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Solution of the Differential Equation $y^{(k)} = e^{ay}$, Special Values of Bell Polynomials, and (k, a)-Autonomous Coefficients

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Abstract

We give special values of Bell polynomials by using the power series solution of the equation $y^{(k)} = e^{ay}$. In addition, we define complete and partial exponential autonomous functions, exponential autonomous polynomials, autonomous polynomials, and (k, a)-autonomous coefficients. Finally, we demonstrate the relationship between various numbers counting combinatorial objects and binomial coefficients, Stirling numbers of the second kind, and autonomous coefficients.

1 Introduction

It is a known fact that Bell polynomials are closely related to derivatives of the composition of functions. For example, Faà di Bruno [5], Foissy [6], and Riordan [10] showed that Bell polynomials are a very useful tool in mathematics to represent the n-th derivative of the composition of functions. Also, Bernardini and Ricci [2], Yildiz et al. [12], Cayley [3], and Wang [13] showed the relationship between Bell polynomials and differential equations. On the other hand, Orozco [9] studied the convergence of the analytic solution of the autonomous differential equation $y^{(k)} = f(y)$ by using the formula of Faà di Bruno. We can then look at differential equations as a source for investigating special values of Bell polynomials.

In this paper, we focus on finding special values of Bell polynomials when the vector field $f(x)$ of the autonomous differential equation $y^{(k)} = f(y)$ is the exponential function. We do not consider the convergence of the solutions, but we will show that well-known numbers such as reduced tangent numbers, Bernoulli numbers, Euler zigzag numbers, Blasius numbers, among others, can be constructed using Bell polynomials. In general, a special class of numbers, which have not yet been studied, are constructed using Bell polynomials. On the other hand, a new family of numbers called (k, a)-autonomous coefficients is obtained for each value of k. Four conjectures about these numbers are established.

This paper is divided as follows. We begin with a summary of results on complete and partial Bell polynomials, which are used to demonstrate the main results presented here. Next, we introduce the complete and partial exponential autonomous functions, the recurrence relations of these are constructed using Bell polynomials, and some recurrence relations of solutions of various initial value problems are given. In the fourth section, the (k, a)-autonomous coefficients are introduced. From these numbers we can obtain the triangular numbers, the 8-sequence numbers of $[1, n]$ with 2 contiguous pairs, among others. We finish this work by studying the cases $k = 2, 3, 4$ for the autonomous differential equation $y^{(k)} = e^{ay}$.

2 Bell exponential polynomials

The following basic results can be found at Comtet [4, pp. 135–136], and Riordan [11, pp. 35–36, 49]. Exponential Bell polynomials are used to encode information on the ways in which a set can be partitioned, hence they are a very useful tool in combinatorial analysis. Bell polynomials are obtained from the derivatives of composite functions and are given by the formula of Faà Di Bruno [5]. Bell [1], Gould [7], Mihoubi [8] and Qi et al. [15, 16], provided important results on these polynomials. We start with the definition of partial Bell polynomials.

Definition 1. The exponential partial Bell polynomials are the polynomials

$$B_{n,k}(x_1, x_n, \ldots, x_{n-k+1})$$

in the variables $x_1, x_2, \ldots$ defined by the series expansion

$$\exp\left(u \sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{k=1}^{n} u^k B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}). \tag{1}$$

The following result gives the explicit way to calculate the partial Bell polynomials.

Theorem 2. *The partial or incomplete exponential Bell polynomials are given by*

$$B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum \frac{n!}{c_1! c_2! \cdots c_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{c_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{c_{n-k+1}}$$

where the summation takes place over all integers $c_1, c_2, \ldots, c_{n-k+1} \geq 0$*, such that*

$$c_1 + 2c_2 + \cdots + (n-k+1)c_{n-k+1} = n,$$
$$c_1 + c_2 + \cdots + c_{n-k+1} = k.$$

The following are special cases of partial Bell polynomials and will be very useful for proving results in this paper:

$$B_{n,1}(x_1, \ldots, x_n) = x_n,$$
$$B_{n,2}(x_1, \ldots, x_{n-1}) = \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}x_k x_{n-k},$$
$$B_{n,n-a}(x_1, \ldots, x_{a+1}) = \sum_{j=a+1}^{2a}\frac{j!}{a!}\binom{n}{j}x_1^{n-j}B_{a,j-a}\left(\frac{x_2}{2}, \cdots, \frac{x_{2(a+1)-j}}{2(a+1)-j}\right),$$
$$1 \leq a < n,$$

$$B_{n,n}(x_1) = x_1^n, \tag{2}$$
$$B_{n,n-1}(x_1, x_2) = \binom{n}{2}x_1^{n-2}x_2, \tag{3}$$
$$B_{n,n-2}(x_1, x_2, x_3) = \binom{n}{3}x_1^{n-3}x_3 + 3\binom{n}{4}x_1^{n-4}x_2^2, \tag{4}$$
$$B_{n,n-3}(x_1, x_2, x_3, x_4) = \binom{n}{4}x_1^{n-4}x_4 + 10\binom{n}{5}x_1^{n-5}x_2x_3 + 15\binom{n}{6}x_1^{n-6}x_2^3, \tag{5}$$
$$B_{n,n-4}(x_1, x_2, x_3, x_4, x_5) = \binom{n}{5}x_1^{n-5}x_5 + 5\binom{n}{6}x_1^{n-6}(3x_2x_4 + 2x_3^2)$$
$$+ 105\binom{n}{7}x_1^{n-7}x_2^2x_3 + 105\binom{n}{8}x_1^{n-8}x_2^4. \tag{6}$$

Some values of partial Bell polynomials are

$$B_{n,k}(0!, 1!, \ldots, (n-k)!) = \begin{bmatrix} n \\ k \end{bmatrix} \quad \text{(Unsigned Stirling number of the first kind)},$$
$$B_{n,k}(1!, \ldots, (n-k)!) = \binom{n-1}{k-1}\frac{n!}{k!} \quad \text{(Lah number)},$$
$$B_{n,k}(1, 1, \ldots, 1) = \begin{Bmatrix} n \\ k \end{Bmatrix} \quad \text{(Stirling number of the second kind)},$$
$$B_{n,k}(1, 2, \ldots, n-k+1) = \binom{n}{k}k^{n-k} \quad \text{(Idempotent number)}.$$

Then we can see the beautiful relationship that exists between Bell polynomials and numbers like the above.

On the other hand, the partial Bell polynomials can be efficiently computed by means of the recurrence relation

$$B_{n,k}(x_1,\ldots,x_{n-k+1}) = \sum_{i=1}^{n-k+1} \binom{n-1}{i-1} x_i B_{n-i,k-1}(x_1,\ldots,x_{n-i-k+2}). \tag{7}$$

The definition of complete Bell polynomials is as follows.

Definition 3. The sum

$$B_n(x_1,x_2,\ldots,x_n) = \sum_{k=1}^{n} B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})$$

is called the n-th complete exponential Bell polynomial with exponential generating function given by to make $u=1$ in Eq. (1)

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right) = \sum_{n=0}^{\infty} B_n(x_1,x_2,\ldots,x_n)\frac{t^n}{n!}$$

and $B_0 = 1$.

Some complete Bell polynomials are as follows:

$$\begin{aligned}
B_1(x_1) &= x_1,\\
B_2(x_1,x_2) &= x_1^2 + x_2,\\
B_3(x_1,x_2,x_3) &= x_1^3 + 3x_1x_2 + x_3,\\
B_4(x_1,x_2,x_3,x_4) &= x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4,\\
B_5(x_1,x_2,x_3,x_4,x_5) &= x_1^5 + 10x_1^3x_2 + 15x_1x_2^2 + 10x_1^2x_3 + 10x_2x_3 + 5x_1x_4 + x_5.
\end{aligned}$$

Theorem 4. *The complete Bell polynomials B_n satisfy the identity*

$$B_{n+1}(x_1,\ldots,x_{n+1}) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(x_1,\ldots,x_{n-i})x_{i+1}. \tag{8}$$

From this it follows that $B_{2n+1}(0,x_2,0,\ldots,0,x_{2n+1}) = 0$ for all $n \geq 0$.
Another useful identity that Bell polynomials fulfill is as follows

$$B_n(-x_1,x_2,-x_3,\ldots,(-1)^{n-1}x_n) = (-1)^n B_n(x_1,x_2,x_3,\ldots,x_n). \tag{9}$$

3 Exponential autonomous functions

We study the solution of the equation

$$y^{(k)} = e^{ay} \tag{10}$$

for $a \in \mathbb{C}$. Setting $y = u/a$ we obtain the equivalent equation

$$u^{(k)} = ae^{u}. \tag{11}$$

Then without loss of generality we focus on Eq. (11). Now by taking the derivative of Eq. (11), we obtain another equation equivalent to Eq. (10), namely

$$u^{(k+1)} = ae^{u}u' = u^{(k)}u'.$$

Let $(x_1, x_2, \ldots, x_k)$ be the initial value problem $y(0) = x_1$, $y'(0) = x_2, \ldots, y^{(k-1)}(0) = x_k$. In this section we give the general solution and solutions with initial values $(x, 0, 0, \ldots, 0)$, $(x_1 + k \ln c, cx_2, \ldots, c^{k-1}x_k)$, and $(x_1, -x_2, \ldots, x_{2k-1}, -x_{2k})$ of Eq. (11). We define the complete and partial exponential autonomous functions and the exponential autonomous polynomials, which are the coefficients of the power series solution of Eq. (11). Moreover, we find special values of these functions by using Bell polynomials. We begin with the following definition.

