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On Noncentral Tanny-Dowling Polynomials and Generalizations of Some Formulas for Geometric Polynomials

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Abstract

In this paper, we establish some formulas for the noncentral Tanny-Dowling polynomials, such as sums of products and explicit formulas. Some special cases are also presented and discussed.

1 Introduction

The geometric polynomials [19], denoted by $w_n(x)$, are defined by

$$w_n(x) = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} x^k, \tag{1}$$

where $\left\{ {n \atop k} \right\}$ are the celebrated Stirling numbers of the second kind [7, 18]. These polynomials are known to satisfy the exponential generating function

$$\sum_{n=0}^{\infty} w_n(x) \frac{z^n}{n!} = \frac{1}{1 - x(e^z - 1)} \tag{2}$$

and the recurrence relation [9, Proposition 7]

$$w_{n+1}(x) = x\frac{d}{dx}\left(w_n(x) + xw_n(x)\right). \tag{3}$$

The case when $x = 1$ yields

$$w_n := w_n(1) = \sum_{k=0}^{n} k!\left\{ {n \atop k} \right\}, \tag{4}$$

the geometric numbers (or ordered Bell numbers) whose values form the sequence A000670. Recall that the numbers $\left\{ {n \atop k} \right\}$ count the number of partitions of a set X with n elements into k non-empty subsets. These numbers can also be interpreted as the number of ways to distribute n distinct objects into k identical boxes such that no box is empty. On the other hand, the numbers $k!\left\{ {n \atop k} \right\}$ can be combinatorially interpreted as the number of distinct ordered partitions of X with k blocks, or the numbers of ways to distribute n distinct objects into k distinct boxes. It follows immediately that the geometric numbers count the number of distinct ordered partitions of the n-set X.

The study of geometric polynomials and numbers has a long history. Aside from the work of Tanny [19], one may also see the papers written by Boyadzhiev [4], Dil and Kurt [9], Boyadzhiev and Dil [5], Kargın and Corcino [12], and the references therein for further readings. Benoumhani [3] studied two equivalent generalizations of $w_n(x)$ given by

$$F_{m,1}(n;x) = \sum_{k=0}^{n} m^k k! W_m(n,k) x^k \tag{5}$$

and

$$F_{m,2}(n;x) = \sum_{k=0}^{n} k! W_m(n,k) x^k, \tag{6}$$

where $W_m(n,k)$ denote the Whitney numbers of the second kind of Dowling lattices [2]. These are called Tanny-Dowling polynomials and are known to satisfy the following exponential generating functions:

$$\sum_{n=0}^{\infty} F_{m,1}(n;x)\frac{z^n}{n!} = \frac{e^z}{1 - x(e^{mz} - 1)}, \tag{7}$$

$$\sum_{n=0}^{\infty} F_{m,2}(n;x)\frac{z^n}{n!} = \frac{e^z}{1 - \frac{x}{m}(e^{mz} - 1)}. \tag{8}$$

More properties can be seen in [2, 3]. In a recent paper, Kargın [10] established a number of explicit formulas and formulas involving products of geometric polynomials, viz.

$$(x+1)\sum_{k=0}^{n}\binom{n}{k} w_k(x) w_{n-k}(x) = w_{n+1}(x) + w_n(x), \tag{9}$$

$$\sum_{k=0}^{n}\binom{n}{k}w_k(x_1)w_{n-k}(x_2)=\frac{x_2w_n(x_2)-x_1w_n(x_1)}{x_2-x_1}, \tag{10}$$

$$w_n(x)=x\sum_{k=1}^{n}\left\{ {n \atop k} \right\}(-1)^{n+k}k!(x+1)^{k-1}, \tag{11}$$

and

$$w_n(x)=\sum_{k=0}^{n}\left\{ {n \atop k} \right\}k!x^k\frac{2^{n+1}(x+1)x^k+(-1)^{k+1}}{(2x+1)^{k+1}}. \tag{12}$$

