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A Note on the Andrews-Ericksson-Petrov-Romick Bijection for MacMahon's Partition Theorem

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Abstract

Andrews' generalization of MacMahon's partition theorem states that the number of partitions of n in which odd multiplicities are at least $2r+1$ is equal to the number of partitions in which odd parts are congruent to $2r+1 \pmod{4r+2}$. In this note, we give a bijective proof of this generalization. Our result naturally extends the bijection of Andrews, Ericksson, Petrov, and Romik for MacMahon's partition theorem.

1 Introduction

Let $n \in \mathbb{Z}_{\geq 1}$. A *partition* of n is a sequence of positive integers $(\lambda_1, \lambda_2, \ldots)$ where $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum\limits_{j\geq 1} \lambda_j = n$. The components λ_i are called *parts* of a partition. If further restrictions are imposed on the parts, one gets some interesting partition functions. One such example is the number of partitions of n into distinct parts. This is interesting because it relates to the number of partitions of n into odd parts, a case of Euler's partition identity [5]. A partition identity relates two partition functions. One of the partition identities of interest in this paper is Theorem 1, which is due to MacMahon.

Theorem 1 (MacMahon, [4]). *The number of partitions of n wherein no part appears with multiplicity one is equal to the number of partitions of n where parts are even or congruent to* 3 (mod 6)*.*

Andrews, Ericksson, Petrov, and Romik [2] gave the first bijective proof of Theorem 1 which we describe in the sequel.

Let C_n denote the set of partitions of n wherein no part appears with multiplicity one and let D_n denote the set of partitions of n wherein parts are even or congruent to 3 (mod 6). The bijection goes as follows: let $\lambda = (l^{h_l}, (l-1)^{h_{l-1}}, \ldots, 3^{h_3}, 2^{h_2}, 1^{h_1}) \in C_n$ where h_i is the multiplicity of i. Since no part appears exactly once, it follows that $h_i \in \{0, 2, 3, 4, \ldots\}$. Uniquely decompose h_i as

$$h_i = k_i + g_i, \text{ where } k_i \in \{0, 3\}, \; g_i \in \{0, 2, 4, 6, 8, \ldots\}.$$

For $j \geq 1$, define d_j as follows:

$$\begin{aligned}
d_{6t+1} &= d_{6t+5} = 0, \\
d_{6t+2} &= \frac{1}{2} g_{3t+1}, \\
d_{6t+4} &= \frac{1}{2} g_{3t+2}, \\
d_{6t+3} &= \frac{1}{3} k_{2t+1} + g_{6t+3}, \\
d_{6t+6} &= \frac{1}{3} k_{2t+2} + g_{6t+6},
\end{aligned}$$

where $t = 0, 1, 2, \ldots$

The partition $(f^{d_f}, (f-1)^{d_{f-1}}, \ldots, 2^{d_2}, 1^{d_1})$ is in D_n. We shall call the above mapping the *Andrews-Ericksson-Petrov-Romik bijection.*

Andrews later gave a generalization for Theorem 1 which we recall below.

Theorem 2 (Andrews, [1]). *Let $A_r(n)$ denote the set of partitions of n wherein parts appearing an odd number of times actually appear at least $2r+1$ times and let $B_r(n)$ denote the set of partitions of n wherein odd parts are congruent to $2r+1$* (mod $4r+2$)*. Then*

$$|A_r(n)| = |B_r(n)|.$$

However, there is no bijective proof that naturally extends Andrews-Ericksson-Petrov-Romik bijection to prove Theorem 2. For this, you may consult [3] and the references therein. Our aim in this short note is to supply a bijection for Theorem 2, which naturally generalizes Andrews-Ericksson-Petrov-Romik bijection and is different from the ones given in the literature.

2 The bijection

We now describe our bijection for Theorem 2. We need to establish the one-to-one correspondence between the sets $A_r(n)$ and $B_r(n)$.

Let $(l^{h_l}, (l-1)^{h_{l-1}}, \ldots, 3^{h_3}, 2^{h_2}, 1^{h_1}) \in A_r(n)$. In this notation, h_i is the multiplicity of i. Note that h_i can be zero for some i. Clearly, $h_i \in \{0, 2, 4, \ldots, 2r, 2r+1, 2r+2, \ldots\}$.

