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Quasi-Injectivity of Some Arithmetic Functions

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Abstract

Let a and b be positive integers and f an arithmetic function. In this article, we investigate whether or not a certain condition on the value of f implies $a = b$. For example, if f is the sum of divisors function and $f(an) = f(bn)$ for all positive integers n, then $a = b$.

1 Introduction

Let φ be the Euler function, which counts the number of positive integers $k \le n$ with $(k, n) = 1$. For each nonnegative integer s and a positive integer n, let $\sigma_s(n) = \sum_{d|n} d^s$, where d runs over the positive divisors of n, $\sigma(n) = \sigma_1(n)$, and $\tau(n) = \sigma_0(n)$. Problems on the ranges of arithmetic functions have been a popular area of research. For example, it is easy to see that if n is a prime, then $\varphi(n) = n-1$; Lehmer asked whether $\varphi(n) \mid n-1$ implies that n is a prime, but this question is still open. In addition, Carmichael's longstanding open problem on the range of φ states that if $\varphi(x) = n$, then there exists $y \in \mathbb{N}$ distinct from x such that $\varphi(y) = n$ too. Moreover, whether or not there are infinitely many $n \in \mathbb{N}$ with $\sigma(n) = 2n$ has been an open question for a very long time.

Many mathematicians have worked on these problems and made some progress. For example, Pomerance [7] obtained a result concerning the maximal order of $A(n)$, where $A(n)$ is the number of solutions to $\varphi(x) = n$. Ford [1] gave a comprehensive study of the range of φ including $A(n)$. In particular, Ford [2] solved Sierpiński's conjecture, and partially solved Carmichael's problem stated above. That is, Ford showed that for each integer $k \geq 2$, there exists a positive integer n for which the equation $\varphi(x) = n$ has exactly k solutions. Furthermore, Ford, Luca, and Pomerance [3] completely answered Erdös' question on the range of φ and σ by showing that $\varphi(x) = \sigma(y)$ has infinitely many solutions in $x, y \in \mathbb{N}$. Ford and Pollack [4] also gave a result complementary to that of Ford, Luca, and Pomerance [3]. For more information on the range of φ and σ, we refer the reader to the sequences A000010, A007617, and A000396 in OEIS [10]. Finally, in a recent Thailand Online Mathematical Olympiad TOMO 2021, an interesting arithmetic problem [9] was to show that if a, $b \in \mathbb{N}$ and $\tau(\tau(an)) = \tau(\tau(bn))$ for all $n \in \mathbb{N}$, then $a = b$. This naturally suggests various generalizations, where τ may be replaced by σ_s or by other arithmetic functions.

Definition 1. We call a function $f : \mathbb{N} \to \mathbb{C}$ a quasi-injective function if for all a, $b \in \mathbb{N}$, the condition $f(an) = f(bn)$ for all $n \in \mathbb{N}$ implies $a = b$. In addition, if $f : \mathbb{N} \to \mathbb{N}$ and $\ell \in \mathbb{N}$, then we say that f is quasi-injective of order ℓ if $f, f^{(2)}, f^{(3)}, \ldots, f^{(\ell)}$ are quasi-injective, that is, for any a, b, $k \in \mathbb{N}$ with $1 \leq k \leq \ell$,

$$\text{if } f^{(k)}(an) = f^{(k)}(bn) \text{ for all } n \in \mathbb{N}, \text{ then } a = b. \tag{1}$$

In the above definition and throughout this article, if $k \in \mathbb{N}$, then $f^{(k)}$ is the k-fold composition of f. In addition, $\mathbb{N}$ is the set of positive integers but we may need to replace $\mathbb{N}$ by the set $\mathbb{N}_0$ of nonnegative integers. Therefore, the problem in TOMO 2021 mentioned above asks to show that (1) holds when $k = 2$ and $f = \tau$. In fact, (1) also holds when $k = 1$ and $f = \tau$. So τ is quasi-injective of order 2. In general, if f is quasi-injective of order ℓ, then it is also quasi-injective of order $m \leq \ell$. For the concept of quasi-injectivity in algebra, see, for example, the articles by Yavari [11] and Yavari and Ebrahimi [12].

In this article, we study quasi-injectivity of f when $f = \sigma_s$ and other popular arithmetic functions such as the Euler totient function, the Jordan totient function, functions counting prime divisors, the Möbius function, and the Alladi-Erdös function. We also give some open problems at the end of this paper.

