$ there exist finite distinct natural numbers $n_1, n_2, \ldots, n_k$ such that $m = \sum_{i=1}^{k} a_{n_i}$. They also showed that the Fibonacci sequence (F_n), defined by the recurrence $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, is complete. Moreover, they proved that any deletion of an element from (F_n) results in a non-complete sequence.

[1]Thanakorn Prinyasart is the corresponding author.

In this case, we say that (F_n) is a *minimal* complete sequence. A necessary and sufficient condition for a sequence to be complete was given by Brown [1]. A nontrivial example of a non-complete sequence is the sequence of Fibonacci squares, which is $1, 1, 4, 9, 25, 64, \dots$. However, O'Connell [7] showed that the 2-fold of the sequence of Fibonacci squares, which is $1, 1, 1, 1, 4, 4, 9, 9, 25, 25, 64, 64, \dots$, is complete. Hunsucker and Wardlaw [6] considerably generalized the result of O'Connell by proving that, for each $m \in \mathbb{N}$, the 2^{m-1}-fold, but not $(2^{m-1} - 1)$-fold, of (F_n^m) is complete. In their work, they also provided an example of an increasing complete sequence, which is minimal, and mapped it onto $\{F_n^m : n \in \mathbb{N}\}$.

Our aim in this paper is to extend the results of Hunsucker and Wardlaw to the *weighted r-generalized Fibonacci sequences*. As far as we know, these sequences were firstly described by Dubeau et al. [2]. Let $c_1, c_2, \dots, c_r$, where $r \geq 2$, be positive integers and let $A_1, A_2, \dots, A_r$ be nonnegative integers with $A_r > 0$. A *weighted r-generalized Fibonacci sequence (with positive integral initial conditions and nonnegative coefficients)* (a_n) is defined by

$$a_n := \begin{cases} c_n, & \text{if } n = 1, 2, \dots, r; \\ \sum_{k=1}^{r} A_k a_{n-k}, & \text{if } n > r. \end{cases} \tag{1}$$

We call $c_1, c_2, \dots, c_r$ *initial values* and $A_1, A_2, \dots, A_r$ *coefficients* of (a_n). Notice that when $r = 2$, and $c_1 = c_2 = A_1 = A_2 = 1$, this sequence is the usual well-known Fibonacci sequence. We are concerned with not only the completeness of the L-fold of a weighted r-generalized Fibonacci sequence but also a particular minimal sequence, called a distinguished sequence, of a weighted r-generalized Fibonacci power as introduced by Hunsucker and Wardlaw [6].

2 Weighted r-generalized Fibonacci sequences

According to Dubeau et al. [2], a weighted r generalized Fibonacci sequence (a_n) is defined as the sequence generated by the recurrence (1) with $c_k, A_k \in \mathbb{C}$ for all $k \in \{1, 2, \dots, r\}$ and $A_r \neq 0$. The characteristic polynomial of (a_n) is $p(x) := x^r - \sum_{k=1}^{r} A_k x^{r-k}$. It is well-known that (a_n) can be written in an exponential polynomial form, $a_n = \sum_{k=1}^{l} C_k(n)\lambda_k^n$, where $\lambda_1, \lambda_2, \dots, \lambda_l \in \mathbb{C}$ are all distinct roots of $p(x)$ and $C_k(x) \in \mathbb{C}[x]$ with $\deg(C_k(x)) + 1 \leq$ multiplicity of λ_k for all $k \in \{1, 2, \dots, l\}$. This exponential polynomial form is convenient to use when we want to determine the limit of the consecutive terms of the sequence, which is one of the crucial ingredients we need to obtain most of our results. Because of the appearance of the roots of the characteristic polynomials in the exponential polynomial forms, it is useful to gather the results about these roots from the work of Dubeau et al. [2] and Ostrowski [8] in this section.

The first upcoming theorem provides a useful result on the largest modulus roots of $p(x)$, when $A_k \geq 0$ for all $k \in \{1, 2, \dots, r-1\}$ and $A_r > 0$, which is the case we are interested in.

Theorem 1. *[8, Theorem 12.2] Let* $p(x) := x^r - \sum_{k=1}^{r} A_k x^{r-k} \in \mathbb{R}[x]$ *for some integer* $r \geq 2$, *and nonnegative real numbers* $A_1, A_2, \dots, A_r$ *with* $A_r > 0$. *If* $\gcd\{k : A_k > 0\} = 1$, *then $p(x)$ has a unique largest modulus root in $\mathbb{C}$. Moreover, this root is a positive real root with multiplicity* 1.

According to Dubeau et al. [2], a polynomial $p(x)$ is called *asymptotically simple* if there exists a unique root λ among its roots of maximum modulus that has maximal multiplicity. The root λ is then called the *dominant root* and its multiplicity is called the *dominant multiplicity*. By Theorem 1, if (a_n) is a weighted r-generalized Fibonacci sequence generated by (1) with $A_k \geq 0$ for all $k \in \{1, 2, \ldots, r-1\}$, $A_r > 0$, and $\gcd\{k : A_k > 0\} = 1$, then its characteristic polynomial is asymptotically simple with a simple positive dominant root λ [2, Theorem 9]. Even though λ is a simple positive dominant root, it does not imply the existence of $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ in general [4, Example 4.6]. However, if we assume further that the initial values $c_1, c_2, \ldots, c_r$ are nonnegative real numbers, not all zero, then we can show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda$ with the aid of the following theorem.

Theorem 2. *[2, Theorem 5] Let $A_1, A_2, \ldots, A_r \in \mathbb{C}$ with $A_r \neq 0$, and λ be an arbitrary nonzero complex number. We denote the sequence (a_n) generated by the recurrence relation (1) with initial values $\boldsymbol{c} := (c_1, c_2, \ldots, c_r) \in \mathbb{C}^r$ by $(a_n(\boldsymbol{c}))$. Fix $\boldsymbol{c}_0 := (0, 0, \ldots, 0, 1) \in \mathbb{C}^r$. Then $\lim_{n\to\infty} \frac{a_n(\boldsymbol{c})}{\lambda^n}$ exists for all $\boldsymbol{c} \in \mathbb{C}^r$ if and only if $\lim_{n\to\infty} \frac{a_n(\boldsymbol{c}_0)}{\lambda^n}$ exists. Moreover, when all the limits exist,*

$$\lim_{n\to\infty} \frac{a_n(\boldsymbol{c})}{\lambda^n} = \left(c_r + \sum_{k=1}^{r-1} c_{r-k} \lambda^k \sum_{l=k}^{r-1} \frac{A_{1+l}}{\lambda^{1+l}} \right) \cdot \lim_{n\to\infty} \frac{a_n(\boldsymbol{c}_0)}{\lambda^n} \tag{2}$$

for all $\boldsymbol{c} := (c_1, c_2, \ldots, c_r) \in \mathbb{C}^r$.

By the above theorem, we have the following result.

Proposition 3. *Let (a_n) be a weighted r-generalized Fibonacci sequence defined by (1), with $c_k, A_k \geq 0$ for all $k \in \{1, 2, \ldots, r\}$, $A_r > 0$, and $c_{k'} > 0$ for some $k' \in \{1, 2, \ldots, r\}$. Assume that $\gcd\{k : A_k > 0\} = 1$. If λ is the dominant root, which is a simple positive root of its characteristic polynomial, then there exists $C > 0$ such that*

$$a_n = C\lambda^n + o(\lambda^n)$$

as $n \to \infty$. Consequently, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda$.

