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Sets of Flattened Partitions with Forbidden Patterns

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Abstract

The study of pattern avoidance in permutations, and specifically in flattened partitions is an active area of current research. In this paper, we count the number of distinct flattened partitions over $[n]$ avoiding a single pattern, as well as a pair of two patterns. Several counting sequences, namely Catalan numbers, powers of two, Fibonacci numbers and Motzkin numbers arise. We also consider other combinatorial statistics, namely runs and inversions, and establish some bijections in situations where the statistics coincide.

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1 Introduction and preliminaries

Counting permutations based on avoidance of a given pattern has been studied from various perspectives in both enumerative and algebraic combinatorics [3, 5, 7, 8, 9, 11, 19]. It provides an easier way for understanding the properties of different combinatorial objects through bijective proofs.

For a fixed positive integer n, we define the set $[n] \coloneqq \{1, 2, \ldots, n\}$. A permutation σ over $[n]$ will be represented as a word $\sigma(1)\sigma(2)\cdots\sigma(n)$, where $\sigma(i)$ is the image of i under σ. We say that σ has an occurrence of a pattern τ, if there exists a subsequence in σ which is order-isomorphic to τ, else we say that σ is τ-avoiding [15]. The elements of an occurrence τ may be consecutive or non consecutive in σ. For example, $\sigma = 7345612$ contains several 231 occurrences among which include: $561, 352, 362, 452, 461$ and 463, and is 213-avoiding. A *run* in σ is a subsequence of the form $\sigma(i)\sigma(i+1)\cdots\sigma(i+p)\sigma(i+p+1)$ where $i, i+1, \ldots, i+p$ are consecutive ascents, $i-1$ (if it does exist) and $i+p+1$ are non-ascents, where $i \in [n]$ [17]. We call $\sigma(i)$ the starting point of the run. A *flattened partition* is a permutation consisting of runs arranged from left to right such that their starting points are in increasing order [17]. Notice that if σ is a flattened partition, then $\sigma(1) = 1$. For example, the permutation $\sigma = 139278456$ is a flattened partition with three runs namely $139, 278, 456$ whose starting points are $1, 2,$ and 4 respectively. Given a non-empty finite subset S of positive integers, a *set partition* P of S is a collection of disjoint non-empty subsets $B_1, B_2, \ldots, B_k$ of S (called blocks) such that $\cup_{i=1}^{k} B_i = S$ [13, 18]. We shall maintain the name and notion of "flattened partition" introduced by Callan [4]. Callan borrowed the notion "`Flatten`" from the *Mathematica* programming language, where it acts by taking lists of sets arranged in increasing order, removes their parentheses, and writes them as a single list [20]. However, different set partitions can have the same resulting flattened partition under the command "`Flatten`" from Mathematica. For example $\texttt{Flatten}(1|2|3) = 123 = \texttt{Flatten}(12|3)$. In his work, Callan [4] studied partitions of a set $[n]$, whose flattening avoids a single 3-letter pattern. Along the same direction, Mansour et al. [15] also studied avoidance of a single 3-letter pattern in set partitions of size n. For more details on flattened partitions and pattern avoidance, see [12, 14, 16]. We study the number of distinct flattened partitions avoiding a pattern τ, using fairly similar methods as those used by Mansour et al. [15], though we work on different sets of permutations.

Definition 1. Let $(i_1, i_2, i_3), (j_1, j_2, j_3) \in \mathbb{N}^3$ where $\mathbb{N}$ denotes the set of positive integers. We say that (i_1, i_2, i_3) is *lexicographically smaller* than (j_1, j_2, j_3), denoted by $(i_1, i_2, i_3) \leq_{lex} (j_1, j_2, j_3)$, if we have

(i) $i_1 < j_1$ or

(ii) $i_1 = j_1$ and $i_2 < j_2$ or

(iii) $i_1 = j_1$, $i_2 = j_2$ and $i_3 \leq j_3$.

A permutation σ over $[n]$ has an *ascent* or *descent* at position i if $\sigma(i) < \sigma(i+1)$ or $\sigma(i) > \sigma(i+1)$, respectively, for $i \in [n-1]$.

A permutation σ over $[n]$ has an *inversion* if there exists a pair $(\pi(i), \pi(j))$ for $i < j$ such that $\pi(i) > \pi(j)$.

In this paper, we count the number of *distinct* flattened partitions over $[n]$ avoiding an occurrence of a single pattern τ, as well as a pair of patterns (τ_1, τ_2).