2 Preliminaries and lemmas

In this section, we recall some basic terminologies and give some useful results for the reader's convenience. From this point on, p is always a prime, s and n are positive integers, μ is the Möbius function, φ is the Euler totient function, $\omega(n)$ is the number of distinct prime factors of n, $\Omega(n)$ is the number of prime divisors of n counted with multiplicity, and J_s is the Jordan totient function. So $J_s(n)$ is the number of s-tuples $(a_1, a_2, \ldots, a_s)$ such that $1 \leq a_i \leq n$ for all $i = 1, 2, \ldots, s$ and $(a_1, a_2, \ldots, a_s, n) = 1$. Therefore $\varphi = J_1$. Furthermore, it is well known that

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \quad \text{and} \quad J_s(n) = n^s \prod_{p|n} \left(1 - \frac{1}{p^s}\right).$$

An arithmetic function f is called multiplicative if f is not the zero function and $f(mn) = f(m)f(n)$ for all m, $n \in \mathbb{N}$ with $(m,n) = 1$; and f is called additive if $f(mn) = f(m) + f(n)$ for all m, $n \in \mathbb{N}$ with $(m,n) = 1$. In addition, $\mu(1) = 1$, $\mu(n) = (-1)^{\omega(n)}$ if n is squarefree, and $\mu(n) = 0$ if n is not squarefree. It is well known that the functions τ, σ_s, μ, and J_s are multiplicative, while ω and Ω are additive. Furthermore,

$$\text{if } c \in \mathbb{N}, \text{ then } \sigma_s(p^c) = \frac{p^{(c+1)s} - 1}{p^s - 1}.$$

For more details about this, see for instance, the books by Hardy and Wright [5] and McCarthy [6].

Recall that the p-adic valuation of n, denoted by $v_p(n)$, is the exponent of p in the prime factorization of n. A useful formula for $v_p(x^n - y^n)$, sometimes called the "Lifting the Exponent Lemma", is well known and popular among students taking mathematical Olympiad exams. For the proof, see, for example, [8, pp. 14–15].

Lemma 2 ("Lifting the Exponent Lemma")**.** *Let x, $y \in \mathbb{Z}$, $n \in \mathbb{N}$, and let p be a prime such that $p \nmid x$ and $p \nmid y$. Then the following statements hold:*

(i) *If p is odd and $p \mid x - y$, then $v_p(x^n - y^n) = v_p(x - y) + v_p(n)$.*

(ii) *If p is odd, n is odd, and $p \mid x + y$, then $v_p(x^n + y^n) = v_p(x + y) + v_p(n)$.*

When $p = 2$ (so x and y are odd integers), we have

(iii) *if n is odd, then $v_2(x^n - y^n) = v_2(x - y)$ and $v_2(x^n + y^n) = v_2(x + y)$;*

(iv) *if n is even, then $v_2(x^n - y^n) = v_2(x^2 - y^2) + v_2(n) - 1$.*

The conditions (A), (B), (C) in the following lemma are used throughout this article.

Lemma 3. *Let a, $b \in \mathbb{N}$ and let $f : \mathbb{N} \to \mathbb{N}$ be a multiplicative function satisfying the following conditions:*

(A) *$f(p^k) > f(p^r)$ for all primes p and nonnegative integers $k > r$;*

(B) *for any prime p and nonnegative integers x, y, c_1, c_2, if $c_1 \neq c_2$ and $f(p^{x+c_2})f(p^{y+c_1}) = f(p^{y+c_2})f(p^{x+c_1})$, then $x = y$;*

(C) *for each prime p dividing ab, there are c_1, $c_2 \in \mathbb{N} \cup \{0\}$ such that $c_1 \neq c_2$ and $f(ap^c) = f(bp^c)$ for $c \in \{c_1, c_2\}$.*

Then $a = b$.

Proof. Since $f : \mathbb{N} \to \mathbb{N}$, $f(n) \geq 1$ for all n. If $a = 1$ but $b \geq 2$, then there are a prime p and $k \in \mathbb{N}$ such that $b = p^k b_1$, $p \nmid b_1$, and so by the condition (C), there exists $c \in \mathbb{N}$ such that

$$f(p^c) = f(ap^c) = f(bp^c) = f(p^{c+k}b_1) = f(p^{c+k})f(b_1) \geq f(p^{c+k}) > f(p^c),$$

which is a contradiction. So if $a = 1$, then $b = 1 = a$. Similarly, if $b = 1$, then $a = 1 = b$. So assume throughout that a, $b \geq 2$. Let

$$a = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} \quad \text{and} \quad b = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k},$$