Write h_i as

$$h_i = k_i + g_i \text{ where } k_i \in \{0, 2r+1\} \text{ and } g_i \in \{0, 2, 4, 6, 8, 10, \ldots\}$$

This decomposition is unique since k_i and g_i can be made explicit, i.e.,

$$k_i = (2r+1)\left(\frac{1-(-1)^{h_i}}{2}\right) \text{ and } g_i = h_i - (2r+1)\left(\frac{1-(-1)^{h_i}}{2}\right).$$

For $j \geq 1$, define d_j as follows:

$$\begin{aligned} d_{(4r+2)t+2j-1} &= 0 \text{ for } j \in \{1, 2, \ldots, 2r+1\} \setminus \{r+1\}, \\ d_{(4r+2)t+2j} &= \frac{1}{2} g_{(2r+1)t+j} \text{ for } j \in \{1, 2, \ldots, 2r\}, \\ d_{(4r+2)t+(2r+1)j} &= \frac{1}{2r+1} k_{2t+j} + g_{(4r+2)t+(2r+1)j} \text{ for } j \in \{1, 2\}, \end{aligned}$$

where $t = 0, 1, 2, \ldots$.

The image is thus given by

$$(\ldots, f^{d_f}, (f-1)^{d_{f-1}}, \ldots, 2^{d_2}, 1^{d_1}).$$

We claim that $(\ldots, f^{d_f}, (f-1)^{d_{f-1}}, \ldots, 2^{d_2}, 1^{d_1}) \in B_r(n)$. In order to show that

$$(l^{h_l}, (l-1)^{h_{l-1}}, \ldots, 3^{h_3}, 2^{h_2}, 1^{h_1}) \mapsto (\ldots, f^{d_f}, (f-1)^{d_{f-1}}, \ldots, 2^{d_2}, 1^{d_1})$$

defines a bijection from $A_r(n)$ onto $B_r(n)$, it suffices to show that $\sum_{i\geq 1} i d_i = n$.

Thus

$$\begin{aligned} \sum_{i\geq 1} i d_i &= \sum_{j=1}^{r}\sum_{t=0}^{\infty}((4r+2)t+2j-1)d_{(4r+2)t+2j-1} + \sum_{j=r+2}^{2r+1}\sum_{t=0}^{\infty}((4r+2)t+2j-1)d_{(4r+2)t+2j-1} \\ &\quad + \sum_{j=1}^{2r}\sum_{t=0}^{\infty}((4r+2)t+2j)d_{(4r+2)t+2j} + \sum_{j=1}^{2}\sum_{t=0}^{\infty}((4r+2)t+(2r+1)j)d_{(4r+2)t+(2r+1)j} \\ &= \sum_{j=1}^{2r}\sum_{t=0}^{\infty}((2r+1)t+j)g_{(2r+1)t+j} + \sum_{j=1}^{2}\sum_{t=0}^{\infty}((4r+2)t+(2r+1)j)g_{(4r+2)t+(2r+1)j} \\ &\quad + \sum_{j=1}^{2}\sum_{t=0}^{\infty}(2t+j)k_{2t+j}. \end{aligned}$$

Using the notation

$$S(a,b)=\sum_{t=0}^{\infty}(at+b)g_{at+b} \quad \text{and} \quad T(a,b)=\sum_{t=0}^{\infty}(at+b)k_{at+b},$$

observe that

$$\begin{aligned}\sum_{t=0}^{\infty}((2r+1)t+j)g_{(2r+1)t+j} &= \sum_{\ell=0,2r+1}\sum_{t=0}^{\infty}((4r+2)t+j+\ell)g_{(4r+2)t+j+\ell} \\ &= S(4r+2,j)+S(4r+2,2r+1)\end{aligned}$$

and

$$\begin{aligned}\sum_{t=0}^{\infty}(2t+j)k_{2t+j} &= \sum_{t=0}^{\infty}\sum_{\ell=0}^{2r}((4r+2)t+j+2\ell)k_{(4r+2)t+j+2\ell} \\ &= \sum_{\ell=0}^{2r}T(4r+2,j+2\ell).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{i\geq 1} id_i &= \sum_{j=1}^{2r}S(4r+2,j)+\sum_{j=1}^{2r}S(4r+2,j+2r+1)+\sum_{j=1}^{2}S(4r+2,(2r+1)j) \\ &\quad +\sum_{\ell=0}^{2r}T(4r+2,1+2\ell)+\sum_{\ell=0}^{2r}T(4r+2,2+2\ell) \\ &= \sum_{j=1}^{2r}(S(4r+2,j)+S(4r+2,j+2r+1)+T(4r+2,1+2j)+T(4r+2,2+2j)) \\ &\quad +\sum_{j=1}^{2}S(4r+2,(2r+1)j)+T(4r+2,1)+T(4r+2,2) \\ &= \sum_{j=1}^{2r+1}(S(4r+2,j)+S(4r+2,j+2r+1))+\sum_{j=0}^{2r}(T(4r+2,1+2j)+T(4r+2,2+2j)) \\ &= \sum_{j=1}^{2r+1}(S(4r+2,j)+S(4r+2,j+2r+1))+\sum_{j=1}^{2r+1}(T(4r+2,2j-1)+T(4r+2,2j)) \\ &= \sum_{j=1}^{4r+2}S(4r+2,j)+\sum_{j=1}^{4r+2}T(4r+2,j)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{4r+2} (S(4r+2,j) + T(4r+2,j)) \\
&= \sum_{j=1}^{4r+2} \sum_{t=0}^{\infty} ((4r+2)t + j)(g_{(4r+2)t+j} + k_{(4r+2)t+j}) \\
&= \sum_{t=0}^{\infty} \sum_{j=1}^{4r+2} ((4r+2)t + j) h_{(4r+2)t+j} \\
&= \sum_{i=1}^{\infty} i h_i \\
&= n.
\end{aligned}
$$