Proof. Notice that, if $\boldsymbol{c}' := (\lambda, \lambda^2, \ldots, \lambda^r) \in \mathbb{R}^r$, then $a_n(\boldsymbol{c}') = \lambda^n$ for all $n \in \mathbb{N}$, and so $\lim_{n\to\infty} \frac{a_n(\boldsymbol{c}')}{\lambda^n} = \lim_{n\to\infty} \frac{\lambda^n}{\lambda^n} = 1$. Since λ is the dominant root, $\lim_{n\to\infty} \frac{a_n(\boldsymbol{c}_0)}{\lambda^n}$ exists, where $\boldsymbol{c}_0 := (0, 0, \ldots, 0, 1) \in \mathbb{R}^r$. Moreover, by (2), since $\boldsymbol{c}' \neq \boldsymbol{0}$ and $A_r > 0$, we have that $\lim_{n\to\infty} \frac{a_n(\boldsymbol{c}_0)}{\lambda^n} > 0$. Again, by (2), since $\boldsymbol{c} := (c_1, c_2, \ldots, c_r) \neq \boldsymbol{0}$ and $A_r > 0$, we obtain that $\lim_{n\to\infty} \frac{a_n}{\lambda^n} = \lim_{n\to\infty} \frac{a_n(\boldsymbol{c})}{\lambda^n} > 0$. Hence, there exists $C > 0$ such that $a_n = C\lambda^n + o(\lambda^n)$ as $n \to \infty$. □

3 Notation

In this section, we provide some notation that will be used in the following sections. From now on, the weighted r-generalized Fibonacci sequences that we are interested in have positive integral initial values and nonnegative coefficients. Since the completeness of a sequence is invariant under rearrangements, it is sometimes easier to consider its nondecreasing rearrangement to justify the completeness of the sequence. So, we define the following equivalence relation for later use.

Definition 4. Let (a_n) and (b_n) be sequences of real numbers. We say that (a_n) and (b_n) are *equivalent*, denoted by $(a_n) \sim (b_n)$, if there exists a bijection $\rho : \mathbb{N} \to \mathbb{N}$ such that $a_n = b_{\rho(n)}$ for all $n \in \mathbb{N}$. We denote the equivalence class containing (a_n) by $[a_n]$.

From Proposition 3, we see that if $\gcd\{k : A_k > 0\} = 1$, then the weighted r-generalized Fibonacci sequence has a nice exponential polynomial form and the limit of consecutive terms exists. This leads us to give a name to the recurrence relation of this form.

Definition 5. Let (a_n) be defined by (1). We say that (a_n) is in a *primitive form* if $\gcd\{k : A_k > 0\} = 1$.

The next notation will be handy in the last section when we study a sequence that may not be in a primitive form. Moreover, we can also use this notation to define an L-fold of any sequence.

Definition 6. For sequences (a_n) and (b_n) of real numbers, we define $[a_n] \oplus [b_n]$ to be the equivalence class containing the sequence (c_n) satisfying $c_{2k-1} = a_k$ and $c_{2k} = b_k$ for all $k \in \mathbb{N}$.

Definition 7. Let (a_n) be a sequence of real numbers. For $L \in \mathbb{N}$, we call each sequence in the equivalence class $\underbrace{[a_n] \oplus [a_n] \oplus \cdots \oplus [a_n]}_{L \text{ times}}$ an *L-fold* of (a_n).

If a weighted r-generalized Fibonacci sequence does not have a nondecreasing rearrangement, then this sequence is periodic and easy to understand as shown in the following theorem.

Theorem 8. *Let (a_n) be a weighted r-generalized Fibonacci sequence. Then (a_n) does not have a nondecreasing rearrangement if and only if there exist $s \geq 2$ distinct constant sequences $(b_n^{(1)}), (b_n^{(2)}), \ldots, (b_n^{(s)})$ such that*

$$[a_n] = [b_n^{(1)}] \oplus [b_n^{(2)}] \oplus \cdots \oplus [b_n^{(s)}].$$

Moreover, when this happens, $a_n = a_{n-r}$ for all $n > r$ and, for $m \in \mathbb{N}$, we have that (a_n^m) is complete if and only if $a_k = 1$ for some $k \leq r$.

Proof. The converse of the theorem is obvious. Now, assume that (a_n) does not have a nondecreasing rearrangement. Since $c_i \geq 1$ for all $i \leq r$, if $A_k > 0$ for some $k < r$ or $A_r > 1$, then $\lim_{n\to\infty} a_n = \infty$. So (a_n) has a nondecreasing rearrangement, which is a contradiction. This implies that $A_k = 0$ for all $k < r$ and $A_r = 1$. Thus, $a_n = a_{n-r}$ for all $n > r$, and we are done. □

Hence, from now on, we will consider only weighted r-generalized Fibonacci sequences that have a nondecreasing rearrangement.

Definition 9. A sequence of real numbers (a_n) is *increasable* if it has a nondecreasing rearrangement, i.e., there exists a nondecreasing sequence (b_n) such that $(a_n) \sim (b_n)$.

4 L-folds of weighted r-generalized Fibonacci powers

In this section, for each $x \in \mathbb{R}$, we let $\lceil x \rceil$ denote the least integer greater than or equal to x. A well-known and widely used criterion for complete sequences is the following result given by Brown [1].

Theorem 10. *[1, Theorem 1] Let (a_n) be a sequence of positive integers. If $a_1 = 1$ and $\sum_{k=1}^{n} a_k \geq a_{n+1} - 1$ for all $n \in \mathbb{N}$, then (a_n) is complete. The converse holds for increasing sequences of positive integers.*

By the above theorem, for any sequence of positive integers (a_n), we see that if $a_1 = 1$, and L is a positive integer such that $L \geq \lceil \sup_{n\in\mathbb{N}} \frac{a_{n+1}-1}{\sum_{k=1}^{n} a_k} \rceil$, then the L-fold sequence of (a_n) is complete. Moreover, the converse holds when (a_n) is increasing. Then, for each sequence (a_n) of positive integers, we define $L_{(a_n)} := \lceil \sup_{n\in\mathbb{N}} \frac{a_{n+1}-1}{\sum_{k=1}^{n} a_k} \rceil$. For an increasable sequence (a_n) of positive integers, we also let $L_{[a_n]} := \lceil \sup_{n\in\mathbb{N}} \frac{b_{n+1}-1}{\sum_{k=1}^{n} b_k} \rceil$, where (b_n) is the nondecreasing sequence equivalent to (a_n).

Hunsucker and Wardlaw [6] showed that $L_{[F_n^m]} = L_{(F_n^m)} = 2^{m-1}$, where (F_n^m) is the usual Fibonacci mth power. Computing $L_{[a_n^m]}$ for an increasable weighted r-generalized Fibonacci mth power is not simple in general. So, in this section, we provide bounds for this value and give its exact value depending on finitely many terms of the sequence in some special cases.

The next lemma provides the least upper bound for the consecutive ratios of a weighted r-generalized Fibonacci sequence. Notice that the supremum depends on only the first r terms of the ratios, which is crucial information to obtain the upper bound for $L_{[a_n^m]}$ in the following theorem.