Let $\tau \in \{123, 132, 213, 231, 312, 321\}$, $S(n;\tau)$ the set of all permutations over $[n]$ which are τ-avoiding, $\mathcal{F}(n;\tau)$ the set of τ-avoiding flattened partitions over $[n]$, and $\mathcal{F}(n;\tau_1,\tau_2)$ the set of (τ_1,τ_2)-avoiding flattened partitions over $[n]$. Let $|\mathcal{F}(n;\tau)|$ and $|\mathcal{F}(n;\tau_1,\tau_2)|$ denote the cardinalities of the sets $\mathcal{F}(n;\tau)$ and $\mathcal{F}(n;\tau_1,\tau_2)$ respectively. Let $\texttt{inv}(\pi) = |\{(i,j) : i < j, \pi(i) > \pi(j)\}|$ denote the number of inversions in a permutation π and $\texttt{runs}(\pi)$ the number of runs in π. In Table 1, we give the first few values of the numbers of 3-letter pattern avoiding flattened partitions and the On-Line Encyclopedia of Integer Sequences (OEIS) they correspond to (for detailed discussions on these sequences, see Section 2).

Pattern τ	$\lvert\mathcal{F}(n;\tau)\rvert_{n\geq 1}$	OEIS sequence
123	$1, 1, 1, 0, 0, 0, 0, \ldots.$	$\cdots$
132	$1, 1, 1, 1, 1, 1, 1, \ldots.$	A000012
213	$1, 1, 2, 4, 8, 16, 32, \ldots.$	A011782
231	$1, 1, 2, 4, 9, 21, 51, \ldots.$	A001006
312	$1, 1, 2, 4, 8, 16, 32, \ldots.$	A011782
321	$1, 1, 2, 5, 14, 42, 132, \ldots.$	A000108

Table 1: The numbers of 3-letter pattern avoiding flattened partitions

We say that a flattened partition avoids two patterns τ_1 and τ_2 if it does not contain an occurrence of either τ_1 or τ_2 or both. In Section 3, we explain the combinatorics behind the sequences in Table 2 by giving their recurrence relations and corresponding combinatorial proofs. The sequences in Table 2 were obtained by computing for the first few values of n (for detailed discussions on these sequences, see Section 3).

Pattern τ	$\lvert\mathcal{F}(n;\tau)\rvert_{n\geq 1}$	OEIS sequence
$(213,231)$, $(231,312)$	$1,1,2,3,5,8,13,\ldots$	A000045
$(132,213)$, $(132,231)$, $(132,312)$, $(132,321)$	$1,1,1,1,1,1,1,\ldots$	A000012
$(213,321)$, $(231,321)$, $(312,321)$	$1,1,2,4,8,16,32,\ldots$	A011782
$(213,312)$	$1,1,2,3,4,5,6\ldots$	A028310
$(123,132)$	$1,1,0,0,0,0,0\ldots$	$\cdots$
$(123,213)$, $(123,231)$, $(123,312)$, $(123,321)$	$1,1,1,0,0,0,0\ldots$	$\cdots$

Table 2: Summary of (τ_1,τ_2)-avoiding flattened partitions

In Section 2, we explain the combinatorics behind the sequences in Table 1 by finding recurrence relations, as well as establishing bijections between flattened partitions avoiding certain patterns and other combinatorial structures counted by the same sequences. One such bijection is given in Theorem 6, where we show that the lengths of runs of the permutations in the involved sets are preserved. Another interesting result is Theorem 4, where we show, using runs, that 213-avoiding flattened partitions are counted by powers of two. We also describe the recurrence relation for 312-avoiding flattened partitions in terms of the number of inversions preserved or created. In Section 3, we find recurrence relations for flattened partitions in the set $\mathcal{F}(n;\tau_1,\tau_2)$ and their corresponding combinatorial proofs.

2 Three letter pattern-avoiding flattened partitions

We shall consider avoidance of patterns $\tau \in \{123,132,213,231,312,321\}$. The cases of 123-avoiding and 132-avoiding flattened partitions are not very interesting: the counting sequences are $(\lvert\mathcal{F}(n;123)\rvert)_{n\geq 1} = (1,1,1,0,0,\ldots)$ and $(\lvert\mathcal{F}(n;132)\rvert)_{n\geq 1} = (1,1,1,1,1,\ldots)$ respectively.

2.1 213-avoiding

Lemma 2. *For any positive integer $n \geq 1$, a 213-avoiding flattened partition over $[n]$ has the integer n at the end of its first run.*

Proof. Let $\sigma \in \mathcal{F}(n;213)$. Suppose n is not in the first run of σ. Then n would be the last element in of any of the remaining runs of σ. For some $j > 2$, let $\sigma(j)$ be the starting point of the second run. Then we would have a 213-occurrence $\sigma(j-1)\sigma(j)n$, a contradiction. □

Proposition 3. *Let $p_1, p_2, \ldots, p_r, q_{r-1}, \ldots, q_1$ be non-empty words such that the concatenation*

$$p_1 p_2 \cdots p_r q_{r-1} q_{r-2} \cdots q_1 = 123 \cdots n. \tag{1}$$

Then $\pi = p_1q_1|p_2q_2| \cdots |p_{r-1}q_{r-1}|p_r$ is an element in $\mathcal{F}(n; 213)$ and all elements in $\mathcal{F}(n; 213)$ are of this form.

