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GCD of Sums of k Consecutive Fibonacci, Lucas, and Generalized Fibonacci Numbers

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Abstract

We explore the sums of k consecutive terms in the generalized Fibonacci sequence $(G_n)_{n\geq 0}$ given by the recurrence $G_n = G_{n-1} + G_{n-2}$ for all $n \geq 2$ with integral initial conditions G_0 and G_1. In particular, we give precise values for the greatest common divisor (GCD) of all sums of k consecutive terms of $(G_n)_{n\geq 0}$. When $G_0 = 0$ and $G_1 = 1$, we yield the GCD of all sums of k consecutive Fibonacci numbers, and when $G_0 = 2$ and $G_1 = 1$, we yield the GCD of all sums of k consecutive Lucas numbers. Denoting the GCD of all sums of k consecutive generalized Fibonacci numbers by the symbol $\mathcal{G}_{G_0,G_1}(k)$, we give two tantalizing characterizations for these values, one involving a simple formula in k and another involving generalized Pisano periods:

$$\mathcal{G}_{G_0,G_1}(k) = \gcd(G_{k+1} - G_1,\, G_{k+2} - G_2) \text{ and}$$
$$\mathcal{G}_{G_0,G_1}(k) = \operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } k\},$$

where $\pi_{G_0,G_1}(m)$ denotes the generalized Pisano period of the generalized Fibonacci sequence modulo m. The fact that these vastly different-looking formulas coincide leads to some surprising and delightful new understandings of the Fibonacci and Lucas numbers.

1 Introduction

In the inaugural issue of the *Fibonacci Quarterly* in 1963, I. D. Ruggles proposed the following problem in the Elementary Problems section: "Show that the sum of twenty consecutive Fibonacci numbers is divisible by the 10^{th} Fibonacci number $F_{10} = 55$." [13] Since the Ruggles problem, there have been numerous papers studying sums of consecutive Fibonacci numbers or Lucas numbers [9, 19, 20, 4, 3, 14]. However, with all this work on consecutive sums of Fibonacci and Lucas numbers, one related topic seems to be missing from the literature, namely that of the greatest common divisor (GCD) of sums of Fibonacci and Lucas numbers. That being said, the On-Line Encyclopedia of Integer Sequences (OEIS) does have two entries, A210209 and A229339, which give the GCDs of the sums of k consecutive Fibonacci (respectively, Lucas) numbers [15]. But in those entries, no references are given to any existing papers in the literature providing rigorous proofs that confirm these sequences. More precisely, two references are given in the entry A210209 but they appear to have little connection to the actual sequence, and the entry A229339 contains no references at all. Our paper serves to fill this deficiency in the literature.

Motivated by the Ruggles problem, we observed the surprising fact that not only is the sum of any twenty consecutive Fibonacci numbers divisible by F_{10}, but also that F_{10} is the greatest of all the divisors of these sums. This became a main motivation for us to explore sums of any finite length of consecutive Fibonacci numbers, then for Lucas numbers, and then eventually for all possible generalized Gibonacci sequences. Appearing in the literature as early as 1901 by Tagiuri [18], the generalized Fibonacci numbers (or so-called *Gibonacci numbers*[1]) are defined by the recurrence

$$G_i = G_{i-1} + G_{i-2} \text{ for all } i \geq 2$$

with initial conditions $G_0, G_1 \in \mathbb{Z}$. We examine the GCD of the sums of k consecutive Gibonacci numbers, and consequently k consecutive Fibonacci and Lucas numbers. More precisely, given $k \in \mathbb{N}$ we explore the GCD of an infinite number of finite sums

$$\sum_{i=1}^{k} G_i\,,\ \ \sum_{i=2}^{k+1} G_i\,,\ \ \sum_{i=3}^{k+2} G_i\,,\ \ \ldots$$

That is, we compute the GCD of the terms in the sequence $\left(\sum_{i=0}^{k-1} G_{n+i}\right)_{n\geq 1}$. By a slight abuse of notation, we write this value as $\gcd\left\{\left(\sum_{i=0}^{k-1} G_{n+i}\right)_{n\geq 1}\right\}$.

[1] Thomas Koshy attributes Art Benjamin and Jennifer Quinn for coining this term "Gibonacci" in their 2003 book *Proofs that Really Count: The Art of Combinatorial Proof* [1].

Convention 1. For brevity, we use the symbols $\mathcal{F}(k)$, $\mathcal{L}(k)$, and $\mathcal{G}_{G_0,G_1}(k)$, respectively, to denote the three values

$$\gcd\left\{\left(\sum_{i=0}^{k-1} F_{n+i}\right)_{n\geq 1}\right\},\ \gcd\left\{\left(\sum_{i=0}^{k-1} L_{n+i}\right)_{n\geq 1}\right\},\ \text{and}\ \gcd\left\{\left(\sum_{i=0}^{k-1} G_{n+i}\right)_{n\geq 1}\right\}.$$

For reasons to be explained in Theorem 19 and Convention 20, it suffices to only consider Gibonacci sequences with relatively prime initial conditions G_0 and G_1.

Remark 2. Observe that when $G_0 = 0$ and $G_1 = 1$ we have $\mathcal{G}_{G_0,G_1}(k) = \mathcal{F}(k)$, and when $G_0 = 2$ and $G_1 = 1$ we have $\mathcal{G}_{G_0,G_1}(k) = \mathcal{L}(k)$. Hence in the symbols $\mathcal{F}(k)$ and $\mathcal{L}(k)$, we suppress writing the initial values since those are well known in the Fibonacci and Lucas setting.

To compute $\mathcal{G}_{G_0,G_1}(k)$, we establish two very different yet equivalent characterizations for this value. One is a simple formula in k, namely $\mathcal{G}_{G_0,G_1}(k) = \gcd(G_{k+1} - G_1, G_{k+2} - G_2)$. Another is a formula utilizing the generalized Pisano period $\pi_{G_0,G_1}(m)$ of the Gibonacci sequence modulo m, namely $\mathcal{G}_{G_0,G_1}(k) = \operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } k\}$. We summarize our main results in Table 1.

k	$\mathcal{F}(k)$	$\mathcal{L}(k)$	$\mathcal{G}_{G_0,G_1}(k)$	Proof in this paper
$k \equiv 0, 4, 8 \pmod{12}$	$F_{k/2}$	$5F_{k/2}$	$F_{k/2}$[a] or $5F_{k/2}$[b]	Theorem 32
$k \equiv 2, 6, 10 \pmod{12}$	$L_{k/2}$	$L_{k/2}$	$L_{k/2}$	Theorem 33
$k \equiv 3, 9 \pmod{12}$	2	2	2[c]	Theorem 38
$k \equiv 1, 5, 7, 11 \pmod{12}$	1	1	1[c]	Theorem 39

Table 1: Summary of our main results

[a] This value holds if and only if $\gcd(G_0 + G_2, G_1 + G_3) = 1$.
[b] This value holds if and only if $\gcd(G_0 + G_2, G_1 + G_3) \neq 1$.
[c] These values hold if $G_1^2 - G_0G_1 - G_0^2 = \pm 1$. The case when $G_1^2 - G_0G_1 - G_0^2 \neq \pm 1$ is addressed in Section 6

The paper is broken down as follows. In Section 2, we give a brief overview of necessary definitions and identities; in particular, we prove a few known results whose proofs seem to be missing in the literature. In Section 3, we provide proofs of our two characterizations for $\mathcal{G}_{G_0,G_1}(k)$. In Sections 4 and 5, we prove our main results for the values $\mathcal{G}_{G_0,G_1}(k)$ when k is even and odd, respectively. In Section 6, we explore three tantalizing applications of our $\mathcal{G}_{G_0,G_1}(k)$ characterizations. Finally, in Section 7, we provide five open questions motivated by results in this paper.

2 Definitions and preliminary identities

Many results in this section are well known, and we provide references to where a proof of each result can be found. Some other lesser "well-known" results have no proofs in the literature as far as we have exhaustively searched, and for those results we do provide our own proofs. We use the convention of denoting these well-known results as propositions.

Definition 3. The *generalized Fibonacci sequence* $(G_n)_{n\geq 0}$ is defined by the recurrence relation

$$G_n = G_{n-1} + G_{n-2}$$

for all $n \geq 2$ and with arbitrary initial conditions $G_0, G_1 \in \mathbb{Z}$. The *Fibonacci sequence* $(F_n)_{n\geq 0}$ is recovered when $G_0 = 0$ and $G_1 = 1$, and the *Lucas sequence* $(L_n)_{n\geq 0}$ is recovered when $G_0 = 2$ and $G_1 = 1$. For brevity, we use the term *Gibonacci sequence* to refer to any generalized Fibonacci sequence.

The following closed form expression for the Fibonacci sequence in Proposition 4 was derived and first published by Jacques Binet in 1843, but it was known at least a century earlier by Abraham de Moivre in 1718. We include this proposition and the related Proposition 5 that follows it because we use them to prove Identities (6), (7), and (8) in Lemma 9. In these two propositions, we set $\alpha := \frac{1+\sqrt{5}}{2}$ and $\beta := \frac{1-\sqrt{5}}{2}$.