Lemma 11. *Let (a_n) be a weighted r-generalized Fibonacci sequence. Then*

$$\sup_{n\in\mathbb{N}} \frac{a_{n+1}}{a_n} = \max_{1\leq n\leq r} \frac{a_{n+1}}{a_n}.$$

Proof. Let $\alpha := \max_{1\le n\le r} \frac{a_{n+1}}{a_n}$. Since

$$\alpha - \frac{a_{n+1}}{a_n} = \frac{1}{a_n}\sum_{k=1}^{r} A_k a_{n-k}\left(\alpha - \frac{a_{n-k+1}}{a_{n-k}}\right)$$

for all $n > r$ and $\alpha - \frac{a_{k+1}}{a_k} \ge 0$ for all $k \le r$, we have that $\alpha - \frac{a_{n+1}}{a_n} \ge 0$ for all $n \in \mathbb{N}$ by induction. □

Theorem 12. *Let (a_n) be an increasable weighted r-generalized Fibonacci sequence with some initial value equal to 1. Let k be a nonnegative integer and let $b_1 \le b_2 \le \ldots \le b_{k+1}$ be a nondecreasing rearrangement of $a_1, a_2, \ldots, a_{k+1}$. Then, for all $m \in \mathbb{N}$,*

$$L_{[a_n^m]} \le \max(S_k \cup T_k),$$

where $S_k := \left\{\left\lceil \frac{b_{i+1}^m - 1}{\sum_{l=1}^{i} b_l^m}\right\rceil : 1 \le i \le k\right\}$ and $T_k := \left\{\left\lceil \frac{a_{k+i+1}^m}{a_{k+i}^m}\right\rceil - 1 : 1 \le i \le r\right\}$.

Proof. The theorem is obvious when $a_n = 1$ for all $n \in \mathbb{N}$. Then assume that (a_n) is not a constant sequence. Fix $m \in \mathbb{N}$, and let $\rho : \mathbb{N} \to \mathbb{N}$ be a bijection such that $(a_{\rho(n)})$ is nondecreasing. We show that

$$\frac{a_{\rho(n+1)}^m - 1}{\sum_{i=1}^{n} a_{\rho(i)}^m} \le \max(S_k \cup T_k)$$

by induction on $n \in \mathbb{N}$. Notice that if $\rho(1) < k+1$, then $a_{\rho(1)} = b_1$ and $a_{\rho(2)} \le b_2$. So

$$\frac{a_{\rho(2)}^m - 1}{a_{\rho(1)}^m} \le \frac{b_2^m - 1}{b_1^m} \le \max S_k.$$

If $\rho(1) \ge k+1$, then by Lemma 11,

$$\frac{a_{\rho(2)}^m - 1}{a_{\rho(1)}^m} = \frac{a_{\rho(2)}^m}{a_{\rho(1)}^m} - 1 \le \frac{a_{\rho(1)+l}^m}{a_{\rho(1)+l-1}^m} - 1 \le \max T_k,$$

where l is the least positive integer such that $a_{\rho(1)+l} > a_{\rho(1)}$. Now, let $n \in \mathbb{N}$ be such that $\frac{a_{\rho(n+1)}^m - 1}{\sum_{i=1}^{n} a_{\rho(i)}^m} \le \max(S_k \cup T_k)$.

Case 1: Assume that there is no $i > k$ such that $a_{\rho(n+1)} = a_i$ and $a_{\rho(n+2)} = a_{\rho(n+1)}$. Let $i' \le k$ be minimal such that $a_{\rho(n+1)} = b_{i'}$. Then $a_{\rho(n+2)}^m \le b_{i'+1}^m$ and $\sum_{i=1}^{n+1} a_{\rho(i)}^m \ge \sum_{i=1}^{i'} b_i^m$. Thus,

$$\frac{a_{\rho(n+2)}^m - 1}{\sum_{i=1}^{n+1} a_{\rho(i)}^m} \le \frac{b_{i'+1}^m - 1}{\sum_{i=1}^{i'} b_i^m} \le \max S_k.$$

Case 2: Assume that there is no $i > k$ such that $a_{\rho(n+1)} = a_i$ and $a_{\rho(n+2)} > a_{\rho(n+1)}$. Let $i' \leq k$ be maximal such that $a_{\rho(n+1)} = b_{i'}$. Then $a^m_{\rho(n+2)} \leq b^m_{i'+1}$ and $\sum_{i=1}^{n+1} a^m_{\rho(i)} \geq \sum_{i=1}^{i'} b^m_i$. Thus,

$$\frac{a^m_{\rho(n+2)} - 1}{\sum_{i=1}^{n+1} a^m_{\rho(i)}} \leq \frac{b^m_{i'+1} - 1}{\sum_{i=1}^{i'} b^m_i} \leq \max S_k.$$

Case 3: Assume that there exists $i > k$ such that $a_{\rho(n+1)} = a_i$. Then, by Lemma 11,

$$\frac{a^m_{\rho(n+2)} - a^m_{\rho(n+1)}}{a^m_{\rho(n+1)}} = \frac{a^m_{\rho(n+2)}}{a^m_{\rho(n+1)}} - 1 \leq \frac{a^m_{\rho(n+1)+l}}{a^m_{\rho(n+1)+l-1}} - 1 \leq \max T_k,$$

where l is the least positive integer such that $a_{\rho(n+1)+l} > a_{\rho(n+1)}$. After combining this result with the inductive hypothesis, we get

$$\frac{a^m_{\rho(n+2)} - 1}{\sum_{i=1}^{n+1} a^m_{\rho(i)}} = \frac{(a^m_{\rho(n+1)} - 1) + (a^m_{\rho(n+2)} - a^m_{\rho(n+1)})}{\left(\sum_{i=1}^{n} a^m_{\rho(i)}\right) + a^m_{\rho(n+1)}} \leq \max(S_k \cup T_k).$$

□

If we let $k = 0$ in the previous theorem, we will get a nicer upper bound for $L_{[a_n^m]}$.

Corollary 13. *Let (a_n) be an increasable weighted r-generalized Fibonacci sequence with at least one initial value equal to 1. Then, for all $m \in \mathbb{N}$,*

$$L_{[a_n^m]} \leq \max_{1 \leq i \leq r} \left\lceil \left(\frac{a_{i+1}}{a_i}\right)^m \right\rceil - 1.$$

Theorem 12 can also be used to find the exact value of $L_{[a_n^m]}$ as shown in the following corollary and examples.

Corollary 14. *Let (a_n) be an increasable weighted r-generalized Fibonacci sequence with at least one initial value equal to 1. Assume that there exists a rearrangement $b_1 \leq b_2 \leq \ldots \leq b_{k+1}$ of $a_1, a_2, \ldots, a_{k+1}$ for some $k \in \mathbb{N}$ such that $b_{k+1} \leq a_n$ for all $n \geq k+1$, and*

$$\max\left\{\left\lceil \frac{a^m_{k+i+1}}{a^m_{k+i}} \right\rceil - 1 : 1 \leq i \leq r\right\} \leq \max\left\{\left\lceil \frac{b^m_{i+1} - 1}{\sum_{l=1}^{i} b^m_i} \right\rceil : 1 \leq i \leq k\right\}.$$

Then, $L_{[a_n^m]} = \max\left\{\left\lceil \frac{b^m_{i+1} - 1}{\sum_{l=1}^{i} b^m_i} \right\rceil : 1 \leq i \leq k\right\}$.

Example 15. Let (a_n) be a generalized Fibonacci sequence with $r = 2$, $A_1 = A_2 = 1$, and $(c_1, c_2) = (1, c)$ for some $c \geq 2$. Then

$$L_{[a_n^m]} = \left\lceil \frac{a_2^m - 1}{\sum_{i=1}^{1} a_i^m} \right\rceil = c^m - 1$$

for all $m \in \mathbb{N}$.

Example 16. Let (a_n) be a weighted r-generalized Fibonacci sequence with $r = 2$, $A_1 = A_2 = 1$, and $(c_1, c_2) = (c, 1)$ for some $c \geq 2$. If (b_n) is a rearrangement of (a_n) that is increasing, then

$$L_{[a_n^m]} = \left\lceil \frac{b_2^m - 1}{\sum_{i=1}^{1} b_i^m} \right\rceil = c^m - 1$$

for all $m \in \mathbb{N}$. In particular, if $c = 2$, then (a_n) is the Lucas sequence.