Proof. It is obvious from Equation (1) that $p_1q_1, p_2q_2, \ldots, p_{r-1}q_{r-1}, p_r$ are runs. The permutation π is flattened because the starting points of the runs from Equation (1) appear in increasing order. If we had a 213-occurrence, then there would exist integers $m < j < k$ such that $\pi(j) < \pi(m) < \pi(k)$. Then $\pi(m)$ and $\pi(j)$ can not be in the same run. Let $\pi(m) \in p_iq_i$ and $\pi(j) \in p_{i+s}q_{i+s}$. Then $\pi(m)$ belongs to q_i (because by Equation (1) if it belonged to p_i, we would have an increasing sequence). Moreover, $\pi(k)$ can belong to p_{i+t} or q_{i+t} for some $t \geq s$. Using Equation (1), we would have $\pi(k)$ appearing to the left of $\pi(m)$ in the identity permutation i.e., $\pi(k) < \pi(m)$, which is against the assumptions. Thus indeed $\pi \in \mathcal{F}(n; 213)$.

Next we prove that all $\mathcal{F}(n; 213)$ have the form $p_1q_1p_2q_2 \cdots p_{r-1}q_{r-1}p_r$ where the subwords p_i and q_i are non-empty and have consecutive elements and satisfy Equation (1). Suppose $\sigma \in \mathcal{F}(n; 213)$ and consider its first run R_1. From Lemma 2, R_1 ends with the integer n. Since σ is flattened, then it is obvious that R_1 starts with 1. If σ is the identity permutation, `id`, then $\texttt{id} = p_1$. If $\sigma \neq \texttt{id}$, let $R_1 = c_1c_2 \cdots c_l$ where $l \geq 2$ and each c_i consists of consecutive numbers which are grouped in non-empty maximal words of R_1. In particular, $c_1 = 123 \cdots k$, and $c_l = m(m+1) \cdots n$ for some $1 \leq k < m \leq n$. We note that $k+1$ is the first element of R_2 (the second run of σ). Now suppose $l \geq 3$ and let k' be the last element of c_2. Then $k'+1$ is not in R_1. Thus we have

$$\sigma = 12 \cdots k \cdots k' \cdots n|(k+1) \cdots (k'+1),$$

where $k+1 < k'$. Then $k'(k+1)(k'+1)$ would be a 213-occurrence, a contradiction. Hence $l = 2$ and $R_1 = c_1c_2$ where $c_1 = 12 \cdots k$ and $c_2 = m(m+1) \cdots n$. Let the remaining runs be denoted as $R_2R_3 \cdots R_r = \sigma'$. Then σ' is a permutation of the elements $\{k+1, \ldots, m-1\}$. We note that the permutation $\sigma' - k$ is 213-avoiding and flattened. By an inductive argument, the claim is indeed true. $\square$

Theorem 4. *For any integer $n \geq 2$, we have*

$$\sum_{\pi \in \mathcal{F}(n;213)} q^{\texttt{runs}(\pi)} = \sum_{r \geq 1} q^r \binom{n-1}{2r-2}.$$

Consequently, $|\mathcal{F}(n; 213)| = 2^{n-2}$ with $|\mathcal{F}(1; 213)| = 1$.

Proof. Let us construct a flattened partition π over $[n]$ having r runs, and with $2r-1$ subwords $p_1, \ldots, p_r, q_{r-1}, \ldots, q_1$ such that $p_1p_2 \cdots p_rq_{r-1} \cdots q_1 = 123 \cdots n$, as in Equation (1). Then π is constructed uniquely from $123 \cdots n$ by choosing $2r-2$ spaces from the $n-1$ spaces between the numbers, and then use them as demarcations between the $2r-1$ sub-words. There are $\binom{n-1}{2r-2}$ such choices. Summing over $r \geq 1$ gives the desired result. $\square$

Example 5. Let us construct a flattened partition $\pi \in \mathcal{F}(9;213)$ having 3 runs. Consider the sequence 1␣2␣3␣4␣5␣6␣7␣8␣9, which has 8 spaces. Then π is determined from this sequence by choosing 4 of them as demarcations. We may for instance choose 12␣3␣45␣67␣89. Since there are three runs, we then label the first three blocks as $p_1 = 12$, $p_2 = 3$, and $p_3 = 45$. Then the remaining blocks are $q_2 = 6$ and $q_1 = 78$. Thus we have $\pi = p_1q_1p_2q_2p_3 = 12783645 \in \mathcal{F}(9, 213)$.