Proposition 4. *For $n \in \mathbb{Z}$, the Fibonacci number F_n has the closed form*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Proposition 5. *For $n \in \mathbb{Z}$, the Lucas number L_n has the closed form*

$$L_n = \alpha^n + \beta^n.$$

Proposition 6. *The following five identities hold:*

$$L_n = F_{n+1} + F_{n-1} \qquad \text{for all } n \in \mathbb{Z}; \tag{1}$$

$$F_{2n} = F_n L_n \qquad \text{for all } n \in \mathbb{Z}; \tag{2}$$

$$G_{m+n} = F_{m-1}G_n + F_m G_{n+1} \qquad \text{for all } m, n \geq 1; \tag{3}$$

$$G_i = G_0 F_{i-1} + G_1 F_i \qquad \text{for all } i \geq 1; \tag{4}$$

$$\sum_{i=1}^{n} G_i = G_{n+2} - G_2 \qquad \text{for all } n \geq 1. \tag{5}$$

Proof. Identities (1), (2), (3), and (5), respectively, are proven by Vajda in his Identities (6), (13), (8), and (33), respectively [21, pp. 24,25,38]. Identity (4) follows from Identity (3) if we set $m := i$ and $n := 0$. □

Proposition 7. *(Generalized Cassini's Identity) For all $n \geq 0$, the following equality holds: $G_{n+1}G_{n-1} - G_n^2 = (-1)^n \cdot D_{G_0,G_1}$, where $D_{G_0,G_1} = G_1^2 - G_0G_1 - G_0^2$.*

Proof. A generalization of this well-known identity is stated in Rabinowitz [12, Theorem 8]. □

Though many encyclopedic resources such as Vajda [21] and Koshy [11] give nice closed forms for $F_{j-1} + F_{j+1}$ and $L_{j-1} + L_{j+1}$, the literature surprisingly lacks a closed form for $G_{j-1} + G_{j+1}$. We fill this gap in the literature with Lemma 8 below, and this lemma along with the three identities in Lemma 9 helps us prove the four Gibonacci propositions to follow in Subsection 2.1.

Lemma 8. *For all $j \geq 1$, the following identity holds:*

$$G_{j-1} + G_{j+1} = G_0L_{j-1} + G_1L_j.$$

Proof. Let $j \geq 1$ be given. Observe the sequence of equalities

$$\begin{aligned} G_{j-1} + G_{j+1} &= (G_0F_{j-2} + G_1F_{j-1}) + (G_0F_j + G_1F_{j+1}) && \text{(by Proposition 6, Identity (4))} \\ &= G_0(F_{j-2} + F_j) + G_1(F_{j-1} + F_{j+1}) \\ &= G_0L_{j-1} + G_1L_j. && \text{(by Proposition 6, Identity (1))} \end{aligned}$$

Hence the identity holds for all $j \geq 1$. □

To prove the propositions in Subsection 2.1, we also utilize three identities given in Lemma 9. Identities (6) and (7) of this lemma can be found in Koshy [11, Identities 70 and 71, p. 90], but he provides no proofs. It turns out that these two identities were originally stated in 1971 (though unfortunately again without proofs) in Dudley and Tucker [6]. The related Identity (8), in the form we provide and utilize in Subsection 2.1, does not appear to be in the literature. Due to the lack of proofs for any of these identities in the literature, for completion we prove these three identities in Lemma 9 by proving a single identity in which these three identities hold as a consequence (see Remark 10).

Lemma 9. *For all $j \geq 0$, the following three identities hold:*

$$F_{4j+1} - 1 = F_{2j}L_{2j+1} \tag{6}$$

$$F_{4j+3} - 1 = F_{2j+2}L_{2j+1} \tag{7}$$

$$F_{4j+4} - 1 = F_{2j+3}L_{2j+1}. \tag{8}$$

Proof. Utilizing the closed forms for F_n and L_n in Propositions 4 and 5, for $r, j \in \mathbb{Z}$ we have the sequence of equalities

$$\begin{aligned}
F_{2j+r}L_{2j+1} &= \frac{\alpha^{2j+r}-\beta^{2j+r}}{\alpha-\beta}\cdot\left(\alpha^{2j+1}+\beta^{2j+1}\right) \\
&= \frac{\alpha^{4j+r+1}-\beta^{4j+r+1}+\alpha^{2j+r}\beta^{2j+1}-\alpha^{2j+1}\beta^{2j+r}}{\alpha-\beta} \\
&= \frac{\alpha^{4j+r+1}-\beta^{4j+r+1}}{\alpha-\beta}+\frac{(\alpha\beta)^{2j+1}\left(\alpha^{r-1}-\beta^{r-1}\right)}{\alpha-\beta} \\
&= F_{4j+r+1}+(-1)^{2j+1}\cdot\frac{\alpha^{r-1}-\beta^{r-1}}{\alpha-\beta} \qquad \text{(since } \alpha\beta=-1\text{)} \\
&= F_{4j+r+1}-F_{r-1}.
\end{aligned}$$

If we set $r := 0$, then we have $F_{2j+0}L_{2j+1} = F_{4j+0+1} - F_{0-1}$ so Identity (6) holds since $F_{-1} = 1$. And if we set $r := 2$, then we have $F_{2j+2}L_{2j+1} = F_{4j+2+1} - F_{2-1}$ so Identity (7) holds since $F_1 = 1$. Lastly if we set $r := 3$, then we have $F_{2j+3}L_{2j+1} = F_{4j+3+1} - F_{3-1}$ so Identity (8) holds since $F_2 = 1$. □

Remark 10. In proving Lemma 9, we actually proved the much stronger result that an infinite family of identities of the following form holds:

$$F_{4j+r+1} - F_{r-1} = F_{2j+r}L_{2j+1},$$

for all $r, j \in \mathbb{Z}$. This follows since the closed formulas for F_n and L_n, given in Propositions 4 and 5, work for all integer values of n.

2.1 Four Gibonacci propositions

The following four Gibonacci identities (along with our characterizations for the values $\mathcal{G}_{G_0,G_1}(k)$ given in Subsections 3.1 and 3.2 to follow) are used in the proofs of our main results in Sections 4 and 5:

$$\begin{aligned}
G_{4j+1} - G_1 &= F_{2j}(G_{2j} + G_{2j+2}) && \text{(Proposition 11)} \\
G_{4j+2} - G_2 &= F_{2j}(G_{2j+1} + G_{2j+3}) && \text{(Proposition 12)} \\
G_{4j+3} - G_1 &= L_{2j+1}G_{2j+2} && \text{(Proposition 13)} \\
G_{4j+4} - G_2 &= L_{2j+1}G_{2j+3} && \text{(Proposition 14).}
\end{aligned}$$

These identities are stated in Koshy but without proof [11, p. 214], so for completeness we provide proofs for each proposition.

Proposition 11. *For all $j \geq 0$, the following identity holds:*

$$G_{4j+1} - G_1 = F_{2j}(G_{2j} + G_{2j+2}).$$

Proof. Let $j \geq 0$ be given. Observe the sequence of equalities

$$\begin{aligned}
F_{2j}(G_{2j}+G_{2j+2}) &= F_{2j}(G_0L_{2j}+G_1L_{2j+1}) && \text{(by Lemma 8)}\\
&= G_0\cdot F_{2j}L_{2j}+G_1\cdot F_{2j}L_{2j+1}\\
&= G_0F_{4j}+G_1\cdot F_{2j}L_{2j+1} && \text{(by Proposition 6, Identity (2))}\\
&= G_0F_{4j}+G_1(F_{4j+1}-1) && \text{(by Lemma 9, Identity (6))}\\
&= (G_0F_{4j}+G_1F_{4j+1})-G_1\\
&= G_{4j+1}-G_1, && \text{(by Proposition 6, Identity (4))}
\end{aligned}$$

as desired. Hence $G_{4j+1}-G_1 = F_{2j}(G_{2j}+G_{2j+2})$ for all $j \geq 0$. □

Proposition 12. *For all $j \geq 0$, the following identity holds:*

$$G_{4j+2}-G_2 = F_{2j}(G_{2j+1}+G_{2j+3}).$$

Proof. Let $j \geq 0$ be given. Observe the sequence of equalities

$$\begin{aligned}
F_{2j}(G_{2j+1}+G_{2j+3}) &= F_{2j}(G_0L_{2j+1}+G_1L_{2j+2}) && \text{(by Lemma 8)}\\
&= G_0\cdot F_{2j}L_{2j+1}+G_1\cdot F_{2j}L_{2j+2}\\
&= G_0\cdot F_{2j}L_{2j+1}+G_1\cdot F_{2j}(L_{2j}+L_{2j+1})\\
&= (G_0\cdot F_{2j}L_{2j+1}+G_1\cdot F_{2j}L_{2j+1})+G_1\cdot F_{2j}L_{2j}\\
&= (G_0+G_1)\cdot F_{2j}L_{2j+1}+G_1\cdot F_{2j}L_{2j}\\
&= G_2\cdot F_{2j}L_{2j+1}+G_1\cdot F_{2j}L_{2j}\\
&= G_2\cdot(F_{4j+1}-1)+G_1\cdot F_{2j}L_{2j} && \text{(by Lemma 9, Identity (6))}\\
&= G_2\cdot(F_{4j+1}-1)+G_1F_{4j} && \text{(by Proposition 6, Identity (2))}\\
&= (G_1F_{4j}+G_2F_{4j+1})-G_2\\
&= G_{4j+2}-G_2,
\end{aligned}$$

where the last equality holds since for all $i \geq 1$, the value G_i can be written in the following form $G_i = G_1F_{i-2}+G_2F_{i-1}$ by Identity (3) of Proposition 6, if we set $m := i-1$ and $n := 1$. Hence $F_{4j+2}-G_2 = F_{2j}(G_{2j+1}+G_{2j+3})$ for all $j \geq 0$. □