Definition 5. Take $a \in \mathbb{C}$. Suppose $\mathbf{x} = (x_1, \ldots, x_k)$. Let $f_n(\mathbf{x}, a)$ denote the *n-th complete exponential autonomous function of order* k, $k \geq 1$, recursively defined as

$$\begin{aligned} f_0(\mathbf{x}, a) &= x_1, \\ f_1(\mathbf{x}, a) &= x_2, \\ &\vdots \\ f_{k-1}(\mathbf{x}, a) &= x_k, \\ f_k(\mathbf{x}, a) &= ae^{x_1}, \\ f_{n+k}(\mathbf{x}, a) &= ae^{x_1} B_n(f_1(\mathbf{x}, a), \ldots, f_n(\mathbf{x}, a)), \quad n \geq 1, \end{aligned} \tag{12}$$

where $B_n(y_1, \ldots, y_n)$ are the complete Bell polynomials. When $x_1 = 0$, we define the *n-th exponential autonomous polynomial* as $q_n(x_2, \ldots, x_k) = f_n(0, x_2, \ldots, x_k)$, for $n \geq 1$.

When $a = 1$ in the above definition, we write $f_n(\mathbf{x}) = f_n(\mathbf{x}, 1)$. In this section we restrict ourselves to exponential autonomous functions. Exponential autonomous polynomials are discussed in the next section.

The following are complete exponential autonomous functions for $k = 1, 2, 3, 4$. They will be very useful in the next section. When $k = 1$, $f_n(x, a) = (n-1)!a^n e^{nx}$. When $k = 2$, we have

$$\begin{aligned}
f_0(x,y,a) &= x,\\
f_1(x,y,a) &= y,\\
f_2(x,y,a) &= ae^x,\\
f_3(x,y,a) &= aye^x,\\
f_4(x,y,a) &= ae^x(ae^x+y^2),\\
f_5(x,y,a) &= ae^x(4aye^x+y^3),\\
f_6(x,y,a) &= ae^x(4a^2e^{2x}+11ay^2e^x+y^4).
\end{aligned}$$

When $k=3$, we have

$$\begin{aligned}
f_0(x,y,z,a) &= x,\\
f_1(x,y,z,a) &= y,\\
f_2(x,y,z,a) &= z,\\
f_3(x,y,z,a) &= ae^x,\\
f_4(x,y,z,a) &= aye^x,\\
f_5(x,y,z,a) &= ae^x(z+y^2),\\
f_6(x,y,z,a) &= ae^x(ae^x+3yz+y^3),\\
f_7(x,y,z,a) &= ae^x(5ye^x+3z^2+6y^2z+y^4).
\end{aligned}$$

And finally, when $k=4$, we have

$$\begin{aligned}
f_0(x,y,z,w,a) &= x,\\
f_1(x,y,z,w,a) &= y,\\
f_2(x,y,z,w,a) &= z,\\
f_3(x,y,z,w,a) &= w,\\
f_4(x,y,z,w,a) &= ae^x,\\
f_5(x,y,z,w,a) &= aye^x,\\
f_6(x,y,z,w,a) &= ae^x(z+y^2),\\
f_7(x,y,z,w,a) &= ae^x(w+3yz+y^3),\\
f_8(x,y,z,w,a) &= ae^x(ae^x+3z^2+4yw+6y^2z+y^4).
\end{aligned}$$

In the following result, we show that the exponential generating function of the complete exponential autonomous functions is the solution of Eq. (11).

Theorem 6. *Let* $\boldsymbol{x}=(x_1,\ldots,x_k)$*. The series*

$$E_k(t,\boldsymbol{x},a)=\sum_{n=0}^{\infty} f_n(\boldsymbol{x},a)\frac{t^n}{n!}$$

is the solution of the differential Eq. (11).

Proof. Taking the k'th derivative of the series $E_k(t,\mathbf{x},a)$ with respect to t, using the definition of the autonomous functions $f_n(\mathbf{x},a)$, and Eq. (1), we get

$$\begin{aligned}\frac{\partial^k E_k(t,\mathbf{x},a)}{\partial t^k} &= \sum_{n=0}^{\infty} f_{n+k}(\mathbf{x},a)\frac{t^n}{n!}\\ &= ae^{x_1} + \sum_{n=1}^{\infty} ae^{x_1} B_n(f_1(\mathbf{x},a),\dots,f_n(\mathbf{x},a))\frac{t^n}{n!}\\ &= e^{ax_1}\left(1+\sum_{n=1}^{\infty} B_n(f_1(\mathbf{x},a),\dots,f_n(\mathbf{x},a))\frac{t^n}{n!}\right)\\ &= ae^{x_1}e^{E_k(t,\mathbf{x},a)-x_1}\\ &= ae^{E_k(t,\mathbf{x},a)}.\end{aligned}$$

□

Now we define the partial exponential autonomous functions.

Definition 7. Let $g_{n,i}(\mathbf{x},a)$ denote the *partial exponential autonomous functions* by

$$g_{n,i}(\mathbf{x},a) = B_{n,i}(f_1(\mathbf{x},a),\dots,f_{n-i+1}(\mathbf{x},a))$$

with $g_{0,0}(\mathbf{x},a)=1$, $g_{n,0}(\mathbf{x},a)=0$, for $n\geq 1$, and $g_{0,i}(\mathbf{x},a)=0$, for $i\geq 1$. Then

$$f_{n+k}(\mathbf{x},a) = ae^{x_1}\sum_{i=1}^{n} g_{n,i}(\mathbf{x},a).$$

In the following result we establish recurrence relations for the functions $f_n(\mathbf{x},a)$ and $g_{n,i}(\mathbf{x},a)$. Many important results of this paper are proved using this theorem.

Theorem 8. *The autonomous functions $f_n(\boldsymbol{x},a)$ and $g_{n,i}(\mathbf{x},a)$ fulfill the recurrence relations*

$$f_{n+k+1}(\boldsymbol{x},a) = \sum_{i=0}^{n}\binom{n}{i} f_{n-i+k}(\boldsymbol{x},a) f_{i+1}(\boldsymbol{x},a) \tag{13}$$

and

$$g_{n,i}(\boldsymbol{x},a) = \sum_{j=1}^{n-i+1}\binom{n-1}{j-1} f_j(\boldsymbol{x},a) g_{n-j,i-1}(\boldsymbol{x},a). \tag{14}$$

Proof. Setting $y_j = f_j(\mathbf{x},a)$ in Eq. (7) and Eq. (8) and multiplying these by ae^{x_1}, we obtain the desired result. □

Now we study the behavior of the functions $f_n(\mathbf{x},a)$ evaluated at $\mathbf{x}=(x_1,0,0,\dots,0)$. From previous result we can construct the first important sequence arising from the differential equation (11).

Theorem 9. *The functions $f_n(\mathbf{x}, a)$ take the following values at $\mathbf{x} = (x_1, 0, \ldots, 0)$*

1. $f_{kn+1}(x_1, 0, \ldots, 0, a) = f_{kn+2}(x_1, 0, \ldots, 0, a) = \cdots = f_{kn+k-1}(x_1, 0, \ldots, 0, a) = 0,\ n \geq 0$;

2. $f_{kn}(x_1, 0, \ldots, 0, a) = A_n^{(k)}(a) e^{nx_1},\ n \geq 1$,

where $A_1^{(k)}(a) = 1$ *and*

$$A_{n+2}^{(k)}(a) = \sum_{i=0}^{n} \binom{kn+k-1}{ki+k-1} A_{n-i+1}^{(k)}(a) A_{i+1}^{(k)}(a), \tag{15}$$

$n \geq 0,\ k \geq 1$.