Theorem 6. *For any integer $n \geq 3$,*

$$\sum_{\pi\in\mathcal{F}(n;312)} x_1^{\alpha_1(\pi)} \cdot x_2^{\alpha_2(\pi)} \cdots x_r^{\alpha_r(\pi)} = \sum_{\pi'\in\mathcal{F}(n;213)} x_1^{\alpha_1(\pi')} \cdot x_2^{\alpha_2(\pi')} \cdots x_r^{\alpha_r(\pi')},$$

where $\alpha_i(\pi)$ is the length of the i^{th} run of π. Consequently, putting $x_i = 1$, we have $|\mathcal{F}(n;213)| = |\mathcal{F}(n;312)|$.

Remark 7. Note that in each π or π', the number of factors corresponds to the number of runs r in π.

Proof. We shall define a mapping $f(\pi) = \pi' \in \mathcal{F}(n;213)$ which associates to each first run R_1 of $\pi \in \mathcal{F}(n;312)$ a corresponding first run R'_1 of π' of the same length as described below. We shall also provide an inverse g to f. If π is the identity permutation, then $f(\pi) = \pi = \pi'$, else by the same arguments as in Proposition 3, we have that R_1 consists of two non-empty sub-words: $p_1 = 123\cdots k$ and $q_1 = m(m+1)\cdots t$, with one gap between them, for some $t \leq n$ and $k < m$. Hence $m \geq k+2$. Suppose $m > k+2$, then $k+1$ would be the starting point of the second run and $k+2$ would be anywhere on the right of $k+1$ in π. Hence we would have a 312 occurrence $m(k+1)(k+2)$, a contradiction. Hence $m = k+2$. By Proposition 3, R'_1 should consist of two non-empty sub-words p'_1 and q'_1 of consecutive elements. We put $p'_1 := p_1$, and the sub-word q'_1 of π' is got by adding a term $(n-t)$ to each element of q_1 i.e., $q'_1 = q_1 + (n-t) = (m+n-t)(m+n+1-t)\cdots n$. Let σ be the resulting sub-word obtained from π after removing R_1 and then writing the remaining elements of π in standard form. Applying the mapping f on σ we obtain σ', for which adding k to each element of its elements gives $\pi' = R'_1(\sigma' + k)$ which indeed is 213-avoiding.

The inverse mapping g could be constructed recursively in an analogous manner as for f. Note that $p_1 = p'_1$ and that q_1 and q'_1 have the same length. It suffices to note that R_1 and R'_1 have the same lengths and hence the mapping f preserves the lengths and the number of the runs in π and π' and is a bijection. □

Example 8. Consider $\pi = 13|246|5 \in \mathcal{F}(6;312)$. Applying the mapping f on the first run $R_1 = 13$ gives $R'_1 = 16$ with $k = 1$. The resulting sub-word $246|5$ when standardized gives $\sigma = 124|3 \in \mathcal{F}(4;312)$. Again applying f on $124|3$ gives $\sigma' = 124|3$ and $\sigma' + 1 = 235|4$. Thus $\pi' = R'_1(\sigma' + 1) = 16|235|4$.

Proposition 9. *A $\pi \in \mathcal{F}(n;213)$ contains at least one* 312 *occurrence if and only if $f(\pi) \in \mathcal{F}(n;312)$ contains at least one* 213 *occurrence.*

Proof. Let $\mathcal{F}(n; 213, 312)$ be the set consisting of all σ which do not contain any 312 and 213 occurrences. Then under the mapping f, we have that $f(\sigma) = \sigma$. From Theorem 6, the sets $\mathcal{F}(n; 213)$ and $\mathcal{F}(n; 312)$ have the same sizes. Thus the sizes of the sets $A = \mathcal{F}(n; 213) \setminus \mathcal{F}(n; 213, 312)$ and $B = \mathcal{F}(n; 312) \setminus \mathcal{F}(n; 213, 312)$ are also the same. This proves the claim. □

2.2 312-avoiding

Proposition 10. *A 312-avoiding flattened partition starts with either 12 or 13.*

Proof. Let π be a 312-avoiding flattened partition. Suppose that π starts with $1i$ where $i \geq 4$. Then 2 is the starting point of the second run. The integer 3 appears on the right of 2. Hence π would contain a 312 occurrence $i23$. Hence $i \leq 3$. □

Proposition 11. *Interchanging the 2 and 3 in a 312-avoiding flattened partition preserves the avoidance property.*

Corollary 12. *For any integer $n \geq 3$, the number of 312-avoiding flattened partitions over $[n]$ starting with 12 is equal to the number of 312-avoiding flattened partitions over $[n]$ starting with 13.*

This is because in each 312-avoiding flattened partitions over $[n]$ starting with 12, interchanging 2 and 3 gives a 312-avoiding flattened partitions over $[n]$ starting with 13, and vice versa.