Proposition 13. *For all $j \geq 0$, the following identity holds:*

$$G_{4j+3}-G_1 = L_{2j+1}G_{2j+2}.$$

Proof. Let $j \geq 0$ be given. Observe the sequence of equalities

$$\begin{aligned}
G_{4j+3}-G_1 &= (G_0F_{4j+2}+G_1F_{4j+3})-G_1 && \text{(by Proposition 6, Identity (4))}\\
&= G_0F_{4j+2}+G_1(F_{4j+3}-1)\\
&= G_0F_{4j+2}+G_1(F_{2j+2}L_{2j+1}) && \text{(by Lemma 9, Identity (7))}\\
&= G_0(F_{2j+1}L_{2j+1})+G_1(F_{2j+2}L_{2j+1}) && \text{(by Proposition 6, Identity (2))}\\
&= L_{2j+1}(G_0G_{2j+1}+G_1F_{2j+2})\\
&= L_{2j+1}G_{2j+2}, && \text{(by Proposition 6, Identity (4))}
\end{aligned}$$

as desired. Hence $G_{4j+3}-G_1 = L_{2j+1}G_{2j+2}$ for all $j \geq 0$. □

Proposition 14. *For all $j \geq 0$, the following identity holds:*

$$G_{4j+4} - G_2 = L_{2j+1}G_{2j+3}.$$

Proof. Let $j \geq 0$ be given. Observe the sequence of equalities

$$\begin{aligned}
G_{4j+4} - G_2 &= (G_0F_{4j+3} + G_1F_{4j+4}) - G_2 && \text{(by Proposition 6, Identity (4))}\\
&= (G_0F_{4j+3} + G_1F_{4j+4}) - (G_0 + G_1)\\
&= G_0(F_{4j+3} - 1) + G_1(F_{4j+4} - 1)\\
&= G_0(F_{2j+2}L_{2j+1}) + G_1(F_{4j+4} - 1) && \text{(by Lemma 9, Identity (7))}\\
&= G_0(F_{2j+2}L_{2j+1}) + G_1(F_{2j+3}L_{2j+1}) && \text{(by Lemma 9, Identity (8))}\\
&= L_{2j+1}(G_0F_{2j+2} + G_1F_{2j+3})\\
&= L_{2j+1}G_{2j+3}, && \text{(by Proposition 6, Identity (4))}
\end{aligned}$$

as desired. Hence $G_{4j+4} - G_2 = L_{2j+1}G_{2j+3}$ for all $j \geq 0$. □

3 Two equivalent formulas used to compute $\mathcal{G}_{G_0,G_1}(k)$

The first two major results of this paper are given in this section. We provide two seemingly different, yet equivalent, formulas that compute the value $\mathcal{G}_{G_0,G_1}(k)$, the GCD of the sums of k consecutive Gibonacci numbers. These two different characterizations not only help prove our main results in Sections 4 and 5, but also lead to some tantalizing applications in Section 6.

3.1 A simple GCD characterization

In this subection, we give our first of two characterizations for the value $\mathcal{G}_{G_0,G_1}(k)$. Moreover, we establish why it suffices to consider only the Gibonacci sequences with relatively prime initial conditions, since the value $\mathcal{G}_{G_0,G_1}(k)$ for a sequence with non-relatively prime initial values G_0 and G_1 turns out to be a multiple of the value $\mathcal{G}_{G'_0,G'_1}(k)$ of a related sequence with relatively prime initial values G'_0 and G'_1.

Theorem 15. *The largest integer that divides every sum of k consecutive Gibonacci numbers is $\gcd(G_{k+1} - G_1, G_{k+2} - G_2)$. That is, $\mathcal{G}_{G_0,G_1}(k) = \gcd(G_{k+1} - G_1, G_{k+2} - G_2)$.*

Proof. Fix $n \in \mathbb{N}$ and consider the arbitrary sum $G_n + G_{n+1} + \cdots + G_{n+(k-1)}$ of k consecutive

Gibonacci numbers. Then we have the sequence of equalities

$$\begin{aligned}
\sum_{i=0}^{k-1} G_{n+i} &= \sum_{i=1}^{n+(k-1)} G_i - \sum_{i=1}^{n-1} G_i \\
&= (G_{(n+k-1)+2} - G_2) - (G_{(n-1)+2} - G_2) && \text{(by Proposition 6, Identity (5))} \\
&= G_{n+k+1} - G_{n+1} \\
&= G_{(k+1)+n} - G_{n+1} \\
&= F_{n-1}G_{k+1} + F_n G_{k+2} - G_{n+1} && \text{(by Proposition 6, Identity (3))} \\
&= F_{n-1}G_{k+1} + F_n G_{k+2} - F_n G_2 - F_{n-1}G_1 && \text{(by Proposition 6, Identity (3))} \\
&= F_{n-1}G_{k+1} - F_{n-1}G_1 + F_n G_{k+2} - F_n G_2 \\
&= F_{n-1}(G_{k+1} - G_1) + F_n(G_{k+2} - G_2).
\end{aligned}$$

Hence our sequence of finite sums of k consecutive Gibonacci numbers can be written as

$$\left(\sum_{i=0}^{k-1} G_{n+i} \right)_{n \geq 1} = \Big(F_{n-1}(G_{k+1} - G_1) + F_n(G_{k+2} - G_2) \Big)_{n \geq 1}. \tag{9}$$

Set $q := \gcd(G_{k+1} - G_1, G_{k+2} - G_2)$. We will show that $q \leq \mathcal{G}_{G_0,G_1}(k)$ and that $\mathcal{G}_{G_0,G_1}(k) \leq q$, and hence $\mathcal{G}_{G_0,G_1}(k) = q$ follows. Since q divides both $G_{k+1} - G_1$ and $G_{k+2} - G_2$, then q divides every term in our sequence, and therefore $q \leq \mathcal{G}_{G_0,G_1}(k)$, as desired. Next we establish that $\mathcal{G}_{G_0,G_1}(k) \leq q$. Observe that the GCD of every term in our sequence is at most the GCD of the first two terms. Consider the GCD of the first two terms. We have the sequence of inequalities and equalities

$$\begin{aligned}
\mathcal{G}_{G_0,G_1}(k) &\leq \gcd\Big(F_0(G_{k+1} - G_1) + F_1(G_{k+2} - G_2)\ ,\ F_1(G_{k+1} - G_1) + F_2(G_{k+2} - G_2) \Big) \\
&= \gcd\big(G_{k+2} - G_2\ ,\ G_{k+1} - G_1 + G_{k+2} - G_2 \big) \\
&= \gcd\big(G_{k+2} - G_2\ ,\ G_{k+1} - G_1 \big) \\
&= q,
\end{aligned}$$

where the second equality holds by the property $\gcd(a, b+a) = \gcd(a, b)$. Thus $\mathcal{G}_{G_0,G_1}(k) \leq q$, as desired. We conclude that $\mathcal{G}_{G_0,G_1}(k) = \gcd(G_{k+1} - G_1, G_{k+2} - G_2)$. □

Corollary 16. *The largest integer to divide every sum of k consecutive Fibonacci numbers is precisely* $\gcd(F_{k+1} - F_1, F_{k+2} - F_2)$. *That is,* $\mathcal{F}(k) = \gcd(F_{k+1} - 1, F_{k+2} - 1)$.

Corollary 17. *The largest integer to divide every sum of k consecutive Lucas numbers is precisely* $\gcd(L_{k+1} - L_1, L_{k+2} - L_2)$. *That is,* $\mathcal{L}(k) = \gcd(L_{k+1} - 1, L_{k+2} - 3)$.

After proving the following two results, Lemma 18 and Theorem 19, we will conclude that it is sufficient to explore only the Gibonacci sequences which have relatively prime initial values.

Lemma 18. *For all $n \in \mathbb{Z}$, the values* $\gcd(G_{n+1}, G_{n+2})$ *and* $\gcd(G_n, G_{n+1})$ *coincide. In particular,* $\gcd(G_0, G_1) = \gcd(G_n, G_{n+1})$ *holds for all* $n \in \mathbb{Z}$

Proof. Observe the following sequence of equalities.

$$\begin{aligned}\gcd(G_{n+1}, G_{n+2}) &= \gcd(G_{n+1}, G_{n+1} + G_n) \\ &= \gcd(G_{n+1}, G_n).\end{aligned}$$

Hence $\gcd(G_0, G_1) = \gcd(G_n, G_{n+1})$ as desired for all $n \in \mathbb{Z}$. □

Theorem 19. *Fix $G_0, G_1 \in \mathbb{Z}$ and set $d := \gcd(G_0, G_1)$. Then the GCD of every sum of k consecutive Gibonacci numbers in the sequence $(G_n)_{n\geq 0}$ is d times the GCD of every sum of k consecutive Gibonacci numbers in the new sequence $\{G'_n\}_{n=0}^{\infty}$ generated by the relatively prime initial conditions $G'_0 = \frac{G_0}{d}$ and $G'_1 = \frac{G_1}{d}$. In particular, we have the following:*

$$\mathcal{G}_{G_0,G_1}(k) = d \cdot \mathcal{G}_{G'_0,G'_1}(k).$$

Proof. Set $d := \gcd(G_0, G_1)$. By Lemma 18, we have $\gcd(G_{k+1}, G_{k+2}) = \gcd(G_0, G_1) = d$ for all $k \in \mathbb{Z}$. By Theorem 15, the largest positive integer that divides every sum of k consecutive Gibonacci numbers is $\gcd(G_{k+1} - G_1, G_{k+2} - G_2)$. Moreover, since d divides G_0 and G_1, then d divides every term in the sequence $(G_n)_{n\geq 0}$. In particular, $\frac{G_{k+1}-G_1}{d}$ and $\frac{G_{k+2}-G_2}{d}$ are integers. Observe the sequence of equalities

$$\begin{aligned}\gcd(G_{k+1} - G_1, G_{k+2} - G_2) &= \gcd\left(d \cdot \frac{G_{k+1} - G_1}{d}, d \cdot \frac{G_{k+2} - G_2}{d}\right) \\ &= d \cdot \gcd\left(\frac{G_{k+1} - G_1}{d}, \frac{G_{k+2} - G_2}{d}\right).\end{aligned}$$

Notice that by Theorem 15, the value $\gcd\left(\frac{G_{k+1}-G_1}{d}, \frac{G_{k+2}-G_2}{d}\right)$ is the GCD of the sum of k consecutive Gibonacci in the new sequence $\{G'_n\}_{n=0}^{\infty}$ generated by the initial values $G'_0 = \frac{G_0}{d}$ and $G'_1 = \frac{G_1}{d}$. Clearly G'_0 and G'_1 are relatively prime. In particular, we have

$$\mathcal{G}_{G_0,G_1}(k) = d \cdot \mathcal{G}_{G'_0,G'_1}(k),$$

as desired. □

Convention 20. In order to give a complete classification of the GCD of every sum of k consecutive Gibonacci numbers, as a consequence of Theorem 19, we need only to consider Gibonacci sequences with relatively prime initial values.