These results were obtained by Kargın [10] with the aid of the two-variable geometric polynomials $w_k(r;x)$ defined by

$$\sum_{n=0}^{\infty}w_n(r;x)\frac{z^n}{n!}=\frac{e^{rz}}{1-x(e^z-1)}. \tag{13}$$

A natural generalization of $F_{m,1}(n;x)$ and $F_{m,2}(n;x)$ are the noncentral Tanny-Dolwing polynomials introduced by Mangontarum et al. [15] defined as

$$\widetilde{\mathcal{F}}_{m,a}(n;x)=\sum_{k=0}^{n}k!\widetilde{W}_{m,a}(n,k)x^k, \tag{14}$$

where $\widetilde{W}_{m,a}(n,k)$ are the noncentral Whitney numbers of the second kind. The polynomials $\widetilde{\mathcal{F}}_{m,a}(n;x)$ satisfy the exponential generating function given by [15, Theorem 18]

$$\sum_{n=k}^{\infty}\widetilde{\mathcal{F}}_{m,a}(n;x)\frac{z^n}{n!}=\frac{me^{-az}}{m-x(e^{mz}-1)}. \tag{15}$$

Looking at (15), it is readily observed that

$$\widetilde{\mathcal{F}}_{m,0}(n;x)=m^nw_n\left(\frac{x}{m}\right),$$

$$\widetilde{\mathcal{F}}_{m,-1}(n;x)=F_{m,2}(n;x),$$

and

$$\widetilde{\mathcal{F}}_{1,-r}(n;x)=w_n(r,x).$$

The numbers $\widetilde{W}_{m,a}(n,k)$ admit a variety of combinatorial properties which can be seen in [15]. One of these properties is the triangular recurrence relation [15, Proposition 8]

$$\widetilde{W}_{m,a}(n+1,k)=\widetilde{W}_{m,a}(n,k-1)+(mk-a)\widetilde{W}_{m,a}(n,k) \tag{16}$$

with $\widetilde{W}_{m,a}(n,0) = (-a)^n$ and $\widetilde{W}_{m,a}(n,k) = 1$ when $k = n$. Using this recurrence relation, the following noncentral Tanny-Dowling polynomials can be derived for $n = 0, 1, 2, 3, 4$:

$$\begin{aligned}
\widetilde{\mathcal{F}}_{m,a}(0;x) &= 1 \\
\widetilde{\mathcal{F}}_{m,a}(1;x) &= x - a \\
\widetilde{\mathcal{F}}_{m,a}(2;x) &= 2x^2 + mx + a^2 \\
\widetilde{\mathcal{F}}_{m,a}(3;x) &= 6x^3 + 2(3m - a)x^2 + (m^2 - ma_a^2)x - a^3 \\
\widetilde{\mathcal{F}}_{m,a}(4;x) &= 24x^4 + 6(6m - 2a)x^3 + 2(7m^2 - 6ma + 2a^2)x^2 \\
&\quad + (m^3 - 2m^2a + 2ma^2 - 2a^3)x + a^4.
\end{aligned}$$

These noncentral Whitney numbers of the second kind appear to be a common generalization of $\left\{ {n \atop k} \right\}$ and $W_m(n,k)$, as well as other notable numbers reported by the respective authors in [1, 6, 13, 14, 16]. It is important to note that the noncentral Whitney numbers of the second kind are equivalent to the (r, β)-Stirling numbers by Corcino [8] and the r-Whitney numbers of the second kind by Mező [17]. On the other hand, the higher order generalized geometric polynomials, an even more generalized polynomial, were introduced in the paper of Kargın and Corcino [11]. However, the said polynomials and the noncentral Tanny-Dolwing polynomials were defined using different motivations. Moreover, the results obtained in this paper do not appear as particular cases of the ones seen in [11].

In the present paper, we establish some formulas for the noncentral Tanny-Dowling polynomials such as sums of products and explicit formulas. These formulas are shown to generalize the above-mentioned identities obtained by Kargın [10] for the geometric polynomials when the parameters are assigned with specific values. We also discuss some other identities resulting from the said formulas.