Theorem 13. *For all $n \geq 1$,*

$$\sum_{\pi \in \mathcal{F}(n;312)} q^{\mathtt{inv}(\pi)} = (1+q)^{n-2}.$$

Proof. It suffices to prove that

$$\sum_{\pi \in \mathcal{F}(n;312)} q^{\mathtt{inv}(\pi)} = \sum_{\pi \in \mathcal{F}(n-1;312)} q^{\mathtt{inv}(\pi)}(1+q).$$

For $n = 1, 2$, the identity is the only 312- avoiding flattened partition. From Proposition 10 and Corollary 12, there are only two classes of 312-avoiding flattened partitions: one class starting with 12 and another one starting with 13 and their sizes are equal. To create the first class, we consider a 312-avoiding flattened partition π of length $n-1$ and insert the integer 1 at the beginning of π. We then increase by 1 the remaining terms to get a 312-avoiding flattened partition σ of length n. Let $P_n(q) = \sum_{\pi \in \mathcal{F}(n;312)} q^{\mathtt{inv}(\pi)}$. Then the first class contributes $1 \cdot P_{n-1}(q)$ inversions. To create the second class, we interchange the integers 2 and 3 of the first class. Hence the second class contributes $q \cdot P_n(q)$ inversions. Summing the inversions proves the theorem. □

2.3 321-avoiding

Theorem 14. *For any integer $n \geq 1$, we have $|\mathcal{F}(n;321)| = C_{n-1}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number with $C_0 = 1$.*

Proof. As shown by Knuth in [10], we have that $|\mathcal{S}(n-1;321)| = C_{n-1}$. Thus the proof of Theorem 14 follows from Lemma 15.

Lemma 15. *For $n \geq 1$, there exists a bijection $h : \mathcal{F}(n;321) \to \mathcal{S}(n-1;321)$ defined by removing integer 1 from $\pi \in \mathcal{F}(n;321)$ and reducing the remaining π elements by 1.*

It is easy to see how to construct the inverse mapping $h' : \mathcal{S}(n-1;321) \to \mathcal{F}(n;321)$, and that both h and h' preserve 321 avoidance. One observation worth noting is that $h'(\sigma)$ is a flattened partition even if $\sigma \in \mathcal{S}(n-1;321)$ is not. The only obstruction to this would be if the first entries of the runs (except the first run) of σ are not in increasing order. In this case, it would imply that there exists integers b and a both starting points of such runs such that $b > a$, although a occurs later. Then there would exist an integer $c > b > a$ in the first run such that cba is a 321 occurrence. □

2.4 231-avoiding

Lemma 16. *Let n be a positive integer and π be a 231-avoiding flattened partition of length n. There exists an integer $2 \leq k \leq n$ such that:*

- *(i)* $\pi(i) < k$ *if* $i < k$,
- *(ii)* $\pi(k) = n$,
- *(iii)* $\pi(i) \geq k$ *if* $i > k$.

Proof. Let $k = \pi(n)^{-1}$ and $\pi = 1\pi(2)\cdots\pi(k-1)n\pi(k+1)\cdots\pi(n)$. Let $m = \max\{\pi(j) : 1 \leq j \leq k-1\}$. Necessarily, $\pi(l) > m$ for all $k+1 \leq l \leq n$ since else there would be a 231 occurrence mna where $a = \pi(l) < m$. Hence $k = m+1$. □

Theorem 17. *For any positive integer $n \geq 3$, we have*

$$|\mathcal{F}(n;231)| = |\mathcal{F}(n-1;231)| + \sum_{k=2}^{n-1} |\mathcal{F}(k-1;231)||\mathcal{F}(n-k;231)| \tag{2}$$

with initial values $|\mathcal{F}(1;231)| = 1$, $|\mathcal{F}(2;231)| = 1$.

Thus $|\mathcal{F}(n;231)| = M_{n-1}$, where M_n is the n^{th} Motzkin number (A001006) with $M_0 = 1$ as given by Aigner [1].

Proof. We consider two cases depending on k: one case when $k < n$ and another case when $k = n$. In the latter case, inserting n at the end of each $\pi' \in \mathcal{F}(n-1; 231)$ gives $\pi \in \mathcal{F}(n; 231)$. Hence we have $|\mathcal{F}(n-1; 231)|$ unique flattened partitions having n at the end of each π.

In the case $k < n$, there are two subsequences on the left and right of n for each $\pi \in \mathcal{F}(n; 231)$ i.e., $1\pi(2)\cdots\pi(k-1)$ and $\pi(k+1)\cdots\pi(n)$ of lengths $k-1$ and $n-k$ respectively. Let $\pi_1 = 1\pi(2)\cdots\pi(k-1)$ and $\pi_2 = \pi(k+1)\cdots\pi(n)$. We note that subtracting integer $k-1$ from each element of π_2 gives $\pi \in \mathcal{F}(n-k; 231)$. By Lemma 16, we have that each $\pi_1 \in \mathcal{F}(k-1; 231)$. Multiplying and summing over k indeed gives $\sum_{k=2}^{n-1} |\mathcal{F}(k-1; 231)||\mathcal{F}(n-k; 231)|$.