3.2 A generalized Pisano period characterization

As in the setting of the Fibonacci and Lucas sequences modulo m, it is well known that the Gibonacci sequence modulo m is also periodic. Hence it makes sense to consider the period $\pi_{G_0,G_1}(m)$ of this sequence given in the following definition.

Definition 21. Let $m \geq 2$. The *generalized Pisano period*, $\pi_{G_0,G_1}(m)$, of the Gibonacci sequence $(G_n)_{n\geq 0}$ is the smallest positive integer r such that

$$G_r \equiv G_0 \pmod{m} \quad \text{and} \quad G_{r+1} \equiv G_1 \pmod{m}.$$

In the Fibonacci (respectively, Lucas) setting we denote this period by $\pi_F(m)$ (respectively, $\pi_L(m)$).

Lemma 22. *The value m divides the sum of any $\pi_{G_0,G_1}(m)$ consecutive Gibonacci numbers. That is, m divides $\sum_{i=1}^{\pi_{G_0,G_1}(m)} G_{n+i}$ for any fixed $n \in \mathbb{Z}$.*

Proof. We need to prove m divides the sum of the terms in a generalized Pisano period of any Gibonacci sequence. However, by the periodicity of generalized Pisano periods, it suffices to show that m divides the sum of the terms in the particular generalized Pisano period given by $(G_1, G_2, \ldots, G_{\pi_{G_0,G_1}(m)})$. By Identity (5) of Proposition 6 we have

$$\sum_{i=1}^{\pi_{G_0,G_1}(m)} G_i = G_{\pi_{G_0,G_1}(m)+2} - G_2.$$

However, by the definition of a generalized Pisano period, $G_{\pi_{G_0,G_1}(m)+2} \equiv G_2 \pmod{m}$. Hence m divides $G_{\pi_{G_0,G_1}(m)+2} - G_2$ and therefore also divides $\sum_{i=1}^{\pi_{G_0,G_1}(m)} G_i$ as desired. It follows that m divides the sum of the terms in the particular generalized Pisano period $(G_1, G_2, \ldots, G_{\pi_{G_0,G_1}(m)})$, which proves that m divides the sum of any $\pi_{G_0,G_1}(m)$ consecutive Gibonacci numbers. □

Remark 23. It can be proven that the value $\pi_{G_0,G_1}(m)$ in Lemma 22 is minimal with respect to the following property: If $s \in \mathbb{N}$ with $s < \pi_{G_0,G_1}(m)$, then m cannot divide the sum of every s consecutive Gibonacci numbers.

Theorem 24. *The value $\pi_{G_0,G_1}(m)$ divides k if and only if m divides $\mathcal{G}_{G_0,G_1}(k)$.*

Proof. Let $k \in \mathbb{N}$ be fixed. Suppose $\pi_{G_0,G_1}(m)$ divides k. By Lemma 22, we know that m divides the sum of any $\pi_{G_0,G_1}(m)$ consecutive Gibonacci numbers. Thus m divides any sum of $t \cdot \pi_{G_0,G_1}(m)$ consecutive Gibonacci numbers for any $t \in \mathbb{N}$. From our assumption that π_{G_0,G_1} divides k, it follows that $k = t_0 \cdot \pi_{G_0,G_1}(m)$ for some $t_0 \in \mathbb{N}$. Hence m is a common divisor of any sum of k consecutive Gibonacci numbers, which proves that m divides the greatest common divisor $\mathcal{G}_{G_0,G_1}(k)$ as desired.

Assume m divides $\mathcal{G}_{G_0,G_1}(k)$. Then m divides $\gcd(G_{k+2} - G_2, G_{k+1} - G_1)$. Thus m divides $G_{k+2} - G_2$ and m divides $G_{k+1} - G_1$. Hence $G_{k+2} \equiv G_2 \pmod{m}$ and $G_{k+1} \equiv G_1 \pmod{m}$. By the periodicity of the sequence $(G_n)_{n\geq 0}$ under a modulus, $\pi_{G_0,G_1}(m)$ divides k. □

Theorem 25. *For all $k \geq 1$, we have $\mathcal{G}_{G_0,G_1}(k) = \operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m)$ divides $k\}$.*

Proof. For ease of notation, set $\ell(k) := \operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m)$ divides $k\}$. Then it suffices to prove that $\mathcal{G}_{G_0,G_1}(k)$ divides $\ell(k)$ and that $\ell(k)$ divides $\mathcal{G}_{G_0,G_1}(k)$. Since both $\mathcal{G}_{G_0,G_1}(k)$ and $\ell(k)$ are strictly greater than 0, we only need to show that any divisor of $\mathcal{G}_{G_0,G_1}(k)$ is a divisor of $\ell(k)$, and vice versa. Let d_0 be a divisor of $\mathcal{G}_{G_0,G_1}(k)$. Then by Theorem 24, it follows that $\pi_{G_0,G_1}(d_0)$ divides k. Hence by definition of $\ell(k)$, we conclude that d_0 divides $\ell(k)$ as desired. Now, suppose that d_1 is a divisor of $\ell(k)$. Then by definition of $\ell(k)$, it must be that $\pi_{G_0,G_1}(d_1)$ divides k. Hence by Theorem 24, we conclude that d_1 divides $\mathcal{G}_{G_0,G_1}(k)$ as desired.

□

4 Main results for $\mathcal{G}_{G_0,G_1}(k)$ when k is even

In this section, we provide our main results for the values $\mathcal{G}_{G_0,G_1}(k)$ when k is even. There are two cases that we consider; namely, when $k \equiv 0, 4$, or $8 \pmod{12}$ given in Subsection 4.1 and when $k \equiv 2, 6$, or $10 \pmod{12}$ given in Subsection 4.2. From Table 1 in Section 1, we see that the second row, which corresponds to $k \equiv 2, 6, 10 \pmod{12}$, gives the same value $L_{k/2}$ regardless if we are considering $\mathcal{F}(k)$, $\mathcal{L}(k)$, or $\mathcal{G}_{G_0,G_1}(k)$; that is, no matter which initial values for the sequence $\{G_i\}_{n=0}^{\infty}$ are chosen, the values $\mathcal{F}(k)$, $\mathcal{L}(k)$, and $\mathcal{G}_{G_0,G_1}(k)$ coincide. However in the first row of this table when $k \equiv 0, 4, 8 \pmod{12}$, it turns out that the value of $\mathcal{G}_{G_0,G_1}(k)$ depends on the initial conditions G_0 and G_1, and hence the values $\mathcal{F}(k)$, $\mathcal{L}(k)$, and $\mathcal{G}_{G_0,G_1}(k)$ may differ. More precisely, for a fixed k such that $k \equiv 0, 4, 8 \pmod{12}$, we will see in the following subsection that the behavior of these latter three values depends on an easily computed parameter which we denote by δ_{G_0,G_1}, defined as $\delta_{G_0,G_1} := \gcd(G_0 + G_2, G_1 + G_3)$.

4.1 The $k \equiv 0, 4, 8 \pmod{12}$ case

Lemmas 27 is used to conclude our penultimate result, Lemma 31, which essentially implies that the value of $\mathcal{G}_{G_0,G_1}(k)$ is determined soley by the value k and the parameter δ_{G_0,G_1}.

Remark 26. It is worth noting that in this subsection, only our main result, Theorem 32, involves the value k. The two lemmas have no mention of the value k, and in fact, say something quite interesting about any Gibonacci sequence $(G_n)_{n\geq 0}$. In particular, as a consequence of Lemma 31, the value $\gcd(G_0 + G_2, G_1 + G_3)$, which is the parameter δ_{G_0,G_1}, equals 1 or 5, and moreover the value $\gcd(G_n + G_{n+2}, G_{n+1} + G_{n+3})$ equals δ_{G_0,G_1} for all $n \geq 0$.