Proof. Let $\mathbf{x} = (x_1, 0, \ldots, 0)$. Clearly, $f_1(\mathbf{x}, a) = 0$, $f_{k+1}(\mathbf{x}, a) = B_1(f_1(\mathbf{x}, a)) = f_1(\mathbf{x}, a) = 0$. Now suppose it is true that $f_{ki+1}(\mathbf{x}, a) = 0$ for $2 \leq i \leq n-1$. By Theorem 8 we have

$$\begin{aligned} f_{kn+1}(\mathbf{x}, a) &= f_{k(n-1)+k+1}(\mathbf{x}, a) \\ &= \sum_{i=0}^{k(n-1)} \binom{k(n-1)}{i} f_{kn-i}(\mathbf{x}, a) f_{i+1}(\mathbf{x}, a). \end{aligned}$$

Since the product $f_{kn-i}(\mathbf{x}, a) f_{i+1}(\mathbf{x}, a)$ contains the functions $f_{kj+1}(\mathbf{x}, a)$, then for all n we get that $f_{kn+1}(\mathbf{x}, a) = 0$. The same conclusion holds for $f_{kn+j}(\mathbf{x}, a)$, $j = 2, \ldots, k-1$.

Now we prove 2. We know that $f_k(\mathbf{x}, a) = ae^{x_1}$, $f_{2k}(\mathbf{x}, a) = a^2 e^{2x_1}$ and suppose that $f_{kn}(\mathbf{x}, a) = A_n^{(k)}(a) e^{nx_1}$. Then

$$\begin{aligned} f_{kn+k}(\mathbf{x}, a) &= f_{(kn-1)+k+1}(\mathbf{x}, a) \\ &= \sum_{i=0}^{kn-1} \binom{kn-1}{i} f_{kn-1-i+k}(\mathbf{x}, a) f_{i+1}(\mathbf{x}, a) \\ &= \sum_{i=0}^{n-1} \binom{kn-1}{ki+1} f_{k(n-i)}(\mathbf{x}, a) f_{ki+k}(\mathbf{x}, a) \\ &= \sum_{i=0}^{n-1} \binom{kn-1}{ki+1} A_{n-i}^{(k)}(a) e^{(n-i)x_1} A_{i+1}^{(k)}(a) e^{(i+1)x_1} \\ &= e^{(n+1)x_1} \sum_{i=0}^{n-1} \binom{kn-1}{ki+1} A_{n-i}^{(k)}(a) A_{i+1}^{(k)}(a) \\ &= e^{(n+1)x_1} A_{n+1}^{(k)}(a). \end{aligned}$$

∎

It is easy to show that $A_n^{(1)}(a) = (n-1)!a^n$ when $k=1$. We use Eq. (15) to prove this result. Suppose the result is true that $A_i^{(1)}(a) = (i-1)!a^i$ for i ranging between 1 and $n+1$. We have

$$\begin{aligned} A_{n+2}^{(1)}(a) &= \sum_{i=0}^{n} \binom{n}{i} A_{n-i+1}^{(1)}(a) A_{i+1}^{(1)}(a) \\ &= \sum_{i=0}^{n} \binom{n}{i} (n-i)! a^{n-i+1} i! a^{i+1} \\ &= a^{n+2} \sum_{i=0}^{n} \binom{n}{i} (n-i)! i! \\ &= a^{n+2} n! (n+1) = a^{n+2}(n+1)!. \end{aligned}$$

We can extend the above result to all $k \geq 1$.

Proposition 10. *For all $k \geq 1$ we have*

$$A_n^{(k)}(a) = a^n A_n^{(k)}(1).$$

Proof. Suppose by induction that $A_i^{(k)}(a) = a^i A_i^{(k)}(1)$ for all i ranging between 1 and $n+1$, then use the same steps as in the previous proof. □

From the above proposition it follows that

$$E_k(t, (x, 0, \ldots, 0), a) = E_k(at, (x, 0, \ldots, 0), 1).$$

Then without loss of generality it is sufficient to study the solution of Eq. (11) with initial conditions $y(0) = x$, $y'(0) = y'(0) = \cdots = y^{(k-1)}(0) = 0$, and $a = 1$ to generate the sequence $A_n^{(k)}(1)$.

The following corollary of Theorem 9 shows us that the numbers $A_n^{(k)}(a)$ can be constructed using Bell polynomials.

Corollary 11. *The numbers $A_n^{(k)}(a)$ satisfy the recurrence relation*

$$A_n^{(k)}(a) = B_{n-k}(\overbrace{0, \ldots, 0}^{k-1}, A_1^{(k)}(a), \ldots, \overbrace{0, \ldots, 0}^{k-1}, A_{n-k}^{(k)}(a)), \quad n \geq 1.$$

In the following theorem we calculate some special values of the functions $g_{n.i}(\mathbf{x}, a)$.

Theorem 12. *Let $\mathbf{x} = (x_1, 0, \ldots, 0)$. For all $a \in \mathbb{R}$ we have*

1. $g_{n,i}(\mathbf{x}, a) = 0$, *if $k \nmid n$.*

2. $g_{lk,1}(\mathbf{x}, a) = A_l^{(k)}(a) e^{lx_1}$.

3. $g_{lk,2}(\mathbf{x},a) = e^{lx_1} \sum_{j=1}^{l} \binom{kl-1}{kj-1} A_j^{(k)}(a) A_{l-j}^{(k)}(a)$.

4. $g_{n,n}(\mathbf{x},a) = g_{n,n-1}(\mathbf{x},a) = g_{n,n-2}(\mathbf{x},a) = g_{n,n-3}(\mathbf{x},a) = g_{n,n-4}(\mathbf{x},a) = 0,\ k > 1$.

Proof. Suppose $k \nmid n$ and $g_{n-j,i-1}(\mathbf{x},a) = 0$ for all k such that $k \nmid j$. Using Theorem 8 and 9 we have that $f_j(\mathbf{x},a) = 0$. This proves 1.

To prove 2 we have

$$\begin{aligned} g_{lk,1}(\mathbf{x},a) &= \sum_{j=1}^{lk} \binom{lk-1}{j-1} f_j(\mathbf{x},a) g_{lk-j,0}(\mathbf{x},a) \\ &= \binom{lk-1}{lk-1} f_{lk}(\mathbf{x},a) g_{0,0}(\mathbf{x},a) \\ &= A_l^{(k)}(a) e^{lx_1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} g_{lk,2}(\mathbf{x},a) &= \sum_{j=1}^{lk-1} \binom{lk-1}{j-1} f_j(\mathbf{x},a) g_{lk-j,1}(\mathbf{x},a) \\ &= \sum_{j=1}^{l} \binom{lk-1}{jk-1} f_{kj}(\mathbf{x},a) g_{lk-kj,1}(\mathbf{x},a) \\ &= \sum_{j=1}^{l} \binom{lk-1}{jk-1} A_j^{(k)}(a) e^{jx_1} A_{l-j}^{(k)}(a) e^{(l-j)x_1} \\ &= e^{lx_1} \sum_{j=1}^{l} \binom{lk-1}{jk-1} A_j^{(k)}(a) A_{l-j}^{(k)}(a). \end{aligned}$$

Then this proves 3.

To prove 4 we use Eqs. (2)–(6). □

We conclude this section with the following properties of the exponential autonomous functions.

Theorem 13. *For all $n \geq 1$, $k \geq 1$ and for all $a, c \in \mathbb{C}$ is fulfilled*

$$f_n(x_1 + k \ln c, cx_2, \ldots, c^{k-1} x_k, a) = c^n f_n(x_1, x_2, \ldots, x_k, a).$$

Proof. Let $\mathbf{y} = (x_1 + k \ln c, cx_2, \ldots, c^{k-1} x_k)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_k)$. Suppose that the result is true for $i \leq n$. Then

$$\begin{aligned} f_{n+1}(\mathbf{y}, a) &= f_{(n+1-k)+k}(\mathbf{y}, a) \\ &= ac^k e^{x_1} B_{n+1-k}(f_1(\mathbf{y}, a), \ldots, f_{n+1-k}(\mathbf{y}, a)) \\ &= ac^k e^{x_1} B_{n+1-k}(cf_1(\mathbf{x}, a), \ldots, c^{n+1-k} f_{n+1-k}(\mathbf{x}, a)) \\ &= ac^k c^{n+1-k} e^{x_1} B_n(f_1(\mathbf{x}, a), \ldots, f_n(\mathbf{x}, a)) \\ &= c^{n+1} f_{n+1}(\mathbf{x}, a). \end{aligned}$$

□

The following is the corollary to Theorem 13 that allows us to calculate the solutions of Eq. (11) when the initial values are $(x_1 + k \ln c, cx_2, \ldots, c^{k-1} x_k)$.