Summing the two cases together proves the claim. $\square$

Alternatively, we also give a bijection between 231-avoiding flattened partitions and the well known Motzkin paths which are also counted by Motzkin numbers. First, we introduce the so called *Motzkin permutations* because they are in bijection with Motzkin paths. This was proved by Mansour et al. [6]. A permutation σ is said to be Motzkin if it avoids pattern 132 and there are no integers $i < j$ for which $\pi(i) < \pi(j) < \pi(j+1)$. The latter condition corresponds to avoidance of a kind of generalized patterns introduced by Babson and Steingrímsson [2]. There is a bijection $\alpha : \mathcal{M}_{n-1} \to \mathcal{F}(n; 231)$ between the set of Motzkin permutations over $[n-1]$ and 231-avoiding flattened partitions over $[n]$ defined by the following: For each $\sigma \in \mathcal{M}_{n-1}$, increase by 1 all elements of σ and reverse their order to obtain π'. Then insert integer 1 at the beginning of π' to obtain $\alpha(\sigma) = \pi \in \mathcal{F}(n; 231)$. We remark that avoidance of pattern 132 in $\sigma \in \mathcal{M}_{n-1}$ corresponds to avoidance of pattern 231 in π. On the other hand, avoidance of the generalized pattern corresponds to $\pi \in \mathcal{F}(n)$.

3 Avoidance of pairs of three letter patterns in flattened partitions

We shall consider avoidance of a pair of patterns (τ_1, τ_2) of length three. The cases of $(123, 132)$-avoiding and $((123, 213)$-, $(123, 231)$-, $(123, 312)$-, $(123, 321))$-avoiding flattened partitions are not very interesting: the counting sequences for $n \geq 1$ are $(1, 1, 0, 0, 0, \ldots)$ and $(1, 1, 1, 0, 0, \ldots)$ for the latter cases respectively. Similarly, the pairs $((132, 213)$, $(132, 231)$, $(132, 312)$, $(132, 321))$ all have a trivial counting sequence $(1, 1, 1, 1, 1, \ldots)_{n\geq 1}$. In the subsections that follow, we consider avoidance of the remaining pairs of patterns.

3.1 $(213, 231)$-avoiding

Here, we consider the problem of avoiding both 213 and 231 patterns.

Proposition 18. *Let $n \geq 1$ be a positive integer. If π is a $(213, 231)$-avoiding flattened partition, then there exists an integer $2 \leq k \leq n$ such that*

(i) $\pi(i) = i$ *for* $1 \leq i \leq k-1$,

(ii) $\pi(k) = n$,

(iii) $\pi(i) \geq k$ *for* $k+1 \leq i \leq n$.

Proof. Let $\pi = 1\pi(2)\cdots\pi(k-1)n\pi(k+1)\cdots\pi(n), x = \max\{\pi(j) : 1 \leq j \leq k-1\}$. By Proposition 2, the integer n must be at the end of the first run. By Theorem 16, the integers $1, 2, \ldots, k-1$ are also elements of the first run. Hence conditions (i) and (ii) are satisfied. Necessarily, $\pi(y) > x$ for all $k+1 \leq y \leq n$, $\pi(y) \geq x$ since else there would be a 231 occurrence xny . Hence $k = x+1$. ☐

Lemma 19. *For any integer* $n \geq 1$, *all elements of* $\mathcal{F}(n; 213, 231)$ *start with either* 12 *or* $1n$.

The proof is similar to that of Proposition 10, just that in this case, we suppose that $\pi \in \mathcal{F}(n; 213, 231)$ starts with $1i$ where $3 \leq i \leq n-1$ and then prove by contradiction that this is not possible.

Proposition 20. *For any integer* $n \geq 1$, *we have* $|\mathcal{F}(n; 213, 231)| = F_n$ *where* F_n *is the Fibonacci number with initial conditions* $F_1 = F_2 = 1$.

Proof. We prove this claim by induction. For $n = 1, 2$, the identity is the only $(213, 231)$-avoiding flattened partition.

For $n \geq 3$, from Lemma 19, there are two classes of $(213, 231)$-avoiding flattened partition: one class starting with 12 and another class starting with $1n$. To create the first class, for each $\pi' \in \mathcal{F}(n-1; 213, 231)$, inserting 1 at the beginning of π' and then increasing by 1 the remaining terms gives $|\mathcal{F}(n-1; 213, 231)|$ unique flattened partitions. To create the second class, for each $\pi' \in \mathcal{F}(n-2; 213, 231)$, inserting the subsequence $1n$ at the beginning of π', and then increasing the remaining elements by 1 gives $|\mathcal{F}(n-2; 213, 231)|$ unique flattened partitions. It is clear that removing 1 or the subsequence $1n$ from each $\pi \in \mathcal{F}(n; 213, 231)$ that starts with 12 or $1n$ respectively gives the elements in the sets $\mathcal{F}(n-1; 213, 231)$ or $\mathcal{F}(n-2; 213, 231)$. Thus inductively,

$$|\mathcal{F}(n; 213, 231)| = |\mathcal{F}(n-1; 213, 231)| + |\mathcal{F}(n-2; 213, 231)| = F_{n-1} + F_{n-2} = F_n.$$