Remark 3. Setting $r = 1$ in the bijection yields the bijection given by Andrews et al. [2].

For example, consider $n = 15$ and $r = 2$. Table 1 demonstrates the correspondence. The inverse of our map is described in the following section.

$A_2(15)$	$\longrightarrow$	$B_2(15)$
(2,2,2,2,2,1,1,1,1,1)	$\mapsto$	(10,5)
(4,4,1,1,1,1,1,1,1)	$\mapsto$	(8,5,2)
(3,3,2,2,1,1,1,1,1)	$\mapsto$	(6,5,4)
(3,3,1,1,1,1,1,1,1,1,1)	$\mapsto$	(6,5,2,2)
(2,2,2,2,1,1,1,1,1,1,1)	$\mapsto$	(5,4,4,2)
(2,2,1,1,1,1,1,1,1,1,1,1,1)	$\mapsto$	(5,4,2,2,2)
(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	$\mapsto$	(5,2,2,2,2,2)

Table 1: The map $A_r(n) \to B_r(n)$ for $r = 2, n = 15$.

3 The inverse mapping

We now describe the inverse to our bijection.

Given that $(\dots, f^{d_f}, (f-1)^{d_{f-1}}, \dots, 2^{d_2}, 1^{d_1}) \in B_r(n)$. Define g_i and k_i as

$$g_{(2r+1)t+j} = 2d_{(4r+2)t+2j}, j = 1, 2, \dots 2r, t = 0, 1, 2, \dots$$

For the remaining cases, we have, for $t = 0, 1, 2, \dots$,

$$g_{(4r+2)t+(2r+1)j} = \begin{cases} d_{(4r+2)t+(2r+1)j}, & \text{if } d_{(4r+2)t+(2r+1)j} \equiv 0 \pmod 2,\ \ j = 1, 2; \\ d_{(4r+2)t+(2r+1)j} - 1, & \text{if } d_{(4r+2)t+(2r+1)j} \equiv 1 \pmod 2,\ \ j = 1, 2, \end{cases}$$

and

$$k_{2t+j} = \begin{cases} 0, & \text{if } d_{(4r+2)t+(2r+1)j} \equiv 0 \pmod{2},\ \ j = 1, 2; \\ 2r+1, & \text{if } d_{(4r+2)t+(2r+1)j} \equiv 1 \pmod{2},\ \ j = 1, 2. \end{cases}$$

Then the partition $(\ldots, 3^{g_3+k_3}, 2^{g_2+k_2}, 1^{g_1+k_1})$ is in $A_r(n)$.

For example, Table 2 demonstrates the inverse mapping when $n = 15$.

$B_2(15)$	$\longrightarrow$	$A_2(15)$
(10,5)	$\mapsto$	(2,2,2,2,2,1,1,1,1,1)
(8,5,2)	$\mapsto$	(4,4,1,1,1,1,1,1,1)
(6,5,4)	$\mapsto$	(3,3,2,2,1,1,1,1,1)
(6,5,2,2)	$\mapsto$	(3,3,1,1,1,1,1,1,1,1,1)
(5,4,4,2)	$\mapsto$	(2,2,2,2,1,1,1,1,1,1,1)
(5,4,2,2,2)	$\mapsto$	(2,2,1,1,1,1,1,1,1,1,1,1,1)
(5,2,2,2,2,2)	$\mapsto$	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)

Table 2: The inverse map $B_r(n) \to A_r(n)$ for $n = 15$, $r = 2$.

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