Corollary 14.

$$E_k(t, (x_1 + k \ln c, cx_2, \ldots, c^{k-1} x_k), a) = k \ln c + E_k(ct, (x_1, x_2, \ldots, x_k), a).$$

Proof. Let $\mathbf{y} = (x_1 + k \ln c, cx_2, \ldots, c^{k-1} x_k)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_k)$. From the above theorem and the definition of the function $E_k(t, \mathbf{x}, a)$ we have

$$\begin{aligned} E_k(t, \mathbf{y}, a) &= x_1 + k \ln c + \sum_{n=1}^{\infty} f_n(\mathbf{y}, a) \frac{t^n}{n!} \\ &= x_1 + k \ln c + \sum_{n=1}^{\infty} c^n f_n(\mathbf{x}, a) \frac{t^n}{n!} \\ &= x_1 + k \ln c + \sum_{n=1}^{\infty} f_n(\mathbf{x}, a) \frac{(ct)^n}{n!} \\ &= k \ln c + E_k(ct, \mathbf{x}, a). \end{aligned}$$

□

Finally, we compute $f_n(\mathbf{x}, a)$ when $\mathbf{x} = (x_1, -x_2, \ldots, x_{2k-1}, -x_{2k})$.

Theorem 15. *For all $n \geq 0$ and for all exponential autonomous functions of order $2k$ we have*

$$f_n((x_1, -x_2, \ldots, x_{2k-1}, -x_{2k}), a) = (-1)^n f_n((x_1, x_2, \ldots, x_{2k-1}, x_{2k}), a).$$

Proof. Let $\mathbf{y} = (x_1, -x_2, \ldots, x_{2k-1}, -x_{2k})$, and let $\mathbf{x} = (x_1, x_2, \ldots, x_{2k-1}, x_{2k})$. Suppose the result is true for all values less than or equal to n. Then

$$\begin{aligned} f_{n+1}(\mathbf{y}, a) &= f_{(n+1-2k)+2k}(\mathbf{y}, a) \\ &= ae^{x_1} B_{n+1-2k}(f_1(\mathbf{y}, a), f_2(\mathbf{y}, a), \ldots, f_{n+1-2k}(\mathbf{y}, a)) \\ &= ae^{x_1} B_{n+1-2k}(-f_1(\mathbf{x}, a), f_2(\mathbf{x}, a), \ldots, (-1)^{n+1-2k} f_{n+1-2k}(\mathbf{x}, a)) \\ &= (-1)^{n+1-2k} ae^{x_1} B_{n+1-2k}(f_1(\mathbf{x}, a), f_2(\mathbf{x}, a), \ldots, f_{n+1-2k}(\mathbf{x}, a)) \\ &= (-1)^{n+1} f_n(\mathbf{x}, a). \end{aligned}$$

□

Finally, we have the corollary to Theorem 15.

Corollary 16.

$$E_{2k}(-t,(x_1,-x_2,\dots,x_{2k-1},-x_{2k}),a)=E_{2k}(t,(x_1,x_2,\dots,x_{2k-1},x_{2k}),a).$$

Proof. From the above theorem and the definition of the function $E_{2k}(t,x,a)$, we have

$$\begin{aligned}E_{2k}(-t,\mathbf{y},a)&=\sum_{n=0}^{\infty}f_n(\mathbf{y},a)\frac{(-t)^n}{n!}\\&=\sum_{n=0}^{\infty}(-1)^n(-1)^n f_n(\mathbf{x},a)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}f_n(\mathbf{x},a)\frac{t^n}{n!}\\&=E_{2k}(t,\mathbf{x},a),\end{aligned}$$

where $\mathbf{y}=(x_1,-x_2,\dots,x_{2k-1},-x_{2k})$ and $\mathbf{x}=(x_1,x_2,\dots,x_{2k-1},x_{2k})$. □

4 (k,a)-autonomous coefficients

When $k=1$ we obtain the equation $y'=ae^y$, which is the easiest to solve for all $a\in\mathbb{R}$. Using the method of separation of variables, we obtain the solution

$$y(t)=-\ln(e^{-x}-at)$$

with initial condition $y(0)=x$. On the other hand, by Theorem 6 the solution in power series becomes

$$\begin{aligned}E_1(t,x,a)&=x+\sum_{n=1}^{\infty}A_n^{(1)}(a)\frac{t^n}{n!}\\&=x+\sum_{n=1}^{\infty}(n-1)!a^ne^{nx}\frac{t^n}{n!}\\&=x+\sum_{n=1}^{\infty}\frac{(ae^x)^n}{n}=x-\ln(1-ae^xt).\end{aligned}$$

Now we can use the results of the previous section to prove some results that are already known. By the definition of complete exponential autonomous functions

$$\begin{aligned}
n!a^{n+1}e^{a(n+1)x} &= ae^x \sum_{i=1}^{n} B_{n,i}(0!a^1e^x, 1!a^2e^{2x}, \ldots, (n-i)!a^{n-i+1}e^{(n-i+1)x}) \\
&= ae^x a^n e^{nx} \sum_{i=1}^{n} B_{n,i}(0!, 1!, \ldots, (n-i)!) \\
&= e^{a(n+1)x} a^{n+1} \sum_{i=1}^{n} B_{n,i}(0!, 1!, \ldots, (n-i)!) = e^{a(n+1)x} a^{n+1} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix}
\end{aligned}$$

from which we get a result relating factorials and unsigned Stirling number of the first kind:

$$n! = \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix}.$$

Furthermore $g_{n,i}(\mathbf{x}, 1) = \begin{bmatrix} n \\ i \end{bmatrix}$ and by Eq. (14) we obtain the following finite-sum identity

$$\begin{aligned}
\begin{bmatrix} n+1 \\ i+1 \end{bmatrix} &= \sum_{j=1}^{n-i+1} \binom{n}{j-1} (j-1)! \begin{bmatrix} n+1-j \\ i \end{bmatrix} \\
&= \sum_{j=n}^{i} \frac{n!}{j!} \begin{bmatrix} j \\ i \end{bmatrix} \\
&= \sum_{j=i}^{n} \frac{n!}{j!} \begin{bmatrix} j \\ i \end{bmatrix} \\
&= \sum_{j=0}^{n} \frac{n!}{j!} \begin{bmatrix} j \\ i \end{bmatrix}.
\end{aligned}$$

On the other hand, from Eq. (13) we obtain the trivial result

$$\begin{aligned}
(n+1)! &= \sum_{i=0}^{n} \binom{n}{i} (n-i)! i! \\
&= \sum_{i=0}^{n} n!.
\end{aligned}$$

The Stirling numbers of the first kind originally arose algebraically from the expansion of the falling factorial

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1)$$

and in polynomial form is as follows

$$(x)_n = \sum_{i=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ i \end{bmatrix} x^i.$$

Analogously, we want to define and study the coefficients of the expansion of the autonomous exponential polynomials $q_n(\mathbf{x}, a)$ with $\mathbf{x} = (0, x, x, \ldots, x)$. First we calculate the degree of $q_n(\mathbf{x}, a)$.

Proposition 17. *Let* $\mathbf{x} = (0, x, x, \ldots, x)$*. Then the degree* gr *of* $q_n(\mathbf{x}, a)$ *is*

$$\mathsf{gr}(q_n(\mathbf{x}, a)) = n - k, \quad n \geq k.$$

Proof. By definition

$$\begin{aligned} q_{n+k}(\mathbf{x}, a) &= \sum_{i=1}^{n-1} a^i g_{n,i}(\mathbf{x}, a) + a^n g_{n,n}(\mathbf{x}, a) \\ &= \sum_{i=1}^{n-1} a^i g_{n,i}(\mathbf{x}, a) + a^n x_1^n. \end{aligned}$$

As $\mathsf{gr}(g_{n,i}(\mathbf{x}, a)) \leq i$, then $\mathsf{gr}(q_{n+k}(\mathbf{x}, a)) = n$. □

We now define the autonomous polynomials and autonomous coefficients.