☐

Example 21. Let us construct flattened partitions $\pi \in \mathcal{F}(6; 213, 231)$. Inserting 1 at the beginning of each elements in $\mathcal{F}(5; 213, 231) = \{15234, 15243, 12345, 12354, 12534\}$, and then increasing the remaining elements by 1 gives π in the class starting with 12. On the other hand, inserting the subsequence 16 at the beginning of each element in the set $\mathcal{F}(4; 213, 231) = \{1234, 1243, 1423\}$, and then increasing the remaining elements by 1 gives π in the second class.

Let us denote by $F(n, k)$ the number of $(213, 231)$-avoiding flattened partitions with r runs, in which the first run R_1 has length k.

Proposition 22. *For all integers k, n such that $1 \leq k < n$, we have $F(n,k) = F_{n-k}$, where F_{n-k} is the $(n-k)^{th}$ Fibonacci number.*

Proof. By Proposition 18, R_1 is unique since given its length k, then $R_1 = 12\cdots(k-2)(k-1)n$. The remaining runs denoted as $R_2R_3\cdots R_r$ thus have length $n-k$. Let $R_2R_3\cdots R_r = \sigma'$. We note that removing integer $k-1$ from each element of σ' gives $\pi \in \mathcal{F}(n-k; 213, 231)$ and vice versa. Thus,

$$F(n,k) = |\mathcal{F}(n-k; 213, 231)| = F_{n-k}$$

□

Proposition 23. *The ordinary generating function $F(u,x)$ for the number of $(213, 231)$-avoiding flattened partitions is given by*

$$F(u,x) = \frac{1-u-u^2+u^3x^2}{(1-ux)(1-u-u^2)}.$$

Proof. Letting

$$F(u,x) = \sum_{n\geq 0}\sum_{k\geq 0} F(n,k)x^k u^n$$

and using Proposition 22 gives the required result. □

3.2 $(312, 231)$-avoiding

Proposition 24. *For any integer $n \geq 1$, we have $|\mathcal{F}(n; 312, 231)| = F_n$ where F_n is the Fibonacci number with initial conditions $F_1 = F_2 = 1$.*

Proof. We prove the claim by induction. For $n = 1, 2$, the identity permutation is the only $(312, 231)$-avoiding flattened partition.

For $n \geq 3$, from Proposition 10, there are two classes of $(312, 231)$-avoiding flattened partition: one class starting with 12 and another class starting with 13. To create the first class, for each $\pi' \in \mathcal{F}(n-1, 312, 231)$, inserting 1 at the beginning of π' and then increasing by 1 the remaining terms gives $|\mathcal{F}(n-1, 312, 231)|$ unique flattened partitions. To create the second class, for each $\pi' \in \mathcal{F}(n-2, 312, 231)$, inserting the subsequence 13 at the beginning of π', and then increasing the first element of π' by 1, and the remaining elements by 2 gives $|\mathcal{F}(n-2, 312, 231)|$ unique flattened partitions. It is clear that removing 1 or the subsequence 13 from each $\pi \in \mathcal{F}(n, 312, 231)$ that starts with 12 or 13 respectively gives elements in the sets $\mathcal{F}(n-1, 312, 231)$ or $\mathcal{F}(n-2, 312, 231)$. Thus inductively,

$$|\mathcal{F}(n, 312, 231)| = |\mathcal{F}(n-1, 312, 231)| + |\mathcal{F}(n-2, 312, 231)| = F_{n-1} + F_{n-2} = F_n.$$

□

Example 25. Let us construct flattened partitions $\pi \in \mathcal{F}(6, 312, 231)$. Inserting 1 at the beginning of each elements in $\mathcal{F}(5, 312, 231) = \{13245, 13254, 12345, 12354, 12435\}$, and then increasing the remaining elements by 1 gives π in the class starting with 12. On the other hand, inserting the subsequence 13 at the beginning of each element in the set $\mathcal{F}(4, 312, 231) = \{1234, 1243, 1324\}$, and then increasing the first element in this set by 1 and the remaining elements by 2 gives π in the second class.

3.3 $(213, 312)$-avoiding

From Proposition 10, we have already seen that 312-avoiding flattened partitions either start with 12 or 13. Hence $(213, 312)$-avoiding flattened partitions also have the same classes. However, there is only one flattened partition in this class that starts with 13.