Example 17. Let (a_n) be a weighted r-generalized Fibonacci sequence with $r \geq 2$, $A_i = P \in \mathbb{N}$, and $c_i = 1$ for all $i \leq r$. Then

$$L_{[a_n^m]} = \left\lceil \frac{a_{r+1}^m - 1}{\sum_{i=1}^{r} a_i^m} \right\rceil = P^m r^{m-1}$$

for all $m \in \mathbb{N}$. In particular, if $r = 2$ and $P = 1$, then this sequence is the usual Fibonacci sequence.

Proof. Notice that $a_i = 1$ for all $i \leq r$ and $a_{r+j} = (Pr-1)(P+1)^{j-1}+1$ for $j = 1, 2, \ldots, r+1$. We apply Corollary 14 with $k = r$. Since

$$\frac{a_{i+1}}{a_i} \leq \frac{a_{j+1}}{a_j}$$

for $r+1 \leq i \leq j \leq 2r$, it suffices to show that

$$\left\lceil \frac{a_{2r+1}^m}{a_{2r}^m} \right\rceil - 1 \leq \frac{a_{r+1}^m}{\sum_{i=1}^{r} a_i^m},$$

which is equivalent to

$$\left(P + 1 - \frac{P}{(Pr-1)(P+1)^{r-1}+1} \right)^m - 1 \leq P^m r^{m-1},$$

for all $m \in \mathbb{N}$. If $m = 1$, then

$$P + 1 - \frac{P}{(Pr-1)(P+1)^{r-1}+1} - 1 \leq P,$$

and we are done. If $m \geq 2$ and $P \geq 2$, then it is easy to see that $(P+1)^m - 1 \leq P^m r^{m-1}$ and we are done. If $m \geq 2$, $P = 1$ and $r \geq 3$, then

$$\left(2 - \frac{1}{2^{r-1}(r-1)+1} \right)^m < 2^m \leq 3^{m-1} + 1 \leq r^{m-1} + 1.$$

Lastly, if $m \geq 2$, $P = 1$ and $r = 2$, then $\left(\frac{5}{3}\right)^m \leq 2^{m-1} + 1$ by induction on m. $\square$

Example 18. Let (a_n) be a weighted r-generalized Fibonacci sequence with $r = 2$, $A_1 = P$, $A_2 = Q$, and $(c_1, c_2) = (1, P)$ for some $P, Q \in \mathbb{N}$. Then

$$L_{[a_n^m]} = \left\lceil \frac{a_3^m - 1}{\sum_{i=1}^{2} a_i^m} \right\rceil = \left\lceil \frac{(P^2+Q)^m - 1}{P^m + 1} \right\rceil$$

for all $m \in \mathbb{N}$. This sequence is called (P, Q)-Fibonacci sequence [9]. In particular, if $Q = 1$, then this sequence is called Fibonacci P-sequence [3].

Proof. Note that in this case $r = 2$, and a_1, a_2, a_3, a_4, a_5 are

$$1, P, P^2 + Q, P^3 + 2PQ, P^4 + 3P^2Q + Q^2,$$

respectively. First, fix $m \geq 2$ and apply Corollary 14 with $k = 2$. Notice that

$$\left\lceil \frac{a_4^m}{a_3^m} \right\rceil - 1 = \left\lceil \left(\frac{P^3 + 2PQ}{P^2 + Q} \right)^m \right\rceil - 1 \leq \left\lceil \left(\frac{P^4 + 3P^2Q + Q^2}{P^3 + 2PQ} \right)^m \right\rceil - 1 = \left\lceil \frac{a_5^m}{a_4^m} \right\rceil - 1$$

because $(P^3 + 2PQ)^2 \leq (P^4 + 3P^2Q + Q^2)(P^2 + Q)$ by comparing coefficients. Also,

$$\left\lceil \frac{a_2^m - 1}{\sum_{i=1}^{1} a_i^m} \right\rceil = P^m - 1 \leq \left\lceil \frac{(P^2+Q)^m - 1}{P^m + 1} \right\rceil = \left\lceil \frac{a_3^m - 1}{\sum_{i=1}^{2} a_i^m} \right\rceil$$

because $(P^m - 1)(P^m + 1) \leq (P^2 + Q)^m - 1$. Then it is left to show that

$$\left\lceil \frac{a_5^m}{a_4^m} \right\rceil - 1 \leq \left\lceil \frac{a_3^m - 1}{\sum_{i=1}^{2} a_i^m} \right\rceil,$$

which is equivalent to

$$\left\lceil \left(\frac{P^4 + 3P^2Q + Q^2}{P^3 + 2PQ} \right)^m \right\rceil \leq \left\lceil \frac{(P^2+Q)^m + P^m}{P^m + 1} \right\rceil.$$

Case $m \geq 4$: Since $(P^4 + 3P^2Q + Q^2)^4(P^4 + 1) \leq (P^2 + Q)^4(P^3 + 2PQ)^4$ by comparing coefficients and $P^m + 1 \leq (P^4 + 1)^{\frac{m}{4}}$, we have that

$$\left(\frac{P^4 + 3P^2Q + Q^2}{P^3 + 2PQ} \right)^m \leq \frac{(P^2+Q)^m}{P^m + 1} \leq \frac{(P^2+Q)^m + P^m}{P^m + 1}.$$

Case $m = 3$: If $Q \geq 2$, then $(P^4 + 3P^2Q + Q^2)^3(P^3 + 1) \leq ((P^2 + Q)^3 + P^3)(P^3 + 2PQ)^3$ by comparing coefficients and we are done. Now, assume that $Q = 1$. Then

$$\left\lceil \frac{(P^2+1)^3 + P^3}{P^3 + 1} \right\rceil = \left\lceil P^3 + 3P + \frac{3P^2 - 3P + 1}{P^3 + 1} \right\rceil = P^3 + 3P + 1.$$

Since $(P^4+3P^2+1)^3 \le (P^3+3P+1)(P^3+2P)^3$ by comparing coefficients, we are done.

Case $m=2$: If $Q \ge 2$, then $(P^4+3P^2Q+Q^2)^2(P^2+1) \le ((P^2+Q)^2+P^2)(P^3+2PQ)^2$ by comparing coefficients and we are done. Now, assume that $Q=1$. Then

$$\left\lceil \frac{(P^2+1)^2+P^2}{P^2+1} \right\rceil = \left\lceil P^2+1+\frac{P^2}{P^2+1} \right\rceil = P^2+2.$$

Since $(P^4+3P^2+1)^2 \le (P^2+2)(P^3+2P)^2$ by comparing coefficients, we are done.

Lastly, assume that $m=1$. If $P \ge 2$ and $Q \ge P+2$, then $(P^4+3P^2Q+Q^2)(P+1) \le (P^2+Q+P)(P^3+2PQ)$ by comparing coefficients, so we are done in this subcase.

If $P \in \mathbb{N}$ and $Q \le P$, then

$$\frac{P^4+3P^2Q+Q^2}{P^3+2PQ} \le P+1 = \left\lceil \frac{P^2+Q+P}{P+1} \right\rceil .$$

For the last two subcases, we cannot use the same method as in the previous cases to obtain the result since the assumption in the Corollary 14 with $k=2$ does not hold. Instead, if $P \in \mathbb{N}$ and $Q = P+1$, then one can show by induction that

$$\left\lceil \frac{a_3-1}{\sum_{i=1}^{2} a_i} \right\rceil = P = \begin{cases} \frac{a_{n+1}-1}{\sum_{i=1}^{n} a_n}, & \text{if } n \text{ is even;} \\ \frac{a_{n+1}}{\sum_{i=1}^{n} a_n} > \frac{a_{n+1}-1}{\sum_{i=1}^{n} a_n}, & \text{if } n \text{ is odd,} \end{cases}$$

and we are done in this subcase.