Lemma 27. *Fix an integer $i \geq 0$. It follows that the value $\gcd(G_i + G_{i+2}, G_{i+1} + G_{i+3})$ is either 1 or 5.*

Proof. Suppose $\gcd(G_0, G_1) = 1$. Let d be any divisor of $\gcd(G_i + G_{i+2}, G_{i+1} + G_{i+3})$. Since d divides the sums $G_i + G_{i+2}$ and $G_{i+1} + G_{i+3}$, we have the congruences

$$G_i + G_{i+2} \equiv 0 \pmod{d} \tag{10}$$

$$G_{i+1} + G_{i+3} \equiv 0 \pmod{d}. \tag{11}$$

We can express the three values G_{i+2}, G_{i+1} and G_{i+3}, respectively, in terms of G_i as follows:

$$G_{i+2} \equiv -G_i \pmod{d} \qquad \text{(by Congruence (10))} \tag{12}$$

$$\begin{aligned} G_{i+1} &= G_{i+2} - G_i \\ &\equiv -G_i - G_i \pmod{d} && \text{(by Congruence (12))} \\ &\equiv -2G_i \pmod{d} \end{aligned} \tag{13}$$

$$\begin{aligned} G_{i+3} &= G_{i+2} + G_{i+1} \\ &\equiv -G_i - 2G_i \pmod{d} && \text{(by Congruences (12) and (13))} \\ &\equiv -3G_i \pmod{d}. \end{aligned} \tag{14}$$

Then by Congruences (11), (13), and (14), we have

$$0 \equiv G_{i+1} + G_{i+3} \equiv -2G_i - 3G_i = -5G_i \pmod{d},$$

and thus $5G_i \equiv 0 \pmod{d}$. Furthermore, observe that by adding Congruences (10) and (11) we get that $G_{i+2} + G_{i+4} \equiv 0 \pmod{d}$. Hence d is a divisor of $G_{i+1} + G_{i+3}$ and $G_{i+2} + G_{i+4}$. Therefore d divides $\gcd(G_{i+1} + G_{i+3}, G_{i+2} + G_{i+4})$. Analogous to our previous work, we can express the three values G_{i+3}, G_{i+2} and G_i, respectively, in terms of G_{i+1} to find that d divides $5G_{i+1}$. Since d divides both $5G_i$ and $5G_{i+1}$ and $\gcd(G_i, G_{i+1}) = 1$, it must be that d divides 5. Thus $d = 1$ or $d = 5$. □

Remark 28. It is worthy to note that both values 1 and 5 are attained in Lemma 27. For $i = 0$ in the Fibonacci sequence, we have $\gcd(F_0 + F_2, F_1 + F_3) = \gcd(1, 2) = 1$. Moreover, for $i = 0$ in the Lucas sequence, we have $\gcd(L_0 + L_2, L_1 + L_3) = \gcd(5, 5) = 5$. However, it is not yet clear that for fixed initial values G_0 and G_1, the values $\gcd(G_n + G_{n+2}, G_{n+1} + G_{n+3})$ will be the same for all n. However, a consequence of Lemma 31 will confirm the latter. But first we need to define what we mean for two Gibonacci sequences to be equivalent (up to shift) modulo m for some $m \geq 2$.

Definition 29. Let $m \geq 2$. Let G and G' denote the Gibonacci sequences $(G_n)_{n\geq 0}$ and $(G'_n)_{n\geq 0}$, respectively, with corresponding generalized Pisano periods $\pi_{G_0,G_1}(m)$ and $\pi_{G'_0,G'_1}(m)$. We say that G modulo m is *equivalent (up to shift)* to G' modulo m if the following two conditions hold:

(i) The values $\pi_{G_0,G_1}(m)$ and $\pi_{G'_0,G'_1}(m)$ coincide.

(ii) For some $r \in \mathbb{Z}$, we have $G_{r+n} \equiv G'_n \pmod{m}$ for all $n \in \mathbb{Z}$.

Remark 30. It can be shown that the value δ_{G_0,G_1} equals 5 if and only if the Gibonacci sequence $(G_n)_{n\geq 0}$ modulo 5 is equivalent (up to shift) to the Lucas sequence $(L_n)_{n\geq 0}$ modulo 5.

Lemma 31. *The value δ_{G_0,G_1} equals 1 or 5, and we have the following:*

$$\gcd(G_n + G_{n+2}, G_{n+1} + G_{n+3}) = \delta_{G_0,G_1}$$

for all $n \geq 0$.

Proof. Let $H_n = G_n + G_{n+2}$. Observe that

$$\begin{aligned} H_n + H_{n+1} &= (G_n + G_{n+2}) + (G_{n+1} + G_{n+3}) \\ &= (G_n + G_{n+1}) + (G_{n+2} + G_{n+3}) \\ &= G_{n+2} + G_{n+4} \\ &= H_{n+2}. \end{aligned}$$

Thus the sequence $(H_n)_{n\geq 0}$ is itself a generalized Fibonacci sequence. By Lemma 18, we have $\gcd(H_0, H_1) = \gcd(H_n, H_{n+1})$ for all $n \geq 0$. □

We are now ready to prove the main theorem of this subsection. We utilize the parameter δ_{G_0,G_1}. Recall from Remark 28 that $\delta_{G_0,G_1} = 1$ for the Fibonacci sequence, $\delta_{G_0,G_1} = 5$ for the Lucas sequence, and $\delta_{G_0,G_1} = 1$ or 5 for Gibonacci sequences.

Theorem 32. *If $k \equiv 0, 4, 8 \pmod{12}$, then $\gcd(G_{k+1} - G_1, G_{k+2} - G_2) = \delta_{G_0,G_1} \cdot F_{k/2}$, where $\delta_{G_0,G_1} = \gcd(G_0 + G_2, G_1 + G_3)$. In particular, we conclude the following:*

$$\begin{aligned} \mathcal{F}(k) &= F_{k/2} \\ \mathcal{L}(k) &= 5F_{k/2} \\ \mathcal{G}_{G_0,G_1}(k) &= \delta_{G_0,G_1} \cdot F_{k/2}. \end{aligned}$$

Proof. Assume $k \equiv 0, 4, 8 \pmod{12}$. Then $k \equiv 0 \pmod{4}$. Thus $k = 4j$ for some $j \in \mathbb{Z}$. Observe the sequence of equalities

$$\begin{aligned} &\gcd(G_{k+1} - G_1, G_{k+2} - G_2) \\ &\qquad = \gcd(G_{4j+1} - G_1, G_{4j+2} - G_2) \\ &\qquad = \gcd(F_{2j}(G_{2j} + G_{2j+2}), F_{2j}(G_{2j+1} + G_{2j+3})) \quad \text{(by Propositions 11 and 12)} \\ &\qquad = F_{2j} \cdot \gcd(G_{2j} + G_{2j+2}, G_{2j+1} + G_{2j+3}) \\ &\qquad = F_{k/2} \cdot \gcd(G_{2j} + G_{2j+2}, G_{2j+1} + G_{2j+3}). \end{aligned}$$

Observe that from Lemma 31, we know $\gcd(G_{2j} + G_{2j+2}, G_{2j+1} + G_{2j+3}) = \delta_{G_0,G_1}$. Thus $\gcd(G_{k+1} - G_1, G_{k+2} - G_2) = \delta_{G_0,G_1} \cdot F_{k/2}$. We conclude that if $k \equiv 0, 4, 8 \pmod{12}$, then $\mathcal{F}(k) = F_{k/2}$ and $\mathcal{L}(k) = 5F_{k/2}$ and $\mathcal{G}_{G_0,G_1}(k) = \delta_{G_0,G_1} \cdot F_{k/2}$. □

4.2 The $k \equiv 2, 6, 10 \pmod{12}$ case

Whereas the $k \equiv 0, 4, 8 \pmod{12}$ case in Subsection 4.1 had variability in the value $\mathcal{G}_{G_0,G_1}(k)$ dependent on the initial values G_0 and G_1, the $k \equiv 2, 6, 10 \pmod{12}$ case in this subsection is more straightforward since all values $\mathcal{F}(k)$, $\mathcal{L}(k)$, and $\mathcal{G}_{G_0,G_1}(k)$ coincide, regardless of the initial values.

Theorem 33. *If $k \equiv 2, 6, 10 \pmod{12}$, then $\gcd(G_{k+1} - 1, G_{k+2} - 3) = L_{k/2}$. In particular, we conclude the following:*

$$\mathcal{F}(k) = L_{k/2}$$
$$\mathcal{L}(k) = L_{k/2}$$
$$\mathcal{G}_{G_0,G_1}(k) = L_{k/2}.$$

Proof. Assume $k \equiv 2, 6, 10 \pmod{12}$. Then $k \equiv 2 \pmod{4}$. Thus $k = 4j + 2$ for some $j \in \mathbb{Z}$. Observe the sequence of equalities

$$\begin{aligned}
&\gcd(G_{k+1} - G_1, G_{k+2} - G_2) \\
&\qquad = \gcd(G_{(4j+2)+1} - G_1, G_{(4j+2)+2} - G_2) \\
&\qquad = \gcd(G_{4j+3} - G_1, G_{4j+4} - G_2) \\
&\qquad = \gcd(L_{2j+1}G_{2j+2}, L_{2j+1}G_{2j+3}) && \text{(by Propositions 13 and 14)} \\
&\qquad = L_{2j+1} \cdot \gcd(G_{2j+2}, G_{2j+3}) \\
&\qquad = L_{2j+1} && (\text{since } \gcd(G_{2j+2}, G_{2j+3}) = 1) \\
&\qquad = L_{k/2}.
\end{aligned}$$

Thus $\gcd(G_{k+1} - G_1, G_{k+2} - G_2) = L_{k/2}$. We conclude that if $k \equiv 2, 6, 10 \pmod{12}$, then we have $\mathcal{F}(k) = \mathcal{L}(k) = \mathcal{G}_{G_0,G_1}(k) = L_{k/2}$. □

5 Main results for $\mathcal{G}_{G_0,G_1}(k)$ when k is odd

The two main results in this section, Theorems 38 and 39, rely on the generalized Pisano period $\pi_{G_0,G_1}(m)$ being even for all $m > 2$. The period $\pi_F(m)$ of the Fibonacci sequence modulo m being even for all $m > 2$ is well known and proven in 1960 by Wall [22], and a clever short proof was given more recently by Elsenhans and Jahnel [8]. Similarly, the period $\pi_L(m)$ of the Lucas sequence modulo m is also even for all $m > 2$; however, this well-known result seems to lack a proof in the literature, though it is stated in a number of sources. A corollary to the following lemmas will not only prove that the Fibonacci and Lucas periods are even, but also provides a sufficiency condition on the initial values G_0 and G_1 that will give the defined $\mathcal{G}_{G_0,G_1}(k)$ values we gave in Table 1 in this case when k is odd.