Lemma 26. *For $n \geq 3$, the only $\pi \in \mathcal{F}(n; 213, 312)$ that starts with 13 has 2 as a singleton second run.*

Proof. Since π starts with 13, then 2 is the starting point of the second run and $3 \in R_1$. Suppose π has at least two runs and that the second run is not singleton. Then there would exist an integer $c > 3 > 2$ to the right of 2 such that we have a 213 occurrence $32c$. Since the position of $1, 2$ and 3 are are known, then the remaining $n - 3$ elements can be arranged in R_1 as an increasing sequence in R_1 after 3 and there is only one way this can be done. $\square$

Proposition 27. *For any integer $n \geq 3$, the number of $(213, 312)$-avoiding flattened partitions satisfies the recurrence relation $|\mathcal{F}(n; 213, 312)| = |\mathcal{F}(n - 1; 213, 312)| + 1$ with initial condition $|\mathcal{F}(2; 213, 312)| = 1$.*

Proof. From Lemma 26, there is exactly one $\pi' \in \mathcal{F}(n - 1; 213, 312)$ that starts with 13. Inserting n at the end of the first run of π' gives 1 unique flattened partition $\pi \in \mathcal{F}(n; 213, 312)$ that starts with 13. By Proposition 10, the second class of $(213, 312)$-avoiding flattened partitions starts with 12. To create this class, we insert 1 at the beginning of each $\pi' \in \mathcal{F}(n - 1; 213, 312)$, and then increase the remaining elements of π' by 1. This gives $|\mathcal{F}(n - 1; 213, 312)|$ unique flattened partitions. It is clear that removing 1 or n from each $\pi \in \mathcal{F}(n; 213, 312)$ that starts with 12 or 13 respectively gives the elements in the set $\mathcal{F}(n - 1; 213, 312)$ or the only element in $\mathcal{F}(n - 1; 213, 312)$ that starts with 13. $\square$

3.4 $(213, 321)$-avoiding

Proposition 28. *For any positive integer $n \geq 2$,*

$$|\mathcal{F}(n; 213, 321)| = |\mathcal{F}(n - 1; 213, 321)| + n - 2 \tag{3}$$

with initial conditions $|\mathcal{F}(1; 213, 321)| = 1$. Consequently,

$$|\mathcal{F}(n; 213, 321)| = \binom{n-1}{2} + 1. \tag{4}$$

Proof. For each $\sigma \in \mathcal{F}(n-1; 213, 321)$ and using Lemma 2, inserting n at the end of the first run preserves the number of runs and gives $\pi \in \mathcal{F}(n; 213, 321)$ with the subsequence $(n-1)n$ at the end of the first run. This gives $|\mathcal{F}(n-1; 213, 321)|$ unique flattened partitions. For the identity flattened partition $id \in \mathcal{F}(n-1; 213, 321)$, there are $(n-2)$ more choices of inserting n into positions $n-1, n-2, \ldots, 2$ respectively in id to create an element $\pi \in \mathcal{F}(n; 213, 321)$. If $\sigma \in \mathcal{F}(n-1; 213, 321)$ is not the identity and we suppose that n appears in the first run before $n-1$, then there exists an integer $a < n-1$, a starting point of the second run such that *n(n-1)a* is a 321 occurrence. Thus such cases can not exist. Summing up gives Equation (3), and solving this easy recursion gives Equation (4). ☐

3.5 $(231, 321)$-avoiding

Proposition 29. *For any* $n \geq 2$, $|\mathcal{F}(n; 231, 321)| = 2^{n-2}$.

Proof. For $n \geq 3$, there are two classes of $(231, 321)$-avoiding flattened partitions: one class that starts with 12 and another class that starts with $1i$ for $3 \leq i \leq n$. Necessarily, in the latter class, 2 is the starting point of the second run, and the first run has exactly two elements. Otherwise there exists integers i, j in the first run, for $1 < i < j$, for which we would have a 231-occurrence $ij2$.

To create the first class, we insert 1 at the beginning of each $\pi' \in \mathcal{F}(n-1; 231, 321)$ and then increase the remaining elements of π' by 1. This gives $|\mathcal{F}(n-1; 231, 321)|$ unique flattened partitions. To create the second class, we increase all elements in $\pi' \in \mathcal{F}(n-1; 231, 321)$ that start with $1i$ except the first element by 1 and then insert 2 in the third position. This gives $|\mathcal{F}(n-1; 231, 321)|$ unique flattened partitions. It is clear how to invert these constructions. Thus we have

$$|\mathcal{F}(n; 231, 321)| = 2|\mathcal{F}(n-1; 231, 321)|,$$

with initial condition $|\mathcal{F}(2; 231, 321)| = 1$. Solving this recursion proves the claim. ☐

Remark 30. All $(312, 321)$-avoiding flattened partitions have similar properties and structure as 312-avoiding flattened partitions studied in Subsection 2.2.

Remark 31. There are many subsequent follow-up questions one can ask about flattened partitions avoiding some patterns, but these will be addressed separately.

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