If $P=1$ and $Q \ge 3$, then we apply Corollary 14 with $k=4$. Notice that in this subcase, $a_1, a_2, \ldots, a_7$ are

$$1, 1, Q+1, 2Q+1, Q^2+3Q+1, 3Q^2+4Q+1, Q^3+6Q^2+5Q+1,$$

respectively. Then

$$\left\lceil \frac{a_6}{a_5} \right\rceil - 1 = \left\lceil \frac{3Q^2+4Q+1}{Q^2+3Q+1} \right\rceil - 1 \le \left\lceil \frac{Q^3+6Q^2+5Q+1}{3Q^2+4Q+1} \right\rceil - 1 = \left\lceil \frac{a_7}{a_6} \right\rceil - 1.$$

Also,

$$\left\lceil \frac{(2Q+1)-1}{1+1+(Q+1)} \right\rceil \le \left\lceil \frac{(Q^2+3Q+1)-1}{1+1+(Q+1)+(2Q+1)} \right\rceil \le \left\lceil \frac{(Q+1)-1}{1+1} \right\rceil,$$

that is,

$$\left\lceil \frac{a_4-1}{\sum_{i=1}^{3} a_i} \right\rceil \le \left\lceil \frac{a_5-1}{\sum_{i=1}^{4} a_i} \right\rceil \le \left\lceil \frac{a_3-1}{\sum_{i=1}^{2} a_i} \right\rceil .$$

Since $\frac{a_7}{a_6} - 1 = \frac{Q^3+6Q^2+5Q+1}{3Q^2+4Q+1} - 1 \le \frac{(Q+1)-1}{1+1} = \frac{a_3-1}{\sum_{i=1}^{2} a_i}$, we are done. □

From the previous example, we see that it is difficult to obtain $L_{[a_n^m]}$ when m is small. However, for a large value of m, we have the following result.

Theorem 19. *Let (a_n) be a weighted r-generalized Fibonacci sequence. Assume that there is only one $k_0 \in \{1, 2, \ldots, 2r\}$ such that $\frac{a_{k_0+1}}{a_{k_0}} = \max_{1\le k\le 2r} \frac{a_{k+1}}{a_k}$. If $a_k \le a_{k_0}$ for all $k \le k_0$, then there exists $M \in \mathbb{N}$ such that*

$$L_{(a_n^m)} = \left\lceil \frac{a_{k_0+1}^m - 1}{\sum_{k=1}^{k_0} a_k^m} \right\rceil$$

for all $m \ge M$.

Proof. Since $a_{k_0+1} > 1$ and $a_i \le a_{k_0}$ for all $i \le k_0$, there exists $\delta > 0$ such that for all $m \in \mathbb{N}$,

$$\frac{a_{k_0+1}^m - 1}{\sum_{i=1}^{k_0} a_i^m} = \left(\frac{1 - \frac{1}{a_{k_0+1}^m}}{\sum_{i=1}^{k_0} \frac{a_i^m}{a_{k_0}^m}} \right) \frac{a_{k_0+1}^m}{a_{k_0}^m} \ge \delta \cdot \frac{a_{k_0+1}^m}{a_{k_0}^m}.$$

Let $k' \in \mathbb{N}$ be such that $\frac{a_{k'+1}}{a_{k'}} = \max_{\substack{1\le k\le 2r \\ k\ne k_0}} \frac{a_{k+1}}{a_k}$. Then, by Lemma 11, for all $m, k \in \mathbb{N}$ with $k \ne k_0$,

$$\frac{a_{k'+1}^m}{a_{k'}^m} \ge \frac{a_{k+1}^m}{a_k^m} \ge \frac{a_{k+1}^m - 1}{\sum_{i=1}^{k} a_i^m}.$$

Since $\frac{a_{k_0+1}}{a_{k_0}} > \frac{a_{k'+1}}{a_{k'}}$, there exists $M \in \mathbb{N}$ such that, for each $m \ge M$,

$$\frac{a_{k_0+1}^m - 1}{\sum_{i=1}^{k_0} a_i^m} \ge \delta \cdot \frac{a_{k_0+1}^m}{a_{k_0}^m} \ge \frac{a_{k'+1}^m}{a_{k'}^m} \ge \frac{a_{k+1}^m - 1}{\sum_{i=1}^{k} a_i^m}$$

for all $k \in \mathbb{N}$. □

We close this section by providing a lower bound for $L_{[a_n^m]}$ related to the dominant root, when (a_n) is in a primitive form.

Theorem 20. *Let (a_n) be a weighted r-generalized Fibonacci sequence in a primitive form, and λ the dominant root of its characteristic polynomial. Then $L_{[a_n^m]} \ge \lceil \lambda^m \rceil - 1$ for all $m \in \mathbb{N}$.*

Proof. The theorem is obvious when $\lambda = 1$. Now, assume that $\lambda > 1$ and fix $m \in \mathbb{N}$. By Proposition 3, there exists $C > 0$ such that $\sum_{k=1}^{n} a_k^m = C\lambda^{nm} + o(\lambda^{nm})$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}^m - 1}{\sum_{k=1}^{n} a_k^m} + 1 = \lim_{n\to\infty} \frac{C\lambda^{(n+1)m} + o(\lambda^{nm})}{C\lambda^{nm} + o(\lambda^{nm})} = \lambda^m.$$

□

5 Generalized distinguished sequences

Before we define the distinguished sequences, let us present some important tools to study a weighted r-generalized Fibonacci sequence that is not in a primitive form. The following result tells us that, for any weighted r-generalized Fibonacci sequence (a_n), the multiset $\{a_n : n \in \mathbb{N}\}$ can be decomposed into a disjoint union of $\{b_n^{(i)} : n \in \mathbb{N}\}$'s, where $(b_n^{(i)})$ is a weighted r-generalized Fibonacci sequence in a primitive form for all i.

Proposition 21. *Let (a_n) be a weighted r-generalized Fibonacci sequence. We also let $d := \gcd\{k : A_k > 0\}$. For $1 \leq i \leq d$, let $b_n^{(i)} := a_{(n-1)d+i}$ for all $n \in \mathbb{N}$. Then the sequences $(b_n^{(1)}), (b_n^{(2)}), \ldots, (b_n^{(d)})$ satisfy a common recurrence relation that is in a primitive form, and*

$$[a_n] = [b_n^{(1)}] \oplus [b_n^{(2)}] \oplus \cdots \oplus [b_n^{(d)}].$$

Proof. Clearly, $[a_n] = [b_n^{(1)}] \oplus [b_n^{(2)}] \oplus \ldots \oplus [b_n^{(d)}]$. Notice that, for $1 \leq i \leq d$ and $n > \frac{r}{d}$,

$$b_n^{(i)} = \sum_{k=1}^{\frac{r}{d}} A_{kd} b_{n-k}^{(i)},$$

and $\gcd\{k : A_{kd} > 0\} = 1$. Hence, $(b_n^{(i)})$ is in a primitive form for all i. □

By the previous result and Proposition 3, we have the following consequence.

Corollary 22. *Let (a_n) be a weighted r-generalized Fibonacci sequence and $d := \gcd\{k : A_k > 0\}$. We also let $b_n^{(i)}$ be as in Proposition 21, for all $1 \leq i \leq d$, and $\lambda > 1$ be the dominant root of the characteristic polynomial of $(b_n^{(1)})$. Then there exist $C_1, C_2, \ldots, C_d \in [1, \lambda)$ and $k_1, k_2, \ldots, k_d \in \mathbb{Z}$ such that*

$$a_{nd+i} = C_i \lambda^{n+k_i} + o(\lambda^n) \text{ for all } i \in \{1, 2, \ldots, d\}, \text{ as } n \to \infty.$$

From the previous section, we have obtained complete sequences of weighted r-generalized Fibonacci mth powers. However, these sequences may not be minimal. Then, in this section, we will study a particular minimal sequence introduced by Hunsucker and Wardlaw in [6], called a distinguished sequence, of a weighted r-generalized Fibonacci power.