where p_1, p_2, …, p_k are distinct primes and a_1, a_2, …, a_k, b_1, b_2, …, b_k are nonnegative integers with $a_j^2 + b_j^2 \neq 0$ for any $j = 1, 2, \dots, k$. To show that $a = b$, it suffices to show that $a_j = b_j$ for all j. So let $j \in \{1, 2, \dots, k\}$ and $p = p_j$. Then $p \mid ab$. By the condition (C), there are c_1, $c_2 \in \mathbb{N} \cup \{0\}$ such that $c_1 \neq c_2$, $f(ap^{c_1}) = f(bp^{c_1})$, and $f(ap^{c_2}) = f(bp^{c_2})$. Then

$$f(p_j^{a_j+c_1}) \prod_{\substack{1 \leq i \leq k \\ i \neq j}} f\left(p_i^{a_i}\right) = f(p_j^{b_j+c_1}) \prod_{\substack{1 \leq i \leq k \\ i \neq j}} f\left(p_i^{b_i}\right) \tag{2}$$

$$f(p_j^{a_j+c_2}) \prod_{\substack{1 \leq i \leq k \\ i \neq j}} f\left(p_i^{a_i}\right) = f(p_j^{b_j+c_2}) \prod_{\substack{1 \leq i \leq k \\ i \neq j}} f\left(p_i^{b_i}\right) \tag{3}$$

Dividing (3) by (2) gives

$$\frac{f(p_j^{a_j+c_2})}{f(p_j^{a_j+c_1})} = \frac{f(p_j^{b_j+c_2})}{f(p_j^{b_j+c_1})} \tag{4}$$

By (4) and the condition (B), we obtain $a_j = b_j$, as required. □

The next lemma is used in the calculation of the p-adic valuation of $\sigma_s(n)$.

Lemma 4. *Assume that p and q are primes, $q \equiv 1 \pmod{p}$, and $c \in \mathbb{N} \cup \{0\}$. Then the following statements hold.*

(i) *If p is odd, then $v_p(\sigma_s(q^c)) = v_p(c+1)$.*

(ii) *If $p = 2$ and s is even, then $v_p(\sigma_s(q^c)) = v_p(c+1)$.*

(iii) *If $p = 2$, s is odd, and c is odd, then $v_p(\sigma_s(q^c)) = v_p(c+1) + v_p(q+1) - 1$.*

(iv) *If $p = 2$, s is odd, and c is even, then $v_p(\sigma_s(q^c)) = 0$.*

Proof. Since $p \nmid q$, $p \nmid 1$, and $p \mid q - 1$, we obtain by Lemma 2 that if p is odd, then

$$\begin{aligned} v_p(\sigma_s(q^c)) &= v_p(q^{(c+1)s} - 1) - v_p(q^s - 1) \\ &= (v_p(q-1) + v_p((c+1)s)) - (v_p(q-1) + v_p(s)) = v_p(c+1). \end{aligned}$$

Similarly, if $p = 2$ and s is even, then

$$v_p(\sigma_s(q^c)) = (v_2(q^2-1) + v_2((c+1)s) - 1) - (v_2(q^2-1) + v_2(s) - 1) = v_2(c+1);$$

if $p = 2$, s is odd, and c is odd, then

$$\begin{aligned} v_p(\sigma_s(q^c)) &= v_2(q^2-1) + v_2((c+1)s) - 1 - v_2(q-1) \\ &= v_2(c+1) + v_2(q+1) - 1; \end{aligned}$$

and if $p = 2$, s is odd, and c is even, then $v_p(\sigma_s(q^c)) = 0$. This completes the proof. □

3 Main results

Theorem 5. *Let a, b, $s \in \mathbb{N}$. Then the functions τ and σ_s satisfy the conditions* (A) *and* (B) *in Lemma 3.*

Proof. If $k > r$, then every divisor of p^r is also a divisor of p^k while p^k is not a divisor of p^r. From this, it is easy to see that τ and σ_s satisfy the condition (A). Let p be a prime, x, y, c_1, $c_2 \in \mathbb{N} \cup \{0\}$, and $c_1 \neq c_2$. Suppose that

$$\tau(p^{x+c_2})\tau(p^{y+c_1}) = \tau(p^{y+c_2})\tau(p^{x+c_1}).$$

Then $(x+c_2+1)(y+c_1+1) = (y+c_2+1)(x+c_1+1)$. Dividing both sides by $(y+c_1+1)(x+c_1+1)$, subtracting both sides by 1, and then dividing both sides by c_2-c_1 leads to $x = y$, as required. Next, suppose that

$$\sigma_s(p^{x+c_2})\sigma_s(p^{y+c_1}) = \sigma_s(p^{y+c_2})\sigma_s(p^{x+c_1}). \tag{5}$$