Definition 18. For all $n \geq k$, let $A_n^{(k)}(x, a) = q_n(0, x, \ldots, x, a)$ denote the *autonomous polynomials* of degree $n - k$.

Using Eq. (12) we note that

$$A_{n+k}^{(k)}(x, a) = aB_n(A_1^{(k)}(x, a), \ldots, A_n^{(k)}(x, a)) \tag{16}$$

for all $n \geq 1$.

Definition 19. We define the (k, a)-*autonomous coefficients*, denoted by $\left[\!\left[n \atop i \right]\!\right]_{(k,a)}$, as the coefficients of the autonomous polynomials $A_{n+k}^{(k)}(x, a)$, i.e.,

$$A_{n+k}^{(k)}(x, a) = \sum_{i=0}^{n} \left[\!\!\left[\begin{matrix} n \\ i \end{matrix} \right]\!\!\right]_{(k,a)} x^i.$$

Now we give some values of the (k, a)-autonomous coefficients.

Theorem 20. *Some values of the coefficients* $\left[\!\left[{n \atop i} \right]\!\right]_{(k,a)}$ *are*

$$\left[\!\!\left[{n \atop 0} \right]\!\!\right]_{(k,a)} = \begin{cases} 0, & \text{if } k \nmid n; \\ a^{n/k} A^{(k)}_{n/k}(1), & \text{if } k|n, \end{cases} \tag{17}$$

$$\left[\!\!\left[{0 \atop i} \right]\!\!\right]_{(k,a)} = 0, \ \text{if } i \geq 1, \tag{18}$$

$$\left[\!\!\left[{n \atop n-l} \right]\!\!\right]_{(k,a)} = a \left\{ {n \atop n-l} \right\}, \ k > l+1, \ 0 \leq l < n. \tag{19}$$

Proof. Eq. (17) follows from Theorem 9. By definition, $B_{0,i} = 0$ for $i \geq 1$. Then Eq. (18) is true. Finally, if $k > l + 1$,

$$\begin{aligned} B_{n,n-l}(A^{(k)}_1(x,a), \ldots, A^{(k)}_{l+1}(x,a)) &= B_{n,n-l}(x, \ldots, x) \\ &= \left\{ {n \atop n-l} \right\} x^{n-l} \end{aligned}$$

and from this Eq. (19) follows. □

We now show the relationship between the $(k, 1)$-autonomous coefficients and the binomial coefficients.

Theorem 21.

$$\begin{aligned} \left[\!\!\left[{n+1 \atop 0} \right]\!\!\right]_{(k,1)} &= \binom{n}{k-1} \left[\!\!\left[{n+1-k \atop 0} \right]\!\!\right]_{(k,1)} \\ &+ \sum_{h=k+1}^{n} \binom{n}{h} \left[\!\!\left[{n-h \atop 0} \right]\!\!\right]_{(k,1)} \left[\!\!\left[{h+1-k \atop 0} \right]\!\!\right]_{(k,1)}, \end{aligned}$$

for $1 \leq i \leq n - k + 3$

$$\begin{aligned} \left[\!\!\left[{n+1 \atop i} \right]\!\!\right]_{(k,1)} &= \sum_{h=0}^{k-2} \binom{n}{h} \left[\!\!\left[{n-h \atop i-1} \right]\!\!\right]_{(k,1)} \\ &+ \binom{n}{k-1} \left[\!\!\left[{n+1-k \atop i} \right]\!\!\right]_{(k,1)} + \binom{n}{k} \left[\!\!\left[{n-k \atop i-1} \right]\!\!\right]_{(k,1)} \\ &+ \sum_{h=k+1}^{n} \binom{n}{h} \sum_{j+l=i} \left[\!\!\left[{n-h \atop j} \right]\!\!\right]_{(k,1)} \left[\!\!\left[{h+1-k \atop l} \right]\!\!\right]_{(k,1)} \end{aligned}$$

and for $n-k+4 \le i \le n+1$

$$\left[\!\left[\begin{matrix} n+1 \\ i \end{matrix}\right]\!\right]_{(k,1)} = \sum_{h=0}^{n-i+1} \binom{n}{h} \left[\!\left[\begin{matrix} n-h \\ i-1 \end{matrix}\right]\!\right]_{(k,1)} + \binom{n}{k-1} \left[\!\left[\begin{matrix} n+1-k \\ i \end{matrix}\right]\!\right]_{(k,1)} + \binom{n}{k} \left[\!\left[\begin{matrix} n-k \\ i-1 \end{matrix}\right]\!\right]_{(k,1)} + \sum_{h=k+1}^{n} \binom{n}{h} \sum_{j+l=i} \left[\!\left[\begin{matrix} n-h \\ j \end{matrix}\right]\!\right]_{(k,1)} \left[\!\left[\begin{matrix} h+1-k \\ l \end{matrix}\right]\!\right]_{(k,1)}.$$

Proof. As

$$\begin{aligned} A^{(k)}_{n+1+k}(x,1) &= \sum_{i=0}^{n} \binom{n}{i} A^{(k)}_{n-i+k}(x,1) A^{(k)}_{i+1}(x,1) \\ &= \sum_{i=0}^{k-2} \binom{n}{i} A^{(k)}_{n-i+k}(x,1)x + \binom{n}{k-1} A^{(k)}_{n+1}(x,1) \\ &+ \binom{n}{k} A^{(k)}_{n}(x,1)x + \sum_{i=k+1}^{n} \binom{n}{i} A^{(k)}_{n-i+k}(x,1) A^{(k)}_{i+1}(x,1), \end{aligned}$$

then

$$\begin{aligned} \sum_{i=0}^{n+1} \left[\!\left[\begin{matrix} n+1 \\ i \end{matrix}\right]\!\right]_{(k,1)} x^i &= \sum_{i=0}^{k-2} \binom{n}{i} \sum_{j=0}^{n-i} \left[\!\left[\begin{matrix} n-i \\ j \end{matrix}\right]\!\right]_{(k,1)} x^{j+1} \\ &+ \binom{n}{k-1} \sum_{j=0}^{n+1-k} \left[\!\left[\begin{matrix} n+1-k \\ j \end{matrix}\right]\!\right]_{(k,1)} x^j + \binom{n}{k} \sum_{j=0}^{n-k} \left[\!\left[\begin{matrix} n-k \\ j \end{matrix}\right]\!\right]_{(k,1)} x^{j+1} \\ &+ \sum_{i=k+1}^{n} \binom{n}{i} \left(\sum_{j=0}^{n-i} \left[\!\left[\begin{matrix} n-i \\ j \end{matrix}\right]\!\right]_{(k,1)} x^j \sum_{l=0}^{i+1-k} \left[\!\left[\begin{matrix} i+1-k \\ l \end{matrix}\right]\!\right]_{(k,1)} x^l \right). \end{aligned}$$

We multiply the two autonomous polynomials within the last sum to get

$$\begin{aligned} \sum_{i=0}^{n+1} \left[\!\left[\begin{matrix} n+1 \\ i \end{matrix}\right]\!\right]_{(k,1)} x^i &= \sum_{i=0}^{k-2} \binom{n}{i} \sum_{j=0}^{n-i} \left[\!\left[\begin{matrix} n-i \\ j \end{matrix}\right]\!\right]_{(k,1)} x^{j+1} \\ &+ \binom{n}{k-1} \sum_{j=0}^{n+1-k} \left[\!\left[\begin{matrix} n+1-k \\ j \end{matrix}\right]\!\right]_{(k,1)} x^j + \binom{n}{k} \sum_{j=0}^{n-k} \left[\!\left[\begin{matrix} n-k \\ j \end{matrix}\right]\!\right]_{(k,1)} x^{j+1} \\ &+ \sum_{i=k+1}^{n} \binom{n}{i} \sum_{h=0}^{n+1-k} \left(\sum_{j+l=h} \left[\!\left[\begin{matrix} n-i \\ j \end{matrix}\right]\!\right]_{(k,1)} \left[\!\left[\begin{matrix} i+1-k \\ l \end{matrix}\right]\!\right]_{(k,1)} \right) x^h. \end{aligned}$$