Definition 23. A *distinguished sequence* of a sequence of positive integers (s_n) is a non-decreasing complete sequence (d_n) that maps $\mathbb{N}$ onto $\{s_n : n \in \mathbb{N}\}$, and for all $n \in \mathbb{N}$, $1 + \sum_{k=1}^{n-1} d_k < d_{n+1}$ whenever $d_n < d_{n+1}$.

Although the distinguished sequence of Fibonacci mth power uniquely exists for each $m \in \mathbb{N}$ [6, Section 4], the distinguished sequence of mth power of a weighted r-generalized Fibonacci (a_n^m) may not exist for all $m \in \mathbb{N}$.

Example 24. Let (a_n) be such that $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$ and $a_n = a_{n-4}$ for all $n \geq 5$. Then the distinguished sequence of (a_n) does not exist since $1 + a_1 + a_2 = a_4$.

Although the distinguished sequence of a weighted r-generalized Fibonacci power may not exist in general, the distinguished sequence of its mth power uniquely exists when the weighted r-generalized Fibonacci sequence is in a primitive form and m is large enough.

Lemma 25. *Let (s_n) be a strictly increasing sequence of positive integers with $s_1 = 1$ such that $s_n + s_{n+1} \leq s_{n+2}$ for all $n \in \mathbb{N}$. Then there exists a unique distinguished sequence of (s_n).*

Proof. Construct a distinguished sequence (d_n) from (s_n) as follows: For each $i \in \mathbb{N}$, start from $i = 1$, add multiple s_i's to the sequence (d_n) so that

(1) (d_n) is nondecreasing,

(2) $1 + \sum_{\substack{k\in\mathbb{N} \\ d_k \leq s_i}} d_k \geq s_{i+1}$, and

(3) $1 - s_i + \sum_{\substack{k\in\mathbb{N} \\ d_k \leq s_i}} d_k < s_{i+1}$.

Since $s_n + s_{n+1} \leq s_{n+2}$ for all $n \in \mathbb{N}$, the sequence (d_n) can be always constructed so that condition (3) is satisfied. Moreover, the uniqueness of the distinguished sequence results from the properties (2) and (3) of (d_n). □

Theorem 26. *Let (a_n) be an increasable weighted r-generalized Fibonacci sequence in a primitive form with at least one initial value equal to 1. Then there exists $M \in \mathbb{N}$ such that the distinguished sequence of (a_n^m) uniquely exists for all $m \geq M$.*

Proof. Let $\lambda \geq 1$ be the dominant root of the characteristic polynomial of (a_n). If $\lambda = 1$, then we are done because $a_n = 1$ for all $n \in \mathbb{N}$. Now, assume that $\lambda > 1$. Let (b_n) be a rearrangement of (a_n) such that (b_n) is nondecreasing. Since $a_n = C\lambda^n + o(\lambda^n)$ for some $C > 0$ and all large $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $a_n = b_n$ for all $n \geq N$. Hence, there exists $M \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, if $b_{n+2} > b_{n+1}$, then

$$b_n^m + b_{n+1}^m \leq b_{n+2}^m \text{ for all } m \geq M$$

because $\lambda > 1$ and $\lim_{n\to\infty} \frac{b_n^m + b_{n+1}^m}{b_{n+2}^m} = \left(\frac{1}{\lambda^{2m}} + \frac{1}{\lambda^m}\right)$ for all $m \in \mathbb{N}$. Let (s_n) be a strictly increasing sequence that maps onto $\{b_n^m : n \in \mathbb{N}\}$. By Lemma 25, we obtain a unique distinguished sequence of (s_n) and we are done. □

The previous theorem does not hold for weighted r-generalized Fibonacci sequences that are not in a primitive form in general.

Example 27. Let $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 5, a_6 = 4, a_7 = 3, a_8 = 2$, and $a_n = a_{n-4} + 2a_{n-8}$ for all $n > 8$. Then, for all $n \in \mathbb{N}$,

$$a_{4(n-1)+i} = \begin{cases} 2^n + (-1)^n, & \text{if } i = 1; \\ 2^n, & \text{if } i = 2; \\ 2^n - (-1)^n, & \text{if } i = 3; \\ 2^n - 2(-1)^n, & \text{if } i = 4. \end{cases}$$

Let $m \in \mathbb{N}$. Notice that in any nondecreasing rearrangement (b_n) of (a_n), there are infinitely many four consecutive terms $b_l < b_{l+1} < b_{l+2} < b_{l+3}$ in the form of $a_{4(2n-1)+4} < a_{4(2n-1)+3} < a_{4(2n-1)+2} < a_{4(2n-1)+1}$, for some $n \in \mathbb{N}$, such that

$$b_l^m + b_{l+1}^m = (2^{2n} - 2)^m + (2^{2n} - 1)^m \geq (2^{2n} + 1)^m = b_{l+3}^m.$$

Hence, there is no distinguished sequence of (a_n^m).

The next theorem provides information on the repetition of a_n^m in its distinguished sequence for large $m, n \in \mathbb{N}$.

Theorem 28. *Let (a_n) be an increasable weighted r-generalized Fibonacci sequence in a primitive form, M be as in the previous theorem, $m \geq M$, and (d_n) be the distinguished sequence of (a_n^m). We also let s be the greatest integer less than or equal to λ^m, where $\lambda := \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. If $\lambda > 1$, then there exists N such that, for each $n \geq N$, $d_k = a_n^m$ for exactly s or $s - 1$ values of k. In particular, if $\lambda = 1$, then $d_n = 1$ for all $n \in \mathbb{N}$.*

Proof. Assume that $\lambda > 1$. Then, by the definition of limits, there exists $N \in \mathbb{N}$ such that $s - \frac{\lambda-1}{2\lambda} < \left(\frac{a_{n+1}}{a_n}\right)^m < s + 1$ and $\left(\frac{a_{n-1}}{a_n}\right)^m < \frac{1}{\lambda} + \frac{\lambda-1}{2\lambda}$ for all $n \geq N$. Fix $n \geq N$ and let $n_0 := \min\{k : d_k = a_n^m\}$. We show that

$$1 + \sum_{k=1}^{n_0-1} d_k + (s-2)a_n^m < a_{n+1}^m \leq 1 + \sum_{k=1}^{n_0-1} d_k + sa_n^m.$$

Since (d_n) is a distinguished sequence,

$$1 + \sum_{k=1}^{n_0-1} d_k < a_{n-1}^m + a_n^m.$$

So

$$1 + \sum_{k=1}^{n_0-1} d_k + (s-2)a_n^m < a_{n-1}^m + (s-1)a_n^m < \left(s - \frac{\lambda-1}{2\lambda}\right) a_n^m < a_{n+1}^m.$$

Also, notice that

$$a_{n+1}^m < (1+s)a_n^m \leq 1 + \sum_{k=1}^{n_0-1} d_k + sa_n^m.$$

□

In general, Theorem 26 does not hold for weighted r-generalized Fibonacci sequences that are not in a primitive form. However, it does hold for weighted r-generalized Fibonacci sequences satisfying some particular conditions as shown in the following theorem and example.