Let $x_1 = x + c_1 + 1$, $x_2 = x + c_2 + 1$, $y_1 = y + c_1 + 1$, and $y_2 = y + c_2 + 1$. Then (5) implies that

$$\frac{p^{x_2 s} - 1}{p^{x_1 s} - 1} = \frac{p^{y_2 s} - 1}{p^{y_1 s} - 1}. \tag{6}$$

Observing that $y_2 - y_1 = c_2 - c_1 = x_2 - x_1$ and subtracting both sides of (6) by 1, we obtain

$$\frac{p^{x_1 s}(p^{(c_2-c_1)s} - 1)}{p^{x_1 s} - 1} = \frac{p^{y_1 s}(p^{(c_2-c_1)s} - 1)}{p^{y_1 s} - 1}. \tag{7}$$

Dividing both sides of (7) by $p^{(c_2-c_1)s} - 1$, and then subtracting both sides by 1 leads to $p^{x_1 s} = p^{y_1 s}$. Therefore $x_1 = y_1$, which implies $x = y$, as required. This shows that τ and σ_s satisfy the condition (B). □

Corollary 6. *Let a, $b \in \mathbb{N}$ and $f = \tau$ or $f = \sigma_s$. Suppose that for each prime $p \mid ab$, we can find distinct nonnegative integers c_1, c_2 such that $f(ap^c) = f(bp^c)$ for $c \in \{c_1, c_2\}$. Then $a = b$. In particular, if $f(an) = f(bn)$ for all $n \in \mathbb{N}$, then $a = b$. In other words, τ and σ_s are quasi-injective.*

Proof. By Theorem 5, f satisfies the conditions (A) and (B), and the above supposition is, in fact, the condition (C). Therefore $a = b$, as required. □

Modifying the proof of Theorem 5, we see that τ is in fact quasi-injective of order ℓ for any $\ell \in \mathbb{N}$, as shown in the next theorem.

Theorem 7. *If m, a, b are positive integers and $\tau^{(m)}(an) = \tau^{(m)}(bn)$ for all $n \in \mathbb{N}$, then $a = b$.*

Proof. We prove by induction on m. If $m = 1$, the result follows from Corollary 6. So let $m \geq 2$ and assume that the result holds for $m - 1$. Let $a, b \in \mathbb{N}$ and

$$\tau^{(m)}(an) = \tau^{(m)}(bn) \text{ for all } n \in \mathbb{N}. \tag{8}$$

Let $k \in \mathbb{N}$ and $d \in \mathbb{N} \cup \{0\}$. Choosing a prime $p \nmid abk$ and substituting $n = p^d k$ in (8), we obtain

$$\tau^{(m-1)}((d+1)\tau(ak)) = \tau^{(m-1)}(\tau(an)) = \tau^{(m-1)}(\tau(bn)) = \tau^{(m-1)}((d+1)\tau(bk)). \tag{9}$$

Since (9) holds for all $d \in \mathbb{N} \cup \{0\}$, we obtain by the induction hypothesis that $\tau(ak) = \tau(bk)$. Since k is arbitrary, we obtain by Corollary 6 that $a = b$. □

We now know that if $\tau^{(m)}(an) = \tau^{(m)}(bn)$ for all n, then $a = b$; and if $\sigma_s(an) = \sigma_s(bn)$ for all n, then $a = b$. We would like to extend the result for σ_s to $\sigma_s^{(m)}$ for any m but it seems much more complicated than that of $\tau^{(m)}$, so we do it only for $\sigma_s^{(2)}$. We conjecture that the result holds for $m \geq 3$ as well but we currently do not have a proof.

Theorem 8. *For each positive integer $x \geq 2$ and for each prime p dividing x, there are positive integers m, n, c_1, c_2, A, B such that $c_2 > c_1$, $(ABmn, x) = 1$, $\sigma_s(m) = p^{c_1}A$, and $\sigma_s(n) = p^{c_2}B$.*

Proof. Let $x = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $a_1, a_2, \ldots, a_k$ are positive integers. Let p be a prime dividing x, say $p = p_j$ for some $j \in \{1, 2, \ldots, k\}$. Let

$$M = \prod_{1 \leq i \leq k} p_i \quad \text{and} \quad M_j = \frac{M}{p}.$$

By Dirichlet's theorem for primes in arithmetic progressions, there exists a prime $q \equiv 1 \pmod{M}$. To find the integers m, n, c_1, c_2, A, B as above, we divide the calculation into two cases:

Case 1: p_i is odd for all $i \neq j$. By the Chinese remainder theorem, there exists a positive integer w such that

$$w \equiv -1 \pmod{p} \quad \text{and} \quad w \equiv 0 \pmod{M_j}. \tag{10}$$

Let L be a positive integer larger than $v_p(w+1)+v_p(q+1)$. Applying the Chinese remainder theorem again to obtain $y \in \mathbb{N}$ such that

$$y \equiv -1 \pmod{p^L} \quad \text{and} \quad y \equiv 0 \pmod{M_j}. \tag{11}$$

Let $m = q^w$ and $n = q^y$. By Lemma 4, we obtain

$$v_{p_i}(\sigma_s(m)) = v_{p_i}(\sigma_s(q^w)) = v_{p_i}(w+1) = 0 \text{ for all } i \neq j.$$

In addition, if p is odd or if $p = 2$ and s is even, then $v_p(\sigma_s(m)) = v_p(\sigma_s(q^w)) = v_p(w+1) > 0$; if $p = 2$ and s is odd, then w is odd, q is odd, $v_p(q+1) \geq 1$, and

$$v_p(\sigma_s(m)) = v_p(\sigma_s(q^w)) = v_p(w+1) + v_p(q+1) - 1 > 0.$$

This shows that $v_p(\sigma_s(m)) > 0$ and $v_{p_i}(\sigma_s(m)) = 0$ for all $i \neq j$. Let $c_1 = v_p(\sigma_s(m))$. Then $c_1 \in \mathbb{N}$ and $\sigma_s(m) = p^{c_1}A$, where $A \in \mathbb{N}$ and $(A, x) = 1$. Similarly, we obtain $v_{p_i}(\sigma_s(n)) = 0$ for all $i \neq j$ and

$$v_p(\sigma_s(n)) \geq v_p(y+1) \geq L > v_p(w+1) + v_p(q+1) \geq v_p(\sigma_s(m)).$$

Let $c_2 = v_p(\sigma_s(n))$. Then $c_2 \in \mathbb{N}$, $c_2 > c_1$, and $\sigma_s(n) = p^{c_2}B$, where $B \in \mathbb{N}$ and $(B, x) = 1$. Since $(m, x) = (n, x) = (A, x) = (B, x) = 1$, we obtain $(mnAB, x) = 1$.

Case 2: $p_i = 2$ for some $i \neq j$. Without loss of generality, assume that $p_1 = 2$. We still choose w, L, y, $m = q^w$ and $n = q^y$ as in Case 1 and apply Lemma 4 to calculate the p-adic valuation of $\sigma_s(m)$ and $\sigma_s(n)$. If $i \neq 1$, then p_i is odd and so

$$v_{p_i}(\sigma_s(m)) = v_{p_i}(w+1) = 0, v_{p_i}(\sigma_s(n)) = v_{p_i}(y+1) = 0.$$

In addition, p is odd, and so

$$v_p(\sigma_s(m)) = v_p(w+1) > 0, v_p(\sigma_s(n)) = v_p(y+1) \geq L > v_p(\sigma_s(m)).$$

Since $p_1 = 2$ and w, $y \equiv 0 \pmod{M_j}$, w and y are even. If s is even, then $v_{p_1}(\sigma_s(m)) = v_2(w+1) = 0 = v_2(y+1) = v_{p_1}(\sigma_s(n))$. If s is odd, then $v_{p_1}(\sigma_s(m))$ and $v_{p_1}(\sigma_s(n))$ are also zero. This shows that $v_p(\sigma_s(n)) > v_p(\sigma_s(m)) > 0$ and $v_{p_i}(\sigma_s(m)) = v_{p_i}(\sigma_s(n)) = 0$ for all $i \neq j$. Therefore we can choose c_1, c_2, A, B as in the previous case. This completes the proof. □

Theorem 9. *Let a and b be positive integers and*

$$\sigma_s(\sigma_s(an)) = \sigma_s(\sigma_s(bn)) \text{ for all } n \in \mathbb{N}. \tag{12}$$

Then $a = b$.