5.1 A sufficiency criterion for when $\pi_{G_0,G_1}(m)$ is even for all $m > 2$

Lemma 34. *Let $D_{G_n,G_{n+1}}$ denote the value $G_{n+1}^2 - G_nG_{n+1} - G_n^2$. Then the following holds:*

$$D_{G_n,G_{n+1}} = (-1)^n \cdot D_{G_0,G_1} \tag{15}$$

for all $n \geq 0$. In particular, we have $|D_{G_n,G_{n+1}}| = |D_{G_0,G_1}|$ for all $n \geq 0$.

Proof. We prove this by induction on n. Identity (15) clearly holds when $n = 0$. So suppose it holds for some $k \geq 0$, and consider $D_{G_{k+1},G_{k+2}}$. Then we have

$$\begin{aligned} D_{G_{k+1},G_{k+2}} &= G_{k+2}^2 - G_{k+1}G_{k+2} - G_{k+1}^2 \\ &= (G_k + G_{k+1})^2 - G_{k+1}(G_k + G_{k+1}) - G_{k+1}^2 \\ &= G_k^2 + 2G_kG_{k+1} + G_{k+1}^2 - G_kG_{k+1} - G_{k+1}^2 - G_{k+1}^2 \\ &= -(G_{k+1}^2 - G_kG_{k+1} - G_k^2) \\ &= -(-1)^k \cdot D_{G_0,G_1} \\ &= (-1)^{k+1} \cdot D_{G_0,G_1}, \end{aligned}$$

where the fifth equality holds by the induction hypothesis. Hence Identity (15) holds for all $n \geq 0$. □

Lemma 35. *For all integers $m > 2$, the following congruence holds:*

$$(-1)^{\pi_{G_0,G_1}(m)} \cdot D_{G_0,G_1} \equiv D_{G_0,G_1} \pmod{m}.$$

Proof. By the generalized Cassini's identity, given in Proposition 7, it follows that

$$G_{n+1}G_{n-1} - G_n^2 = (-1)^n \cdot D_{G_0,G_1}.$$

Substituting $\pi_{G_0,G_1}(m)$ for n in the latter identity, we have

$$G_{\pi_{G_0,G_1}(m)+1} \cdot G_{\pi_{G_0,G_1}(m)-1} - G_{\pi_{G_0,G_1}(m)}^2 = (-1)^{\pi_{G_0,G_1}(m)} \cdot D_{G_0,G_1}.$$

Since $G_{\pi_{G_0,G_1}(m)+i} \equiv G_i \pmod{m}$ for all i, we get the sequence of congruences

$$\begin{aligned} (-1)^{\pi_{G_0,G_1}(m)} \cdot D_{G_0,G_1} &\equiv G_1G_{-1} - G_0^2 \pmod{m} \\ &\equiv G_1(G_1 - G_0) - G_0^2 \pmod{m} \\ &\equiv D_{G_0,G_1} \pmod{m}, \end{aligned}$$

and the claim holds. □

Corollary 36. *For all $m > 2$, the Pisano periods of the Fibonacci and Lucas sequences are even. In the general setting, if $D_{G_0,G_1} = \pm 1$ then $\pi_{G_0,G_1}(m)$ is even for all $m > 2$.*

Proof. By Lemma 35, we have $(-1)^{\pi_{G_0,G_1}(m)} \cdot D_{G_0,G_1} \equiv D_{G_0,G_1} \pmod{m}$ for all $m > 2$. We address the Fibonacci, Lucas, and Gibonacci settings in three separate cases.

Case 1: If $G_0 = 0$ and $G_1 = 1$, then $D_{G_0,G_1} = 1$, and we have $(-1)^{\pi_F(m)} \equiv 1 \pmod{m}$, which implies that $\pi_F(m)$ is even for all $m > 2$.

Case 2: If $G_0 = 2$ and $G_1 = 1$, then $D_{G_0,G_1} = -5$, and we have $(-1)^{\pi_L(m)}5 \equiv 5 \pmod{m}$, which implies that $\pi_L(m)$ is even for all $m > 2$ when $\gcd(5, m) = 1$. If on the other hand $\gcd(5, m) \neq 1$, then $m = 5^s t$ for some $s, t \in \mathbb{N}$ with $\gcd(5, t) = 1$. A consequence of Theorem 2 by Wall yields $\pi_L(m) = \operatorname{lcm}\big(\pi_L(5^s), \pi_L(t)\big)$ [22]. But since $\pi_L(5)$ divides $\pi_L(5^s)$ and $\pi_L(5) = 4$, then 4 divides $\operatorname{lcm}\big(\pi_L(5^s), \pi_L(t)\big)$. Hence $\pi_L(m)$ is even for all $m > 2$.

Case 3: In general, for any initial values G_0 and G_1 with $D_{G_0,G_1} = \pm 1$, it follows that $(-1)^{\pi_{G_0,G_1}(m)} \equiv 1 \pmod{m}$. Hence $\pi_{G_0,G_1}(m)$ is even for all $m > 2$ when $D_{G_0,G_1} = \pm 1$. □

5.2 The $k \equiv 3, 9 \pmod{12}$ case

Lemma 37. *If k is odd and $\pi_{G_0,G_1}(m)$ is even for all $m > 2$, then $\mathcal{G}_{G_0,G_1}(k) \leq 2$.*

Proof. Suppose that $\pi_{G_0,G_1}(m)$ is even for all $m > 2$ and k is odd. By Theorem 25, we have $\mathcal{G}_{G_0,G_1}(k) = \operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } k\}$. Since $\pi_{G_0,G_1}(m)$ is even for all $m > 2$ and k is odd, $\pi_{G_0,G_1}(m)$ cannot divide k for all $m > 2$. Thus $\mathcal{G}_{G_0,G_1}(k) = \operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } k\} \leq 2$ as desired. □

Theorem 38. *If $k \equiv 3, 9 \pmod{12}$ and $\pi_{G_0,G_1}(m)$ is even for all $m > 2$, then $\mathcal{G}_{G_0,G_1}(k) = 2$. In particular, we conclude the following:*

$$\begin{aligned} \mathcal{F}(k) &= 2 \\ \mathcal{L}(k) &= 2 \\ \mathcal{G}_{G_0,G_1}(k) &= 2 \text{ if } D_{G_0,G_1} = \pm 1. \end{aligned}$$

Proof. Suppose $k \equiv 3, 9 \pmod{12}$ and that $\pi_{G_0,G_1}(m)$ is even for all $m > 2$. Then we have $k \equiv 0 \pmod{3}$. Since $\gcd(G_0, G_1) = 1$, the Gibonacci sequence modulo 2 is equivalent (up to shift) to the sequence

$$1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1 \ldots$$

Hence $\pi_{G_0,G_1}(2) = 3$. By assumption, $k \equiv 0 \pmod{3}$ and thus $\pi_{G_0,G_1}(2)$ divides k. Therefore by Theorem 25, we have $\mathcal{G}_{G_0,G_1}(k) = \operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } k\} \geq 2$. However by Lemma 37, $\mathcal{G}_{G_0,G_1}(k) \leq 2$ since k is odd. We conclude that if $k \equiv 3, 9 \pmod{12}$, then we have $\mathcal{F}(k) = \mathcal{L}(k) = 2$. Also if $D_{G_0,G_1} = \pm 1$, then $\pi_{G_0,G_1}(m)$ is even and $\mathcal{G}_{G_0,G_1}(k) = 2$. □

5.3 The $k \equiv 1,5,7,11 \pmod{12}$ case

Theorem 39. *If $k \equiv 1,5,7,11 \pmod{12}$ and $\pi_{G_0,G_1}(m)$ is even for all $m > 2$, then $\mathcal{G}_{G_0,G_1}(k) = 1$. In particular, we conclude the following:*

$$\begin{aligned} \mathcal{F}(k) &= 1 \\ \mathcal{L}(k) &= 1 \\ \mathcal{G}_{G_0,G_1}(k) &= 1 \text{ if } D_{G_0,G_1} = \pm 1. \end{aligned}$$

Proof. Suppose $k \equiv 1,5,7,11 \pmod{12}$ and that $\pi_{G_0,G_1}(m)$ is even for all $m > 2$. As in the proof of Theorem 38, we know the fact that $\gcd(G_0, G_1) = 1$ holds implies that $\pi_{G_0,G_1}(2) = 3$. However, since we have $k \not\equiv 0 \pmod 3$ it cannot be that $\pi_{G_0,G_1}(2)$ divides the value k. Therefore by Lemma 37, it follows that $\mathcal{G}_{G_0,G_1}(k) = \operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } k\} = 1$ as desired. We conclude that if $k \equiv 1,5,7,11 \pmod{12}$, then we have $\mathcal{F}(k) = \mathcal{L}(k) = 1$. Also if $D_{G_0,G_1} = \pm 1$, then $\mathcal{G}_{G_0,G_1}(k) = 1$. □

6 Interesting applications

The Pisano period characterization of the value $\mathcal{G}_{G_0,G_1}(k)$ not only yields the GCD of all sums of k consecutive Gibonacci numbers, but also leads to some interesting applications. In this section, we highlight three such applications.