Then by rearranging the first and fourth sums we obtain

$$\begin{aligned}\sum_{i=0}^{n+1} \left[\!\!\left[{n+1 \atop i} \right]\!\!\right]_{(k,1)} x^i = &\sum_{i=0}^{n-k+3} \left(\sum_{h=0}^{k-2} \binom{n}{h} \left[\!\!\left[{n-h \atop i-1} \right]\!\!\right]_{(k,1)} \right) x^i \\ &+ \sum_{i=n-k+4}^{n+1} \left(\sum_{h=0}^{n+1-i} \binom{n}{h} \left[\!\!\left[{n-h \atop i-1} \right]\!\!\right]_{(k,1)} \right) x^i \\ &+ \binom{n}{k-1} \sum_{i=0}^{n+1-k} \left[\!\!\left[{n+1-k \atop i} \right]\!\!\right]_{(k,1)} x^i \\ &+ \binom{n}{k} \sum_{i=1}^{n-k+1} \left[\!\!\left[{n-k \atop i-1} \right]\!\!\right]_{(k,1)} x^i \\ &+ \sum_{i=0}^{n+1-k} \left(\sum_{h=k+1}^{n} \binom{n}{h} \sum_{j+l=i} \left[\!\!\left[{n-h \atop j} \right]\!\!\right]_{(k,1)} \left[\!\!\left[{h+1-k \atop l} \right]\!\!\right]_{(k,1)} \right) x^i.\end{aligned}$$

For a suitable value of i the desired results are attained. □

Finally, we show without proof the relationship between the $(k,1)$-autonomous coefficients and the Stirling numbers of the second kind.

Conjecture 22. Suppose that $A_1^{(k)}(1,1) = \cdots = A_k^{(k)}(1,1) = 1$. Then

$$B_n(A_1^{(k)}(1,1), \ldots, A_n^{(k)}(1,1)) = \sum_{i=1}^{n} \left\{ {n \atop i} \right\} A_i^{(k)}(1,1), \quad n \geq 1.$$

Then,

$$A_{n+k}^{(k)}(1,1) = \sum_{i=1}^{n} \left\{ {n \atop i} \right\} A_i^{(k)}(1,1), \quad n \geq 1 \tag{20}$$

and

$$\sum_{i=0}^{n} \left[\!\!\left[{n \atop i} \right]\!\!\right]_{(k,1)} = \sum_{j=1}^{k} \left\{ {n \atop j} \right\} + \sum_{j=k+1}^{n} \left\{ {n \atop j} \right\} \sum_{i=0}^{j-k} \left[\!\!\left[{n \atop i} \right]\!\!\right]_{(k,1)}.$$

Eq. (20) corresponds to the number of shifts left $k-1$ places under Stirling transform.

5 Sequences related to Eq. (11)

We conclude this paper by showing sequences related to Eq. (11) for values of $k = 2, 3, 4$. Especially, we show that the numbers known as reduced tangent numbers, Bernoulli numbers, Euler zigzag numbers, Eulerian numbers, Blasius numbers, triangular numbers, number of shifts left 3 places under Stirling transform, and number of 8-sequences of $[1, n]$ with 2 contiguous pairs can be constructed using Bell polynomials, Stirling numbers of the second kind, binomial coefficient and autonomous coefficients.

5.1 Case $k = 2$

The first case to be studied is

$$y'' = ae^y. \tag{21}$$

Eq. (21) is equivalent to the equation $y^{(3)} = y''y'$, whose solution is

$$y' = \sqrt{2}\sqrt{c_1}\tan\left(\frac{1}{2}\sqrt{2}\sqrt{c_1}t + \frac{1}{2}\sqrt{2}\sqrt{c_1}c_2\right)$$

and therefore

$$y = \ln\left(\sec^2\left(\frac{1}{2}\sqrt{2}\sqrt{c_1}t + \frac{1}{2}\sqrt{2}\sqrt{c_1}c_2\right)\right) + c_3$$

where c_1, c_2 and c_3 are constants in $\mathbb{C}$. Since we want $y(0) = x$, $y'(0) = y$ and $y''(0) = ae^x$, then

$$\begin{aligned}
\ln\left(\sec^2\left(\frac{1}{2}\sqrt{2}\sqrt{c_1}c_2\right)\right) + c_3 &= x, \\
\sqrt{2}\sqrt{c_1}\tan\left(\frac{1}{2}\sqrt{2}\sqrt{c_1}c_2\right) &= y, \\
c_1\sec^2\left(\frac{1}{2}\sqrt{2}\sqrt{c_1}c_2\right) &= ae^x.
\end{aligned}$$

Hence

$$\begin{aligned}
c_1 &= ae^x - \frac{y^2}{2}, \\
\frac{1}{2}\sqrt{2}\sqrt{c_1}c_2 &= \arctan\left(\frac{y}{\sqrt{2ae^x - y^2}}\right), \\
c_3 &= x - \ln\left(1 + \frac{y^2}{2ae^x - y^2}\right).
\end{aligned}$$

Thus, the function

$$\begin{aligned}
E_2(t, (x, y), a) = x + \ln\sec^2&\left(\frac{\sqrt{2ae^x - y^2}t}{2} + \arctan\left(\frac{y}{\sqrt{2ae^x - y^2}}\right)\right) \\
&- \ln\left(1 + \frac{y^2}{2ae^x - y^2}\right)
\end{aligned} \tag{22}$$

is the solution of Eq. (21) with initial values $y(0) = x$ and $y'(0) = y$.

The following is a list of particular solutions of Eq. (21) that can be obtained from Eq. (22):

$$E_2(t,(x,0),a) = x + \ln\left(\sec^2\left(\frac{\sqrt{a}e^{x/2}t}{\sqrt{2}}\right)\right), \quad a>0, \tag{23}$$

$$E_2(t,(x,0),-a) = x + \ln\left(\operatorname{sech}^2\left(\frac{\sqrt{a}e^{x/2}t}{\sqrt{2}}\right)\right), \quad a>0,$$

$$\begin{aligned} E_2(t,(0,y),a) &= \ln\sec^2\left(\frac{\sqrt{2a-y^2}t}{2} + \arctan\left(\frac{y}{\sqrt{2a-y^2}}\right)\right) \\ &\quad - \ln\left(1+\frac{y^2}{2a-y^2}\right), \quad a>0, \end{aligned} \tag{24}$$

$$\begin{aligned} E_2(t,(0,y),-a) &= \ln\operatorname{sech}^2\left(\frac{\sqrt{2a+y^2}t}{2} + \operatorname{arctanh}\left(\frac{y}{\sqrt{2a+y^2}}\right)\right) \\ &\quad - \ln\left(1-\frac{y^2}{2a+y^2}\right), \quad a>0. \end{aligned}$$

We now show the relationship between reduced tangent numbers (A002105 in [17]) and Bell polynomials and binomial coefficients.

Theorem 23. *Let*

$$(T_n)_{n\geq 1} = (1, 1, 4, 34, 496, \ldots)$$

denote the sequence of reduced tangent numbers. Then

1. $A_n^{(2)}(a) = a^n T_n$.
2. $T_n = B_n(0, T_1, \ldots, 0, T_{n-1})$, $n \geq 2$.
3. $(-1)^n T_n = B_n(0, -T_1, \ldots, 0, (-1)^{n-1} T_{n-1})$, $n \geq 2$.
4. $T_{n+2} = \sum_{i=0}^{n} \binom{2n+1}{2i+1} T_{n-i+2} T_{i+1}$, $n \geq 0$.

Proof. Another way to write Eq. (23) is

$$E_2(t,(x,0),a) = x + \sqrt{2}\int_0^{\sqrt{a}e^{x/2}t} \tan\left(\frac{u}{\sqrt{2}}\right) du.$$

Then

$$\begin{aligned} E_2(t,(x,0),a) &= x + \int_0^{\sqrt{a}e^{x/2}t} \sum_{n=1}^{\infty} T_n \frac{u^{2n-1}}{(2n-1)!} \\ &= x + \sum_{n=1}^{\infty} T_n \frac{(\sqrt{a}e^{x/2}t)^{2n}}{(2n)!}. \end{aligned}$$

By comparison, $f_{2n}(x,0,a) = a^n T_n e^{nx}$. Thus parts 1, 2, and 3 follow.