Proof. Let p be a prime dividing ab and $d \in \mathbb{N} \cup \{0\}$. Let n_1 be a positive integer such that $(n_1, ab) = 1$. Substituting $n = p^d n_1$ in (12), we obtain

$$\sigma_s(\sigma_s(ap^d)\sigma_s(n_1)) = \sigma_s(\sigma_s(bp^d)\sigma_s(n_1)), \tag{13}$$

which holds for any $n_1 \in \mathbb{N}$ with $(n_1, ab) = 1$. Next, let $a_1 = \sigma_s(ap^d)$, $b_1 = \sigma_s(bp^d)$, $x = aba_1b_1$ and let q be a prime dividing a_1b_1. Then $q \mid x$. By Theorem 8, there are positive integers m, n, c_1, c_2, A, B such that $c_2 > c_1$, $(ABmn, x) = 1$, $\sigma_s(m) = q^{c_1}A$, and $\sigma_s(n) = q^{c_2}B$. Since (13) holds for any $n_1 \in \mathbb{N}$ with $(n_1, ab) = 1$ and $(m, ab) = (n, ab) = 1$, we can substitute $n_1 = m$ and $n_1 = n$ in (13) to obtain

$$\sigma_s(a_1q^{c_1}A) = \sigma_s(b_1q^{c_1}A) \quad \text{and} \quad \sigma_s(a_1q^{c_2}B) = \sigma_s(b_1q^{c_2}B). \tag{14}$$

Since $(AB, x) = 1$, we see that $(A, qa_1b_1) = (B, qa_1b_1) = 1$. Therefore (14) reduces to

$$\sigma_s(a_1q^{c_1})\sigma_s(A) = \sigma_s(b_1q^{c_1})\sigma_s(A) \quad \text{and} \quad \sigma_s(a_1q^{c_2})\sigma_s(B) = \sigma_s(b_1q^{c_2})\sigma_s(B),$$

which imply

$$\sigma_s(a_1q^{c_1}) = \sigma_s(b_1q^{c_1}) \quad \text{and} \quad \sigma_s(a_1q^{c_2}) = \sigma_s(b_1q^{c_2}). \tag{15}$$

This shows that for any prime $q \mid a_1b_1$, we can find distinct c_1, $c_2 \in \mathbb{N}$ such that (15) holds. By Corollary 6, we obtain $a_1 = b_1$. Therefore

$$\sigma_s(ap^d) = \sigma_s(bp^d). \tag{16}$$

Since (16) holds for each prime $p \mid ab$ and each $d \in \mathbb{N} \cup \{0\}$, we apply Corollary 6 again to obtain $a = b$. This completes the proof. □

Before proceeding to the case of J_s, we give an example to show that ω, Ω, and μ are not quasi-injective.

Example 10. Let $k \geq 2$, p_1, p_2, …, p_k distinct primes, $c = p_1p_2 \dots p_k$, $a = p_1c$, and $b = p_2c$. Then $\omega(a) = \omega(b)$, $\Omega(a) = \Omega(b)$, and $\mu(a) = \mu(b)$. Since Ω is completely additive, that is, $\Omega(mn) = \Omega(m) + \Omega(n)$ for all m, $n \in \mathbb{N}$, we see that $\Omega(an) = \Omega(bn)$ for all $n \in \mathbb{N}$ but $a \neq b$. Therefore Ω is not quasi-injective. Similarly, it is not difficult to see that $\omega(an) = \omega(bn)$ and $\mu(an) = \mu(bn)$ for all $n \in \mathbb{N}$ but $a \neq b$, and so ω and μ are not quasi-injective.

We can generalize the idea in Example 10 as follows.

Theorem 11. *Let $f : \mathbb{N} \to \mathbb{C}$ be completely additive. Then f is quasi-injective if and only if f is injective.*

Proof. Suppose f is quasi-injective, a, $b \in \mathbb{N}$, and $f(a) = f(b)$. Then for each $n \in \mathbb{N}$, we have

$$f(an) = f(a) + f(n) = f(b) + f(n) = f(bn). \tag{17}$$

Since (17) holds for all $n \in \mathbb{N}$ and f is quasi-injective, we obtain $a = b$, as required. The converse is obvious. □

Example 12. Let A be the Alladi-Erdös function defined by $A(n) = \sum_{p^\alpha \| n} \alpha p$. It is easy to verify that A is completely additive, that is, $A(mn) = A(m) + A(n)$ for all m, $n \in \mathbb{N}$. In addition, $A(2^3) = 6 = A(3^2)$, so A is not injective. By Theorem 11, A is not quasi-injective.

A variation of Alladi-Erdös function can defined by $A_0(n) = \sum_{p|n} p$. Then $A_0(6) = 2 + 3 = 5 = A_0(12)$, and it is not difficult to see that $A_0(6n) = A_0(12n)$ for all $n \in \mathbb{N}$. Therefore A_0 is not quasi-injective.