Theorem 29. *Let (a_n) be an increasable weighted r-generalized Fibonacci sequence with $a_i = 1$ for some $i \le r$. Assume that $[a_n] = [b_n^{(1)}] \oplus [b_n^{(2)}] \oplus \ldots \oplus [b_n^{(d)}]$, where $d \ge 2$ and $(b_n^{(k)})$ is as in Proposition 21 for all $k \le d$. Let $\lambda > 1$ be the dominant root of the characteristic polynomial of $(b_n^{(1)})$. If $\lim_{n\to\infty} \log_\lambda \left(\frac{b_n^{(p)}}{b_n^{(q)}}\right) \notin \mathbb{Z}$ for all $1 \le p < q \le d$, then for all but finitely many $m \in \mathbb{N}$, there exists a unique distinguished sequence of (a_n^m).*

Proof. Assume that $\lim_{n\to\infty} \log_\lambda \left(\frac{b_n^{(p)}}{b_n^{(q)}}\right) \notin \mathbb{Z}$ for all $1 \le p < q \le s$. Then, by Corollary 22, there exist $k_1, k_2, \ldots, k_d \in \mathbb{Z}$ and distinct numbers $C_1, C_2, \ldots, C_d \in [1, \lambda)$ such that

$$a_{nd+i} = C_i \lambda^{n+k_i} + o(\lambda^n) \text{ for all } i \in \{1, 2, \ldots, d\}, \text{ as } n \to \infty.$$

Without loss of generality, we may assume that $1 \le C_1 < C_2 < \ldots < C_d < \lambda$. Let (b_n) be a nondecreasing rearrangement of (a_n). Then there exists $k \in \mathbb{Z}$ such that, for each $i \in \{1, 2, \ldots, d\}$, we have that $b_{nd+i} = C_i\lambda^{n+k} + o(\lambda^n)$ as $n \to \infty$. Since $1 \le C_1 < C_2 < \ldots < C_d < \lambda$, there exist $\alpha < 1$ and $N \in \mathbb{N}$ such that $\frac{b_n}{b_{n+1}} \le \alpha$ for all $n \ge N$. Thus, there exists $M \in \mathbb{N}$ such that whenever $b_{n+2} > b_{n+1}$ for $n \in \mathbb{N}$ and $b_n^m + b_{n+1}^m \le b_{n+2}^m$ for all $m \ge M$. Let (s_n) be a strictly increasing sequence that maps onto $\{b_n^m : n \in \mathbb{N}\}$. By Lemma 25, we obtain a unique distinguished sequence of (s_n) and we are done. □

Example 30. Let $a_1 = 2$, $a_2 = a_3 = a_4 = 1$, and $a_n = a_{n-2} + a_{n-4}$ for all $n > 4$. Then $[a_n] = [L_n] \oplus [F_n]$, where (L_n) is the sequence of Lucas numbers and (F_n) is the Fibonacci sequence. Notice that the dominant root of the characteristic polynomial of (L_n) is φ, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. It is also well-known that, for all $n \in \mathbb{N}$,

$$L_n = \varphi^{n-1} + (1-\varphi)^{n-1} \text{ and } F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}.$$

Thus

$$\lim_{n\to\infty} \log_\varphi \left(\frac{L_n}{F_n}\right) = \log_\varphi(\sqrt{5}) - 1 \notin \mathbb{Z}.$$

By Theorem 29, there exists a unique distinguished sequence of (a_n^m) for all but finitely many $m \in \mathbb{N}$.

The converse of Theorem 29 does not hold in general. For instance, it does not hold for any sequence that can be decomposed into multiple identical sequences as shown in the next example.

Example 31. Let $a_1 = 1$, $a_2 = 1$, and $a_n = 2a_{n-2}$ for all $n > 2$. Then $[a_n] = [b_n^{(1)}] \oplus [b_n^{(2)}]$, where $b_1^{(i)} = 1$ and $b_n^{(i)} = 2b_{n-1}^{(i)}$, for all $i \in \{1, 2\}$ and $n > 1$. Notice that the dominant root of the characteristic polynomial of $(b_n^{(1)})$ is $\lambda = 2$. Moreover,

$$\lim_{n\to\infty} \log_2 \left(\frac{b_n^{(1)}}{b_n^{(2)}} \right) = 0 \in \mathbb{Z}.$$

However, since $\{a_n : n \in \mathbb{N}\} = \{b_n^{(1)} : n \in \mathbb{N}\}$, by using Theorem 26 with the sequence $(b_n^{(1)})$, there exists a unique distinguished sequence of (a_n^m) for all but finitely many $m \in \mathbb{N}$.

As we see in Example 27, it may happen that the distinguished sequence of the mth powers of a weighted r-generalized Fibonacci does not exist for any $m \in \mathbb{N}$. However, if we generalize Definition 23 as in the following definition, then this generalized distinguished sequence of the weighted r-generalized Fibonacci mth power exists for all but finitely many $m \in \mathbb{N}$.

Definition 32. A *generalized distinguished sequence* of a sequence of positive integers (s_n) is a nondecreasing complete sequence (d_n) that is minimal and maps $\mathbb{N}$ onto $\{s_n : n \in \mathbb{N}\}$.

Theorem 33. *Let (a_n) be an increasable weighted r-generalized Fibonacci sequence with at least one initial value equal to 1. If (a_n) is not a constant sequence, then there exists a generalized distinguished sequence of (a_n^m) for all large $m \in \mathbb{N}$.*

Proof. Assume that (a_n) is not a constant sequence. Then, by Corollary 22, there exist $\lambda > 1$, $C_1, C_2, \ldots, C_d \in [1, \lambda)$, and $k_1, k_2, \ldots, k_d \in \mathbb{Z}$ such that

$$a_{nd+i} = C_i \lambda^{n+k_i} + o(\lambda^n) \text{ for all } i \in \{1, 2, \ldots, d\}, \text{ as } n \to \infty.$$

Without loss of generality, we may assume that $1 \le C_1 \le C_2 \le \ldots \le C_d < \lambda$. Let (b_n) be a nondecreasing rearrangement of (a_n). Then there exists $k \in \mathbb{Z}$ such that, for all $i \in \{1, 2, \ldots, d\}$, we have $b_{nd+i} = C_i \lambda^{n+k} + o(\lambda^n)$ as $n \to \infty$. Since $\frac{C_d}{C_1 \lambda} < 1$ and

$$\lim_{n\to\infty} \frac{(d+1) b_{nd+d}^m}{b_{(n+1)d+1}^m} = (d+1) \left(\frac{C_d}{C_1 \lambda} \right)^m \text{ for all } m \in \mathbb{N},$$

there exist $M, N \in \mathbb{N}$ such that, for all $m \ge M$

- $b_{Nd}^m < b_{Nd+1}^m$,
- if $n \ge N$, then $(d+1) b_{nd+d}^m \le b_{(n+1)d+1}^m$, and
- if $n \le Nd + 1$ and $b_{n-1} < b_n$, then $b_{n-2}^m + b_{n-1}^m \le b_n^m$.

Now, fix $m \ge M$. Let (s_n) be a strictly increasing finite sequence that maps onto $\{b_n^m : n \le Nd + 1\}$. Similarly to Lemma 25, we can construct a finite nondecreasing sequence $d_1, d_2, \ldots, d_l \in \{b_n^m : n \le Nd + 1\}$ such that

- $d_l = b^m_{Nd+1}$,
- $\{d_1, d_2, \dots, d_{l-1}\} = \{b^m_n : n \le Nd\}$,
- $1 + \sum_{i=1}^{j} d_i \ge d_{j+1}$ for all $j < l$, and
- $1 + \sum_{i=1}^{j-1} d_i < d_{j+1}$ for all $j < l$ with $d_j < d_{j+1}$.