The recurrence relation 4 follows from Eq. (15). □

In general, Eq. (23) is the generating function of the sequence

$$(a^n T_n)_{n\geq 1} = (a, a^2, 4a^3, 34a^4, 496a^5, \ldots).$$

On the other hand, it is known that $T_n = \frac{2^n(2^{2n}-1)|b_{2n}|}{n}$, where the b_{2n} are the Bernoulli numbers (A000367, A002445 in [17]). Then Theorem 23 provides a relation between Bell polynomials and Bernoulli numbers, that is

$$\frac{2^n(2^{2n}-1)|b_{2n}|}{n} = B_n\left(0, 6|b_2|, 0, 30|b_4|, \ldots, 0, \frac{2^{n-1}(2^{2n-2}-1)|b_{2n-2}|}{n-1}\right).$$

We now show the relationship between Euler zigzag numbers (A000111 in [17]) and Bell polynomials, binomial coefficients, and Stirling numbers of the second kind.

Theorem 24. *Let*

$$(A_n)_{n\geq 0} = (1, 1, 1, 2, 5, 16, 61, 272, \ldots)$$

denote the sequence of Euler zigzag numbers. Then

1. $A_{n+1} = B_n(A_0, \ldots, A_{n-1})$, $n \geq 1$.
2. $(-1)^n A_{n+1} = B_n(-A_0, A_1, \ldots, (-1)^{n-1}A_{n-1})$, $n \geq 1$.
3. $A_{n+2} = \sum_{i=0}^{n} \binom{n}{i} A_{n-i+1}A_i$, $n \geq 0$.
4. $A_{n+2} = \sum_{i=1}^{n} \left\{ {n \atop i} \right\} A_i$, $n \geq 1$.
5. $A_{2n+2} = T_n + \sum_{i=1}^{n} \left[\!\left[{2n \atop 2i} \right]\!\right]_{(2,1)}$.
6. $A_{2n+3} = \sum_{i=1}^{n} \left[\!\left[{2n+1 \atop 2i+1} \right]\!\right]_{(2,1)}$.

Proof. From Eq. (24) we get

$$\begin{aligned} E_2(t,(0,1),1) &= \ln\left(\sec^2\left(\frac{t}{2}+\frac{\pi}{4}\right)\right) - \ln(2) \\ &= \ln(\sec^2(t) + \sec(t)\tan(t)). \end{aligned}$$

Then

$$\begin{aligned}
E_2(t,(0,1),1) &= \int_0^t (\sec(u)+\tan(u))du \\
&= \int_0^t \left(1+\sum_{n=1}^{\infty} A_n \frac{u^n}{n!}\right) du \\
&= t + \sum_{n=1}^{\infty} A_n \int_0^t \frac{u^n}{n!} du \\
&= t + \sum_{n=1}^{\infty} A_n \frac{t^{n+1}}{(n+1)!} \\
&= \sum_{n=1}^{\infty} A_{n-1} \frac{t^n}{n!}.
\end{aligned}$$

We apply Eq. (12) to obtain 1.

By Corollary 16, it follows that

$$E_2(t,(0,-1),1) = E(-t,(0,1),1) = \ln(\sec^2(t) - \sec(t)\tan(t)).$$

From Eq. (9) we get 2.

Formula 3 follows from Eq. (13).

From Eq. (20) we get 4.

Identities 5 and 6 follow because the Euler zigzag numbers are obtained when $x = 1$ in $A_n^{(2)}(x,1)$. □

When $k = 2$, the exponential autonomous polynomials and the autonomous polynomials match. Some autonomous polynomials of Eq. (21) are as follows:

$$\begin{aligned}
q_1(y,a) &= y, \\
q_2(y,a) &= a, \\
q_3(y,a) &= ay, \\
q_4(y,a) &= a(a+y^2), \\
q_5(y,a) &= a(4ay+y^3), \\
q_6(y,a) &= a(4a^2+11ay^2+y^4), \\
q_7(y,a) &= a(34a^2y+26ay^3+y^5), \\
q_8(y,a) &= a(34a^3+180a^2y^2+57ay^4+y^6).
\end{aligned}$$

From the above we obtain the first $(2,a)$-autonomous coefficients

n \ i	0	1	2	3	4	5	6
0	a						
1	0	a					
2	a^2	0	a				
3	0	$4a^2$	0	a			
4	$4a^3$	0	$11a^2$	0	a		
5	0	$34a^3$	0	$26a^2$	0	a	
6	$34a^4$	0	$180a^3$	0	$57a^2$	0	a

Table 1: $(2, a)$-autonomous coefficients.

Theorem 25. *Some values of $(2, a)$-autonomous coefficients are*

$$\left[\!\!\left[{2n \atop 2i+1} \right]\!\!\right]_{(2,a)} = \left[\!\!\left[{2n+1 \atop 2i} \right]\!\!\right]_{(2,a)} = 0 \tag{25}$$

for all i.

Proof. Eq. (25) follows from Theorem 9. □

Conjecture 26.

$$\left[\!\!\left[{n \atop n-2} \right]\!\!\right]_{(2,a)} = a^2(2^n - n - 1).$$

The sequence

$$2^n - n - 1 = (0, 0, 1, 4, 11, 26, 57, 120, 247, 502, 1013, 2036, 4083, 8178, 16369, \\ 32752, 65519, 131054, 262125, 524268, 1048555, 2097130, \ldots)$$

is known as the Eulerian numbers (A000295 in [17]).

5.2 Case $k = 3$

When $k = 3$, we obtain the equation

$$y^{(3)} = ae^y. \tag{26}$$

Solving Eq. (26) with initial conditions $(0, 0, x)$ and $a = -1$, we get the solution of Blasius equation

$$u^{(3)} + u''u = 0.$$

The Blasius equation [14] describes the velocity profile of the fluid in the boundary layer which forms when fluid flows along with a flat plate. Using Theorem 9 and Corollary 11, we obtain the following result on Blasius numbers (A018893 in [17]).

Theorem 27. *Let*

$$(\mathrm{b}_n)_{n\geq 1} = (1, 1, 11, 375, 27.897, \ldots)$$

denote the sequence of Blasius numbers. Then

1. $\mathrm{b}_n = B_n(0, 0, \mathrm{b}_1, \ldots, 0, 0, \mathrm{b}_{n-1})$, $n \geq 2$.
2. $\mathrm{b}_{n+2} = \sum_{i=0}^{n} \binom{3n+2}{3i+2} \mathrm{b}_{n-i+1} \mathrm{b}_{i+1}$, $n \geq 0$.

On the other hand, the autonomous polynomials for Eq. (26) are

$$\begin{aligned}
A_3^{(3)}(x, a) &= a,\\
A_4^{(3)}(x, a) &= ax,\\
A_5^{(3)}(x, a) &= a(x + x^2),\\
A_6^{(3)}(x, a) &= a(a + 3x^2 + x^3),\\
A_7^{(3)}(x, a) &= a(5ax + 3x^2 + 6x^3 + x^4),\\
A_8^{(3)}(x, a) &= a(11ax + 16ax^2 + 15x^3 + 10x^4 + x^5),\\
A_9^{(3)}(x, a) &= a(11a^2 + 84ax^2 + (42a + 15)x^3 + 45x^4 + 15x^5 + x^6),\\
A_{10}^{(3)}(x, a) &= a(117a^2x + 129ax^2 + 384ax^3 + (99a + 105)x^4 + 105x^5 + 21x^6 + x^7)
\end{aligned}$$

and from here we obtain the following table with the first $(3, a)$-autonomous coefficients

n \ i	0	1	2	3	4	5	6	7
0	a							
1	0	a						
2	0	a	a					
3	a^2	0	$3a$	a				
4	0	$5a^2$	$3a$	$6a$	a			
5	0	$11a^2$	$16a^2$	$15a$	$10a$	a		
6	$11a^3$	0	$84a^2$	$42a^2 + 15a$	$45a$	$15a$	a	
7	0	$117a^3$	$129a^2$	$384a^2$	$99a^2 + 105a$	$105a$	$21a$	a

Table 2: $(3, a)$-autonomous coefficients.

Theorem 28. *Some values of* $(3, a)$*-autonomous coefficients are*

$$\left[\!\!\left[\begin{matrix} n \\ n \end{matrix} \right]\!\!\right]_{(3,a)} = a,$$

$$\left[\!\!\left[\begin{matrix} n \\ n-1 \end{matrix} \right]\!\!\right]_{(3,a)} = a\binom{n}{2}.$$

Proof. The results follow from Theorem 20 with $l = 0, 1$ and by keeping in mind that $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$. □

Conjecture 29.

$$\left[\!\left[{n \atop n-2} \right]\!\right]_{(3,a)} = a\binom{\binom{n}{2}}{2}.$$

The numbers $\left[\!\left[{n \atop n-2} \right]\!\right]_{(3,1)}$ are the triangular numbers

$$(0, 0, 3, 15, 45, 105, 210, 378, 630, 990, 1485, \ldots)$$

(A050534 in [17]).