6.1 Restrictions on the factors of $\mathcal{G}_{G_0,G_1}(k)$ when k is odd

While the Fibonacci and Lucas sequences satisfy the property that both $\pi_F(m)$ and $\pi_L(m)$ are even for all $m > 2$, this is not the case in general. From this, we may exhibit odd values of k for which $\mathcal{G}_{G_0,G_1}(k)$ is greater than two. We provide an example of this below and we place restrictions on the values of m that can make $\pi_{G_0,G_1}(m)$ odd later in this subsection. Using the generalized Pisano period characterization of $\mathcal{G}_{G_0,G_1}(k)$, we place restrictions on the factors of $\mathcal{G}_{G_0,G_1}(k)$.

Example 40. In this example, we will show that $\pi_{1,4}(m) = 5$ only when $m = 11$. Recall that by Theorems 38 and 39, the value $\mathcal{G}_{G_0,G_1}(k)$ is 1 or 2 if k is odd and $D_{G_0,G_1} = \pm 1$, where D_{G_0,G_1} was defined to be $G_1^2 - G_0G_1 - G_0^2$. Hence if $D_{G_0,G_1} \neq \pm 1$, it is interesting to consider what are the possible values of $\mathcal{G}_{G_0,G_1}(k)$ when k is odd. We shall consider the Gibonacci sequence with initial values $G_0 = 1$ and $G_1 = 4$, which we will call the $(1,4)$-Gibonacci sequence. Observe that in this case, we have $G_1^2 - G_0G_1 - G_0^2 = 11 \neq \pm 1$, and hence the value $\mathcal{G}_{1,4}(k)$ when k is odd is not forced to be 1 or 2, necessarily. We consider the value $\mathcal{G}_{1,4}(k)$ when $k = 5$ by computing it in two different ways. First, we write out the $(1,4)$-Gibonacci sequence as follows:

$$1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 245, 402, 647, 849, 1496, \ldots$$

Consider the first four terms of the sequence $\left(\sum_{i=0}^{k-1} G_{n+i}\right)_{n\geq 1}$ when $k = 5$:

$(4+5+9+14+23,\ 5+9+14+23+37,\ 9+14+23+37+60, 14+23+37+60+97, \ldots)$,

or equivalently, $(55, 88, 143, 231, \ldots)$. By inspection, one may suspect that $\mathcal{G}_{1,4}(5)$ is 11. This can be affirmed by our simple GCD characterization as follows:

$$\mathcal{G}_{1,4}(5) = \gcd(G_7 - G_2, G_6 - G_1) = \gcd(60-5, 37-4) = \gcd(55, 33) = 11.$$

On the other hand, by using the generalized Pisano characterization of $\mathcal{G}_{G_0,G_1}(k)$, we know

$$\mathcal{G}_{1,4}(5) = 11 = \operatorname{lcm}\{m \mid \pi_{1,4}(m) \text{ divides } 5\}.$$

By Theorem 24, since 11 divides $\mathcal{G}_{1,4}(5)$, it must be that $\pi_{1,4}(11)$ divides 5. Clearly $\pi_{1,4}(11) \neq 1$. Hence $\pi_{1,4}(11) = 5$ is forced. Furthermore, since 11 is prime, its only divisors are 1 and 11. Hence the only divisors of $\operatorname{lcm}\{m \mid \pi_{1,4}(m) \text{ divides } 5\}$ can be 1 or 11, and again employing Theorem 24, this implies that $\pi_{1,4}(m)$ does not divide 5 for all $m \neq 1, 11$. Hence, we can conclude that for the particular Gibonacci sequence with initial values $G_0 = 1$ and $G_1 = 4$, we know that the only modulus value m that yields $\pi_{G_0,G_1}(m) = 5$ is the value $m = 11$.

Remark 41. Observe that in the previous example the values $m = 11$ and $D_{1,4} = 11$ coincide. When examining the $(1, 24)$-Gibonacci sequence, which yields an odd period for $m = 29$, we do not have $m = D_{1,24}$. However, the value $m = 29$ divides $D_{1,24} = 551$.

Proposition 42 (Wall, Theorem 8)**.** *If p is prime and $p \equiv 3, 7, 13, 17 \pmod{20}$, then it follows that $\pi_{G_0,G_1}(p^e) = \pi_F(p^e)$.*

Since $\pi_F(m)$ is even for all $m > 2$, Proposition 42 yields the immediate corollary.

Corollary 43. *If p is prime and $p \equiv 3, 7, 13, 17 \pmod{20}$, then $\pi_{G_0,G_1}(p)$ is even no matter the choice of (coprime) initial conditions.*

Theorem 44. *There exists no prime $p \equiv 3, 7, 13, 17 \pmod{20}$ that can be a factor of $\mathcal{G}_{G_0,G_1}(k)$ if k is odd.*

Proof. Recall that by Theorem 24, the value $\pi_{G_0,G_1}(m)$ divides k if and only if m divides $\mathcal{G}_{G_0,G_1}(k)$. So if p is a prime such that $p \equiv 3, 7, 13, 17 \pmod{20}$, then by Corollary 43, we know that $\pi_{G_0,G_1}(p)$ is even. However, if k is odd then surely $\pi_{G_0,G_1}(p)$ cannot divide k. Hence no prime of the form $p \equiv 3, 7, 13, 17 \pmod{20}$ can be a factor of $\mathcal{G}_{G_0,G_1}(k)$ if k is odd. □

6.2 Largest modulus m yielding a given Pisano period value $\pi_F(m)$

It is well known that for a given modulus m, the corresponding Pisano period $\pi_F(m)$ is bounded above by $6m$. This problem was proposed by Freyd in 1990 and answered by Brown in 1992 [2]. Moreover, this upper bound is achieved, for instance, when $m = 10$ since

$\pi_F(10) = 60$. Furthermore, since the value $\pi_{G_0,G_1}(m)$ divides $\pi_F(m)$, the value $6m$ also serves as an upper bound on any Gibonacci sequence. Hence, this question of an upper bound for any generalized Pisano period of a given modulus m is answered. However, a different but related question can be considered.

Question 45. For a given period k and a Gibonacci sequence with initial values G_0 and G_1, what is the largest modulus value m such that $\pi_{G_0,G_1}(m) = k$? We answer this question in the Fibonacci setting.[2] We invite the interested reader to explore this problem in the Lucas and Gibonacci setting.

The following example exhibits how this question may be approached in the Fibonacci and Lucas settings, in particular, from the generalized Pisano period characterization of $\mathcal{G}_{G_0,G_1}(k)$.

Example 46. Let us attempt to compute the largest modulus value m that yields a Pisano period $\pi_F(m)$ equal to 60. Setting $k := 60$ in Theorem 32, we know $\mathcal{F}(60) = F_{30} = 832\,040$. By Theorem 24, we know the following:

$$\pi_F(m) \text{ divides } 60 \text{ if and only if } m \text{ divides } \mathcal{F}(60).$$

Hence we can conclude that if $\pi_F(m) = 60$, then m divides $832\,040$. So certainly, we have $832\,040$ as a potential maximum value m that makes $\pi_F(m) = 60$, but the question that remains is "Does $\pi_F(832\,040)$ indeed equal 60?" A simple computer computation reveals that this is so. Hence the largest modulus value m that yields a Pisano period $\pi_F(m)$ equal to 60 is $m = 832\,040$. In Theorem 48, we will prove that in general for $k \equiv 0 \pmod 4$ that $\mathcal{F}(k)$ (or equivalently $F_{k/2}$) is the actual largest modulus that produces a Pisano period equal to k.

Before we prove our main result, Theorem 48, we need the following known results on the periods $\pi_F(m)$ when m is a Fibonacci or Lucas number.

Lemma 47. *The following identities hold:*

$$\pi_F(F_i) = \begin{cases} 2i, & \text{if } i \geq 4 \text{ and even;} \\ 4i, & \text{if } i \geq 5 \text{ and odd.} \end{cases} \tag{16}$$

$$\pi_F(L_i) = \begin{cases} 4i, & \text{if } i \geq 2 \text{ and even;} \\ 2i, & \text{if } i \geq 3 \text{ and odd.} \end{cases} \tag{17}$$

In particular, it follows that $\text{range}(\pi_F) = \{3\} \cup \{n \in 2\mathbb{Z} \mid n \geq 6\}$.

[2] This question was explored in 2018 in the Fibonacci setting by Dishong and Renault from an algorithmic approach that allows a computer to calculate all values m such that $\pi_F(m) = k$ [5]. However, we answer this question from a theoretical approach utilizing the generalized Pisano period characterization of $\mathcal{G}_{G_0,G_1}(k)$.