Theorem 13. *Suppose that $f : \mathbb{N} \to \mathbb{C}$ is strongly additive, that is, f is additive and $f(p^k) = f(p)$ for all primes p and positive integers k. If there are distinct positive integers a, $b \geq 2$ such that $f(a) = f(b)$ and a, b have the same prime factors, then f is not quasi-injective.*

Proof. Since a and b have the same prime factors, we write

$$a = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} \quad \text{and} \quad b = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k},$$

where p_1, p_2, …, p_k are distinct primes and a_1, a_2, …, a_k, b_1, b_2, …, b_k are positive integers. We show that $f(an) = f(bn)$ for all $n \in \mathbb{N}$. For each $n \geq 2$, we write $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k} m$, where $m \in \mathbb{N}$, $(m, ab) = 1$, and n_1, n_2, …, n_k are nonnegative integers, and so

$$f(an) = f(p_1^{a_1+n_1} p_2^{a_2+n_2} \dots p_k^{a_k+n_k} m) = f(a)f(m) = f(b)f(m) = f(bn).$$

Therefore $f(an) = f(bn)$ for all $n \in \mathbb{N}$ but $a \neq b$. This completes the proof. □

Theorem 14. *Let a, b, c, k, m, s be positive integers. Then the following statements hold.*

(i) *If $a \mid b$, then $J_s(ab) = a^s J_s(b)$.*

(ii) *If $a \mid c$ and $b \mid c$, then $b^s J_s(ac) = a^s J_s(bc)$.*

(iii) *If $a \mid b$, then $J_s^{(m)}(a) \mid J_s^{(m)}(b)$.*

(iv) *$J_s(a^k b) = a^{(k-1)s} J_s(ab)$.*

(v) *If $k \geq m + 1$, then $J_s^{(m)}(a^k b) = a^{s^m} J_s^{(m)}(a^{k-1} b)$.*

(vi) *If $k \geq m + 1$, then $a^{s^m} \mid J_s^{(m)}(a^k)$.*

(vii) *If $k \geq m + 1$ and $c \geq m + 1$, then $a^{s^m} \mid J_s^{(m)}(a^k b^c)$ and $b^{s^m} \mid J_s^{(m)}(a^k b^c)$.*

Proof. For (i) and (iii), assume that $a \mid b$. Then $p \mid ab$ if and only if $p \mid b$. Therefore

$$J_s(ab) = (ab)^s \prod_{p|ab} \left(1 - \frac{1}{p^s}\right) = a^s b^s \prod_{p|b} \left(1 - \frac{1}{p^s}\right) = a^s J_s(b),$$

which proves (i). If $a = 1$, then (iii) is obvious. So assume that $a \geq 2$. Let $a = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ and $b = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k} b_0$, where $p_1, p_2, \dots, p_k$ are distinct primes, $a_i, b_i, b_0 \in \mathbb{N}$, $b_i \geq a_i$ for all $i = 1, 2, \dots, k$, and $(b_0, p_1 p_2 \dots p_k) = 1$. Then

$$J_s(a) = \prod_{i=1}^{k} \left(p_i^{(a_i - 1)s} \right) \prod_{i=1}^{k} (p_i^s - 1),$$

$$J_s(b) = \prod_{i=1}^{k} \left(p_i^{(b_i - 1)s} \right) \prod_{i=1}^{k} (p_i^s - 1) J_s(b_0).$$

From this, it is easy to see that $J_s(a) \mid J_s(b)$. So (iii) is proved for $m = 1$. If $m \geq 2$, we can apply the result when $m = 1$ repeatedly and obtain the chain of implication as follows:

$$a \mid b \Rightarrow J_s(a) \mid J_s(b) \Rightarrow J_s^{(2)}(a) \mid J_s^{(2)}(b) \Rightarrow \cdots \Rightarrow J_s^{(m)} \mid J_s^{(m)}(b).$$

Therefore (iii) is proved for every $m \geq 1$. For (ii), if $a \mid c$ and $b \mid c$, then we obtain by (i) that

$$b^s J_s(ac) = b^s a^s J_s(c) = a^s b^s J_s(c) = a^s J_s(bc).$$

Next, we prove (iv) by induction on k. If $k = 1$, then (iv) is obvious. So let $k \geq 1$ and assume that (iv) holds for k. Since $a \mid a^k b$, we obtain by (i) that

$$J_s(a^{k+1} b) = J_s(a(a^k b)) = a^s J_s(a^k b) = a^{ks} J_s(ab),$$

where the last equality is obtained from the induction hypothesis. Next, we prove (v) by induction on m. If $m = 1$ and $k \geq 2$, then we obtain by (i) that $J_s^{(m)}(a^k b) = J_s(a \cdot a^{k-1} b) = a^s J_s(a^{k-1} b)$. So let $m \geq 1$ and assume that the result holds for m. Suppose $k \geq m + 2$. By the induction hypothesis, we obtain

$$J_s^{(m+1)}(a^k b) = J_s(J_s^{(m)}(a^k b)) = J_s(a^{s^m} J_s^{(m)}(a^{k-1} b)). \tag{18}$$