2 Formulas for sum of products

Now the exponential generating function in (15) can be rewritten as

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{F}}_{m,a}(n;x) \frac{z^n}{n!} = \frac{1}{1 - \frac{x}{m}(e^{mz} - 1)} \cdot e^{-az}.$$

Hence, by applying (2) and using Cauchy's product for two series, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \widetilde{\mathcal{F}}_{m,a}(n;x) \frac{z^n}{n!} &= \sum_{n=0}^{\infty} m^n w_n \left(\frac{x}{m}\right) \frac{z^n}{n!} \sum_{n=0}^{\infty} (-a)^n \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} w_k \left(\frac{x}{m}\right) m^k (-a)^{n-k} \right) \frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ yields the result in the next theorem.

Theorem 1. *The noncentral Tanny-Dowling polynomials $\widetilde{\mathcal{F}}_{m,a}(n;x)$ satisfy the following identity:*

$$\widetilde{\mathcal{F}}_{m,a}(n;x) = \sum_{k=0}^{n} \binom{n}{k} m^k w_k\left(\frac{x}{m}\right)(-a)^{n-k}. \tag{17}$$

Alternative proof of Theorem 1. From [15, Theorem 10], the noncentral Whitney numbers of the second kind satisfy the following formula in terms of the Stirling numbers of the second kind:

$$\widetilde{W}_{m,a}(n,k) = \sum_{j=0}^{n} \binom{n}{j} (-a)^{n-j} m^{j-k} \left\{ {j \atop k} \right\}.$$

Multiplying both sides by $k!x^k$ and summing over k gives the desired result. □

Before proceeding, we see that when $m = 1$ and $a = -r$, (17) becomes

$$\widetilde{\mathcal{F}}_{1,-r}(n;x) = \sum_{k=0}^{n} \binom{n}{k} w_k(x) r^{n-k} := w_n(r;x),$$

which is precisely an identity obtained by Kargın [10, Equation (13)].

By applying the exponential generating function in (15),

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\widetilde{\mathcal{F}}_{m,a-m}(n;x) - \widetilde{\mathcal{F}}_{m,a}(n;x)\right)\frac{z^n}{n!} &= \frac{me^{-(a-m)z}}{m - x(e^{mz}-1)} - \frac{me^{-az}}{m - x(e^{mz}-1)} \\
&= \frac{m}{x}\left(\frac{me^{-az}}{m - x(e^{mz}-1)} - e^{-az}\right) \\
&= \frac{m}{x}\sum_{n=0}^{\infty} \widetilde{\mathcal{F}}_{m,a}(n;x)\frac{z^n}{n!} - \sum_{n=0}^{\infty} (-a)^n \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{m}{x}\left(\widetilde{\mathcal{F}}_{m,a}(n;x) - (-a)^n\right)\frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ gives

$$\widetilde{\mathcal{F}}_{m,a-m}(n;x) - \widetilde{\mathcal{F}}_{m,a}(n;x) = \frac{m}{x}\left(\widetilde{\mathcal{F}}_{m,a}(n;x) - (-a)^n\right).$$

The result in the next theorem follows by solving for $x\widetilde{\mathcal{F}}_{m,a-m}(n;x)$.

Theorem 2. *The noncentral Tanny-Dowling polynomials $\widetilde{\mathcal{F}}_{m,a}(n;x)$ satisfy the following recurrence relation:*

$$x\widetilde{\mathcal{F}}_{m,a-m}(n;x) = (m+x)\widetilde{\mathcal{F}}_{m,a}(n;x) - (-a)^n m. \tag{18}$$