Proof. Identity (16) was first proven in 1971 by Stanley [16] (and independently in 1989 by Ehrlich [7], who was apparently unaware of Stanley's result). Identity (17) was proven in 1976 by Stanley [17]. Moreover, in that same paper Stanley states that the range of π_F is all even integers greater than 4 though omits the trivial result that $\pi_F(2) = 3$ and hence we have $\text{range}(\pi_F) = \{3\} \cup \{n \in 2\mathbb{Z} \mid n \geq 6\}$, as desired. □

Theorem 48. *Let $k \geq 6$ be an even integer, and set $m_F := \mathcal{F}(k)$. Then m_F is the largest modulus value yielding a Fibonacci period of k. More precisely, $\pi_F(m_F) = k$ and for all $m > m_F$, we have $\pi_F(m) \neq k$. In particular, we have the following:*

$$m_F = \begin{cases} F_{k/2}, & \text{if } k \equiv 0 \pmod{4}, \\ L_{k/2}, & \text{if } k \equiv 2 \pmod{4}. \end{cases}$$

Proof. Suppose $k \geq 6$ is an even integer. Then either $k \equiv 0 \pmod{4}$ or $k \equiv 2 \pmod{4}$.

Case 1: Suppose $k \equiv 0 \pmod{4}$. Set $m_F := \mathcal{F}(k)$. Then by Theorem 32, we have $m_F = F_{k/2}$. Since $k \equiv 0 \pmod{4}$ and $k \geq 6$ is even, then $k/2 \geq 4$ is even. Thus, Identity (16) of Lemma 47 implies that $\pi_F(F_{k/2}) = k$. Hence we have $\pi_F(m_F) = k$. We now show that there are no larger values $m > m_F$ yielding $\pi_F(m) = k$. Recall that Theorem 24 implies

$$\pi_F(m) \text{ divides } k \text{ if and only if } m \text{ divides } \mathcal{F}(k).$$

Hence $\mathcal{F}(k)$ is the maximum potential modulus value m that could make $\pi_F(m) = k$. Since we have $\pi_F(m_F) = k$ and $m_F = \mathcal{F}(k)$, then we have achieved the maximum modulus, namely $F_{k/2}$, yielding a period of k when $k \equiv 0 \pmod{4}$.

Case 2: Suppose $k \equiv 2 \pmod{4}$. Set $m_F := \mathcal{F}(k)$. Then by Theorem 33, we have $m_F = L_{k/2}$. Since $k \equiv 2 \pmod{4}$ and $k \geq 6$ is even, then $k/2 \geq 3$ is odd. Thus, Identity (17) of Lemma 47 implies that $\pi_F(L_{k/2}) = k$. Hence we have $\pi_F(m_F) = k$. By the exact same argument given in Case 1, we know that $\mathcal{F}(k)$ is the maximum potential modulus value m that could make $\pi_F(m) = k$. So we have achieved the maximum modulus, namely $L_{k/2}$, yielding a period of k when $k \equiv 2 \pmod{4}$. □

6.3 Computing odd-indexed Lucas numbers using $\mathcal{G}_{G_0,G_1}(k)$ characterizations

The fact that the formulas $\gcd(G_{k+1} - G_1, G_{k+2} - G_2)$ and $\text{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } k\}$ for $\mathcal{G}_{G_0,G_1}(k)$ coincide leads to some surprising and delightful new understandings of the Fibonacci and Lucas numbers. One such example can be garnered from looking at the the $k \equiv 2, 6, 10 \pmod{12}$ row in Table 1. Given such a k-value, the Lucas number $L_{k/2}$ can be computed in two new ways. One is by taking a Gibonacci sequence $(G_n)_{n\geq 0}$ with any initial relatively prime initial values G_0 and G_1, then by the first $\mathcal{G}_{G_0,G_1}(k)$ characterization, we have $L_{k/2} = \gcd(G_{k+1} - G_1, G_{k+2} - G_2)$. On the other hand, by the second $\mathcal{G}_{G_0,G_1}(k)$ characterization, we have $L_{k/2} = \text{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } k\}$. In this subsection, we consider the first of these two ways. We have effectively established an easily computable way to find any odd-index Lucas number using any Gibonacci sequence with relatively prime initial values.

Theorem 49. *Let j be an odd positive integer, and suppose that $(G_n)_{n\geq 0}$ is a Gibonacci sequence with relatively prime initial values G_0 and G_1. Then the j^{th} Lucas number L_j is given by $\mathcal{G}_{G_0,G_1}(2j)$. More precisely, we have*

$$L_j = \gcd\left(G_{2j+1} - G_1,\ G_{2j+2} - G_2\right).$$

Proof. This follows from Theorem 33 if we set $k := 2j$ and observe that $k \equiv 2 \pmod 4$ since j is odd. □

The latter theorem is quite surprising. It leads one to ponder if such a GCD-formulation can be discovered which yields the even-index Lucas numbers. But even more intriguing is the fact that we have our second characterization of the $\mathcal{G}_{G_0,G_1}(k)$ formula. More precisely, the odd index Lucas number L_j is given by $\operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } 2j\}$. Admittedly, computing the Lucas number L_j using this LCM formulation is not as easily done as it is using Theorem 49, due to the fact that the periods $\pi_{G_0,G_1}(m)$ are not easily computed. In the open questions section, we ask a question regarding this formulation.

7 Open questions

There are many avenues for further research motivated from the work in this present paper. The following open problems arose from the consideration of our $\mathcal{G}_{G_0,G_1}(k)$ characterizations and other questions related to our research.

Question 50. By examining our two equivalent definitions of $\mathcal{G}_{G_0,G_1}(k)$, we observe that when $k \equiv 2 \pmod 4$ it follows that $L_{k/2} = \operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } k\}$ for every possible choice of G_0 and G_1. Is there an intuitive reason why this must be true?

Question 51. Theorem 44 establishes that no prime $p \equiv 3, 7, 13, 17 \pmod{20}$ can be a factor of $\mathcal{G}_{G_0,G_1}(k)$ if k is odd. Which primes of the form $p \equiv 1, 9, 11, 19 \pmod{20}$ can be factors of $\mathcal{G}_{G_0,G_1}(k)$ when k is odd? Can we place further restrictions on the possible factors of $\mathcal{G}_{G_0,G_1}(k)$ when k is odd?

Question 52. Observe that for the Fibonacci and Lucas sequences we have that $\pi_F(m)$ and $\pi_L(m)$ are even for all $m > 2$. For which initial values G_0 and G_1, does there exist a number N such that $\pi_{G_0,G_1}(m)$ is even for all $m > N$?

Question 53. To prove Theorem 48 for the maximum modulus value m that yields a given period k in the Fibonacci and Lucas settings, respectively, we relied heavily on Lemma 47 which gave the Fibonacci and Lucas Pisano periods for moduli of the form F_i and L_i. Can we generalize this lemma to provide conditions on initial values G_0 and G_1 that can help us predict the precise value of $\pi_{G_0,G_1}(G_i)$ for each i; that is, the generalized Pisano period of the sequence $(G_n)_{n\geq 0}$ modulo the Gibonacci number G_i?

Question 54. Can we extend our work to sums of k consecutive squares of Gibonacci numbers? That is, for a fixed $k \in \mathbb{N}$ and initial values $G_0, G_1 \in \mathbb{Z}$, what is the value of $\mathcal{G}^2_{G_0,G_1}(k)$, which we define to be $\gcd\left\{\left(\sum_{i=0}^{k-1} G^2_{n+i}\right)_{n\geq 1}\right\}$? In the Fibonacci setting, small computational data leads to the following conjectural values of $\mathcal{F}^2(k)$, which are the values $\mathcal{G}^2_{G_0,G_1}(k)$ when $G_0 = 0$ and $G_1 = 1$:

k	$\mathcal{F}^2(k)$
0	$0 = F_0L_0$
4	$1 \cdot 3 = F_2L_2$
8	$3 \cdot 7 = F_4L_4$
12	$2^4 \cdot 3^2 = F_6L_6$
16	$3 \cdot 7 \cdot 47 = F_8L_8$
20	$3 \cdot 5 \cdot 11 \cdot 41 = F_{10}L_{10}$

k	$\mathcal{F}^2(k)$
1	1
5	1
9	2
13	1
17	1
21	2

k	$\mathcal{F}^2(k)$
2	$1 = F_1L_1$
6	$2^3 = F_3L_3$
10	$5 \cdot 11 = F_5L_5$
14	$13 \cdot 29 = F_7L_7$
18	$2^3 \cdot 17 \cdot 19 = F_9L_9$
22	$89 \cdot 199 = F_{11}L_{11}$

k	$\mathcal{F}^2(k)$
3	2
7	1
11	1
15	2
19	1
23	1

The four tables above partition the possible k-values into residue classes modulo 4. Observe that in the third table, namely when $k \equiv 2 \pmod 4$, we highlight in red the fact that $\mathcal{F}^2(k)$ values factor into two distinct primes, namely $F_{k/2}$ and $L_{k/2}$. However it is well known that $F_nL_n = F_{2n}$. Hence, for these aforementioned k-values, we conjecture that $\mathcal{F}^2(k) = F_k$. Looking closer at the conjectural $\mathcal{F}^2(k)$ value when $k = 18$, observe that $F_9 = 2 \cdot 17$ and $L_9 = 2^2 \cdot 19$, and their product is indeed the conjectured $\mathcal{F}^2(k)$ value $2^3 \cdot 17 \cdot 19$. This occurs also for all the values in the table above for $k \equiv 0 \pmod 4$, so the phenomena of $\mathcal{F}^2(k) = F_k$ does seems to hold for all even k values.

Further computational evidence does support the conjecture that $\mathcal{F}^2(k) = F_k$ for all even k values. We feel this is simply too beautiful a result to not be true. Of course, the ultimate goal would be to prove this result and extend it to the Lucas setting to find $\mathcal{L}^2(k)$ and more generally $\mathcal{G}^2_{G_0,G_1}(k)$ for any Gibonacci sequence.

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