Since $k - 1 \geq m + 1$, we apply the induction hypothesis again to conclude that $a^{s^m} \mid J_s^{(m)}(a^{k-1} b)$. Then (i) implies that the right-hand side of (18) is equal to $a^{s^{m+1}} J_s^{(m+1)}(a^{k-1} b)$. Therefore (18) becomes

$$J_s^{(m+1)}(a^k b) = a^{s^{m+1}} J_s^{(m+1)}(a^{k-1} b),$$

which proves (v). Then (vi) is a special case of (v) when $b = 1$. For (vii), if $k \geq m + 1$ and $c \geq m + 1$, then we use (vi) and (iii) to obtain

$$a^{s^m} \mid J_s^{(m)}(a^k) \mid J_s^{(m)}(a^k b^c) \quad \text{and} \quad b^{s^m} \mid J_s^{(m)}(b^c) \mid J_s^{(m)}(a^k b^c).$$

Here $x \mid y \mid z$ means that $x \mid y$ and $y \mid z$. Hence the proof is complete. □

Theorem 15. *Let $a, b, m, s \in \mathbb{N}$ and $J_s^{(m)}(an) = J_s^{(m)}(bn)$ for all $n \in \mathbb{N}$. Then $a = b$. In other words, J_s is quasi-injective of any order.*

Proof. Substituting $n = a^m b^m$ in the above condition, we have $J_s^{(m)}(a^{m+1}b^m) = J_s^{(m)}(b^{m+1}a^m)$. By (v) of Theorem 14, we obtain

$$J_s^{(m)}(a^{m+1}b^m) = a^{s^m} J_s^{(m)}(a^m b^m) \quad \text{and} \quad J_s^{(m)}(b^{m+1}a^m) = b^{s^m} J_s^{(m)}(b^m a^m).$$

Therefore

$$a^{s^m} J_s^{(m)}(a^m b^m) = b^{s^m} J_s^{(m)}(b^m a^m),$$

which implies $a = b$. This completes the proof. □

4 Conclusion and some open problems

We have proved that τ and J_s are quasi-injective of order m for any $m \in \mathbb{N}$ while we only show that σ_s is quasi-injective of order 2. We believe that it can be extended to any order. In addition, Example 10 shows that μ, ω, and Ω are not quasi-injective. This leads us to the following problems.

Question 16. If $a, b, m, s \in \mathbb{N}$, $m \geq 3$, and $\sigma_s^{(m)}(an) = \sigma_s^{(m)}(bn)$ for all $n \in \mathbb{N}$, can we conclude that $a = b$?

Question 17. For each $m \geq 2$, is there a function $f : \mathbb{N} \to \mathbb{N}$ such that f is quasi-injective of order $m - 1$ but not of order m?

We may also consider the mix between τ, σ_s, and J_s.

Question 18. Let $a, b, k, m, s \in \mathbb{N}$. Are the functions $\tau^{(m)} \circ \sigma_s^{(k)}$, $\sigma_s^{(k)} \circ \tau^{(m)}$, $\tau^{(m)} \circ J_s^{(k)}$, $J_s^{(k)} \circ \tau^{(m)}$, $\sigma_s^{(m)} \circ J_s^{(k)}$, and $J_s^{(k)} \circ \sigma_s^{(m)}$ quasi-injective? That is, if f is one of the above functions and $f(an) = f(bn)$ for all n, can we show that $a = b$? This may be easy when k or m is less than 3. Can we say something when both k and m are larger than 2?

Question 19. Suppose f and g are quasi-injective. Is the composition $f \circ g$ quasi-injective? Can we categorize those functions f and g for which $f \circ g$ must be quasi-injective? An obvious sufficient condition for $f \circ g$ to be quasi-injective is that g is both surjective and completely multiplicative, but there may be a weaker condition.

For each $n \in \mathbb{N}$ and $b \geq 2$, let $S_b(n)$ be the sum of digits of n in base b, and let $S(n) = S_{10}(n)$ be the sum of the decimal digits of n.

Question 20. If $a, c \in \mathbb{N}$ and $S(an) = S(cn)$ for all $n \in \mathbb{N}$, is it true that $a = c$? More generally, if $a, c, m \in \mathbb{N}$ and $S^{(m)}(an) = S^{(m)}(cn)$ for all $n \in \mathbb{N}$, can we prove that $a = c$? Can we replace S by S_b for any $b \geq 2$?

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