Finally, by Eqs. (9), (16), (20) we have the following theorem.

Theorem 30. *Let*

$$(\mathrm{e}_n)_{n\geq 1} = (A_n^{(3)}(1,1))_{n\geq 1} = (1, 1, 1, 1, 2, 5, 15, 53, 213, \ldots)$$

denote the number of shifts 3 places left under exponentiation (A007548 in [17]). Then

1. $\mathrm{e}_{n+3} = B_n(\mathrm{e}_1, \ldots, \mathrm{e}_n)$, $n \geq 1$.
2. $(-1)^n \mathrm{e}_{n+3} = B_n(-\mathrm{e}_1, \mathrm{e}_2, \ldots, (-1)^{n-1}\mathrm{e}_n)$, $n \geq 1$.
3. $\mathrm{e}_{n+3} = \sum_{i=1}^{n} \left\{ {n \atop i} \right\} \mathrm{e}_i$, $n \geq 1$.
4. $\mathrm{d}_{3n} = \mathrm{b}_n + \sum_{i=1}^{3n} \left[\!\left[{3n \atop i} \right]\!\right]_{(3,1)}$.
5. $\mathrm{d}_{3n+j} = \sum_{i=1}^{3n+j} \left[\!\left[{3n+j \atop i} \right]\!\right]_{(3,1)}$, $j = 1, 2$.

5.3 Case $k = 4$

The equation to be studied is

$$y^{(4)} = ae^y. \tag{27}$$

This equation is not commonly studied in the literature. Here we show the relation of this equation with the number of shifts left 3 places under Stirling transform, and also the relation with the numbers $A_n^{(4)}(1)$. A list of exponential autonomous polynomials of Eq. (27) is as follows:

$$\begin{aligned}
q_1(y, z, w, a) &= y, \\
q_2(y, z, w, a) &= z, \\
q_3(y, z, w, a) &= w, \\
q_4(y, z, w, a) &= a, \\
q_5(y, z, w, a) &= ay, \\
q_6(y, z, w, a) &= a(z + y^2), \\
q_7(y, z, w, a) &= a(w + 3yz + y^3), \\
q_8(y, z, w, a) &= a(a + 3z^2 + 4yw + 6y^2z + y^4).
\end{aligned}$$

From Eq. (15) we calculate the first few numbers $A_n^{(4)}(1)$,

$$
\begin{aligned}
q_4(0,0,0,1) &= A_1^{(4)}(1) = 1,\\
q_8(0,0,0,1) &= A_2^{(4)}(1) = \binom{3}{3} A_1^{(4)} A_1^{(4)} = 1,\\
q_{12}(0,0,0,1) &= A_3^{(4)}(1) = \binom{7}{3} A_2^{(4)} A_1^{(4)} + \binom{7}{7} A_1^{(4)} A_2^{(4)} = 35,\\
q_{16}(0,0,0,1) &= A_4^{(4)}(1) = \binom{11}{3} A_3^{(4)} A_1^{(4)} + \binom{11}{7} A_2^{(4)} A_2^{(4)} + \binom{11}{11} A_1^{(4)} A_3^{(4)} = 6140.
\end{aligned}
$$

Following Theorem 9, Corollary 11 and Eq. (9) we have the following recurrence relations for the numbers $A_n^{(4)}(1)$.

Theorem 31. *Let* $(\mathrm{c}_n)_{n\geq 1} = (A_n^{(4)}(1))_{n\geq 1} = (1, 1, 35, 6140, \ldots)$. *Then*

1. $\mathrm{c}_n = B_n(0,0,0,\mathrm{c}_1,\ldots,0,0,0,\mathrm{c}_{n-1})$, $n \geq 2$.
2. $\mathrm{c}_{n+2} = \sum_{i=0}^{n} \binom{4n+3}{4i+3} \mathrm{c}_{n-i+1}\mathrm{c}_{i+1}$, $n \geq 0$.
3. $(-1)^n c_n = B_n(0,0,0,-\mathrm{c}_1,\ldots,0,0,0,(-1)^n \mathrm{c}_{n-1})$.

The autonomous polynomials associated with Eq. (27) are

$$
\begin{aligned}
&A_1^{(4)}(x,a) = A_2^{(4)}(x,a) = A_3^{(4)}(x,a) = x,\\
&A_4^{(4)}(x,a) = a,\\
&A_5^{(4)}(x,a) = ax,\\
&A_6^{(4)}(x,a) = a(x + x^2),\\
&A_7^{(4)}(x,a) = a(x + 3x^2 + x^3),\\
&A_8^{(4)}(x,a) = a(a + 7x^2 + 6x^3 + x^4),\\
&A_9^{(4)}(x,a) = a(6ax + 10x^2 + 25x^3 + 10x^4 + x^5),\\
&A_{10}^{(4)}(x,a) = a(16ax + 32ax^2 + 75x^3 + 65x^4 + 15x^5 + x^6),\\
&A_{11}^{(4)}(x,a) = a(36ax + 136ax^2 + (64a + 175)x^3 + 315x^4 + 140x^5 + 21x^6 + x^7).
\end{aligned}
$$

We now derive recurrence relations of the numbers $A_n^{(4)}(1,1)$ using Eqs. (9), (16), (20).

Theorem 32. *Let*

$$(\mathrm{d}_n)_{n\geq 1} = (A_n^{(4)}(1,1))_{n\geq 1} = (1,1,1,1,1,2,5,15,53,222,1115,6698,\ldots)$$

denote the number of shifts left 3 places under Stirling transform (A336020 in [17]). Then

1. $\mathrm{d}_{n+4} = B_n(\mathrm{d}_1, \ldots, \mathrm{d}_n)$, $n \geq 1$.
2. $(-1)^n \mathrm{d}_{n+4} = B_n(-\mathrm{d}_1, \mathrm{d}_2, \ldots, (-1)^{n-1}\mathrm{d}_n)$, $n \geq 1$.
3. $\mathrm{d}_{n+4} = \sum_{i=1}^{n} \left\{ {n \atop i} \right\} \mathrm{d}_i$, $n \geq 1$.
4. $\mathrm{d}_{4n} = \mathrm{c}_n + \sum_{i=1}^{4n} \left[\!\left[{4n \atop i} \right]\!\right]_{(4,1)}$.
5. $\mathrm{d}_{4n+j} = \sum_{i=1}^{4n+j} \left[\!\left[{4n+j \atop i} \right]\!\right]_{(4,1)}$, $j = 1, 2, 3$.

The following is a table of the first $(4, a)$-autonomous coefficients:

$n \backslash i$	0	1	2	3	4	5	6	7
0	a							
1	0	a						
2	0	a	a					
3	0	a	$3a$	a				
4	a^2	0	$7a$	$6a$	a			
5	0	$6a^2$	$10a$	$25a$	$10a$	a		
6	0	$16a^2$	$32a^2$	$75a$	$65a$	$15a$	a	
7	0	$36a^2$	$136a^2$	$64a + 175$	$315a$	$140a$	$21a$	a

Table 3: $(4, a)$-autonomous coefficients.

Theorem 33. *Some values of $(4, a)$-autonomous coefficients are*

$$\left[\!\left[{n \atop n} \right]\!\right]_{(4,a)} = a, \tag{28}$$

$$\left[\!\left[{n \atop n-1} \right]\!\right]_{(4,a)} = a\binom{n}{2}, \tag{29}$$

$$\left[\!\left[{n \atop n-2} \right]\!\right]_{(4,a)} = a\left\{ {n+2 \atop n} \right\}. \tag{30}$$

Proof. Eqs. (28), (29), (30) arise from Theorem 20 with $l = 0, 1, 2$. □

Conjecture 34.

$$\left[\!\left[{n \atop n-3} \right]\!\right]_{(4,a)} = \frac{5a}{2}(n-1)\binom{n}{5}, \; n \geq 5.$$

The sequence

$$\left[\!\left[{n \atop n-3} \right]\!\right]_{(4,1)} = (10, 75, 315, 980, 2520, 5670, 11550, 21780, 38610, 65065,$$
$$105105, 163800, 247520, 364140, 523260, 736440, 1017450,$$
$$1382535, 1850695, 2443980, 3187800, 4111250, 5247450, \ldots)$$

counts the number of 8-sequences of $[1, n]$ with 2 contiguous pairs, (A027778 in [17]).

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