Next, let us append $b^m_{Nd+1}, b^m_{Nd+2}, \dots, b^m_{(N+1)d+1}$ to our sequence (d_n). Notice that $1 + \sum_{i=1}^{l-1} d_i < d_l + d_{l-1} \le 2b^m_{Nd+d}$. Thus,

$$b^m_{(N+1)d+1} - \left(1 + \sum_{i=1}^{l-1} d_i + \sum_{i=Nd+2}^{Nd+d} b^m_i\right) > b^m_{(N+1)d+1} - (d+1)b^m_{Nd+d} \ge 0.$$

Let $p := \left\lceil \frac{1}{b^m_{Nd+1}} \left(b^m_{(N+1)d+1} - \left(1 + \sum_{i=1}^{l-1} d_i + \sum_{i=Nd+2}^{Nd+d} b^m_i\right)\right)\right\rceil \in \mathbb{N}$. Define $d_{l+i} := b^m_{Nd+1}$ for all $i \in \{1, \dots, p-1\}$, and $d_{l+p+j-2} := b^m_{Nd+j}$ for all $j \in \{2, 3, \dots, d+1\}$. Then

$$\begin{aligned} 1 + \left(\sum_{i=1}^{l+p+d-2} d_i\right) - d_j &\le 1 + \left(\sum_{i=1}^{l+p+d-2} d_i\right) - b^m_{Nd+1} \\ &= (p-1)b^m_{Nd+1} + 1 + \sum_{i=1}^{l-1} d_i + \sum_{i=Nd+2}^{Nd+d} b^m_i \\ &< b^m_{(N+1)d+1} \\ &= d_{l+p+d-1} \end{aligned}$$

for $l \le j < l + p + d - 1$. For $j \in \{l, l+1, \dots, l+p-2\}$, since $d_j = d_{j+1}$, we have that

$$1 + \sum_{i=1}^{j} d_i \ge d_{j+1}.$$

Moreover, for all $j \in \{l+p-1, l+p, \dots, l+p+d-3\}$,

$$\begin{aligned} 1 + \sum_{i=1}^{j} d_i &= 1 + pb^m_{Nd+1} + \sum_{i=1}^{l-1} d_i + \sum_{i=Nd+2}^{Nd+j-l-p+2} b^m_i \\ &\ge b^m_{(N+1)d+1} - \sum_{i=Nd+j-l-p+3}^{Nd+d} b^m_i \\ &\ge b^m_{(N+1)d+1} - (d-1)b^m_{Nd+d} \\ &= b^m_{(N+1)d+1} - db^m_{Nd+d} + b^m_{Nd+d} \\ &\ge b^m_{Nd+d} \\ &\ge d_{j+1}, \end{aligned}$$

and

$$1 + \sum_{i=1}^{l+p+d-2} d_i = 1 + pb^m_{Nd+1} + \sum_{i=1}^{l-1} d_i + \sum_{i=Nd+2}^{Nd+d} b^m_i \geq b^m_{(N+1)d+1} = d_{l+p+d-1}.$$

After this process, we have that $b^m_{Nd+1}, b^m_{Nd+2}, \ldots, b^m_{(N+1)d+1} \in \{d_n\}$. After doing this process for $b^m_{nd+1}, b^m_{nd+2}, \ldots, b^m_{(n+1)d+1}$, for all $n \geq N+1$, we obtain a generalized distinguished sequence (d_n) for (a^m_n). □

We illustrate the proof of the previous theorem by the following example.

Example 34. Let (a_n) be as in Example 27 and let (b_n) be its nondecreasing rearrangement, which is

$$1, 2, 2, 3, 3, 4, 4, 5\ ,\ 7, 8, 9, 10\ ,\ 14, 15, 16, 17\ ,\ 31, 32, 33, 34\ ,\ \ldots.$$

According to the proof, $d = 4$, $\lambda = 2$, and $C_1 = C_2 = C_3 = C_4 = 1$. Let $N = 2$ and $M = 5$. Notice that for all $m \geq 5$,

- $b^m_8 < b^m_9$,
- if $n \geq 2$, then $5 \cdot b^m_{4n+4} \leq b^m_{4(n+1)+1}$, and
- if $n \leq 9$ and $b_{n-1} < b_n$, then $b^m_{n-2} + b^m_{n-1} \leq b^m_n$.

Now, fix $m = 5$. We show that there exists a generalized distinguished sequence of (a^5_n).

As in the proof of Theorem 26, there is a finite nondecreasing sequence $d_1, d_2, \ldots, d_{50} \in \{1^5, 2^5, 3^5, 4^5, 5^5, 7^5\}$ with the number of $1^5, 2^5, 3^5, 4^5, 5^5, 7^5$ equal to $31, 7, 4, 2, 5, 1$, respectively, such that

- $1 + \sum_{i=1}^{j} d_i \geq d_{j+1}$ for all $j < 50$ and
- $1 + \sum_{i=1}^{j-1} d_i < d_{j+1}$ for all $j < 50$ with $d_j < d_{j+1}$.

Next, we add $7^5, 8^5, 9^5, 10^5$ to (d_n). In this case, according to the proof,

$$\begin{aligned} p &= \left\lceil \frac{1}{7^5} \left(14^5 - \left(1 + (31 \cdot 1^5 + 7 \cdot 2^5 + 4 \cdot 3^5 + 2 \cdot 4^5 + 5 \cdot 5^5) + (8^5 + 9^5 + 10^5)\right)\right) \right\rceil \\ &= 20. \end{aligned}$$

Then we define $d_{51}, d_{52}, \ldots, d_{69}$ to be 7^5, and define $d_{70}, d_{71}, d_{72}, d_{73}$ to be $8^5, 9^5, 10^5, 14^5$ respectively.

If we repeat the process in the previous paragraph with each of the following 4 consecutive terms, we will obtain a generalized distinguished sequence for (a^5_n).

6 Acknowledgments

The authors wish to thank Department of Mathematics, Faculty of Science, Silpakorn University. We would also like to thank anonymous reviewers for valuable and constructive comments and suggestions.

References

[1] J. L. Brown, Jr., Note on complete sequences of integers, *Amer. Math. Monthly* **68** (1961), 557–560.

[2] F. Dubeau, W. Motta, M. Rachidi, and O. Saeki, On weighted r-generalized Fibonacci sequences, *Fibonacci Quart.* **35** (1997), 102–110.

[3] S. Falcón and Á. Plaza, On the Fibonacci k-numbers, *Chaos Solitons Fractals* **32** (2007), 1615–1624.

[4] A. Fiorenza and G. Vincenzi, Limit of ratio of consecutive terms for general order-k linear homogeneous recurrences with constant coefficients, *Chaos Solitons Fractals* **44** (2011), 145–152.

[5] V. E. Hoggatt, Jr. and C. King, Problem E1424, *Amer. Math. Monthly* **67** (1960), 593.

[6] J. L. Hunsucker and W. P. Wardlaw, Complete sequences of Fibonacci powers, *Fibonacci Quart.* **11** (1973), 387–394.

[7] R. O'Connell, Representations of integers as sums of Fibonacci squares, *Fibonacci Quart.* **10** (1972), 103–111.

[8] A. M. Ostrowski, *Solution of Equations in Euclidean and Banach Spaces*, Academic Press, 1973.

[9] A. Suvarnamani and M. Tatong, Some properties of (p, q)-Fibonacci numbers, *Sci. & Tech. RMUTT J.* **5** (2015), 17–21.

2020 *Mathematics Subject Classification*: Primary 11B39; Secondary 11B37.
Keywords: weighted r-generalized Fibonacci sequence, complete sequence, Fibonacci power, generalized distinguished sequence.

(Concerned with sequences A000045, A000032, and A007598.)

Received September 13 2020; revised versions received September 14 2020; February 10 2021; February 26 2021. Published in *Journal of Integer Sequences*, April 25 2021.

Return to Journal of Integer Sequences home page.