Setting $m=1$ and $a=-r$ in (18) gives

$$x\widetilde{\mathcal{F}}_{1,-r-1}(n;x) = (1+x)\widetilde{\mathcal{F}}_{1,-r}(n;x) - r^n$$

which is exactly the following identity [10, Equation (14)]:

$$xw_n(r+1;x) = (1+x)w_n(r;x) - r^n.$$

On the other hand, when $a=0$ and $a=m$ in (18), we get

$$x\widetilde{\mathcal{F}}_{m,-m}(n;x) = (m+x)m^n w_n\left(\frac{x}{m}\right) \tag{19}$$

and

$$(m+x)\widetilde{\mathcal{F}}_{m,m}(n;x) = xm^n w_n\left(\frac{x}{m}\right) - (-m)^{n+1}, \tag{20}$$

respectively. Substituting (17) to the right hand sides of these equations yields

$$x\sum_{k=0}^{n}\binom{n}{k}w_k\left(\frac{x}{m}\right) = (m+x)w_n\left(\frac{x}{m}\right) \tag{21}$$

and

$$(m+x)\sum_{k=0}^{n}\binom{n}{k}w_k\left(\frac{x}{m}\right)(-1)^{n-k} = xw_n\left(\frac{x}{m}\right) - m(-1)^{n+1}. \tag{22}$$

These identities are generalizations of the results obtained by Dil and Kurt [9, Proposition 3] using the Euler-Seidel matrix method and by Kargın [10, Equations (8) and (9)]. That is, setting $x=1$ and $m=1$ gives

$$\sum_{k=0}^{n}\binom{n}{k}w_k = 2w_n$$

and

$$2\sum_{k=0}^{n}\binom{n}{k}(-1)^k w_k = (-1)^n w_n + 1.$$

The next theorem contains a formula for the sum of product of noncentral Tanny-Dowling polynomials for different values of a.

Theorem 3. *The noncentral Tanny-Dowling polynomials satisfy the following relation:*

$$x\sum_{k=0}^{n}\binom{n}{k}\widetilde{\mathcal{F}}_{m,a_1}(k;x)\widetilde{\mathcal{F}}_{m,a_2}(n-k;x) = \widetilde{\mathcal{F}}_{m,\bar{A}}(n+1;x) + \bar{A}\widetilde{\mathcal{F}}_{m,\bar{A}}(n;x), \tag{23}$$

where $\bar{A} = a_1 + a_2 + m$ for real numbers a_1 and a_2.

Proof. We start by taking the derivative of (15) with respect to z. That is,

$$\frac{\partial}{\partial z}\left(\frac{me^{-az}}{m-x(e^{mz}-1)}\right)=\frac{me^{-az}}{m-x(e^{mz}-1)}\cdot\frac{xme^{mz}}{m-x(e^{mz}-1)}-\frac{ame^{-az}}{m-x(e^{mz}-1)}.$$

Replacing a with $\bar{A}=a_1+a_2+m$ yields

$$\frac{\partial}{\partial z}\left(\frac{me^{-\bar{A}z}}{m-x(e^{mz}-1)}\right)=\sum_{n=k}^{\infty}\widetilde{\mathcal{F}}_{m,\bar{A}}(n+1;x)\frac{z^n}{n!}$$

in the left-hand side while we get

$$\begin{aligned}\frac{me^{-\bar{A}z}}{m-x(e^{mz}-1)}\cdot\frac{xme^{mz}}{m-x(e^{mz}-1)}&=\frac{me^{-a_1z}}{m-x(e^{mz}-1)}\cdot\frac{me^{-a_2z}}{m-x(e^{mz}-1)}\\&=x\sum_{n=k}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\widetilde{\mathcal{F}}_{m,a_1}(k;x)\widetilde{\mathcal{F}}_{m,a_2}(n-k;x)\frac{z^n}{n!}\end{aligned}$$

and

$$\frac{\bar{A}me^{-\bar{A}z}}{m-x(e^{mz}-1)}=\bar{A}\cdot\sum_{n=k}^{\infty}\widetilde{\mathcal{F}}_{m,\bar{A}}(n;x)\frac{z^n}{n!}$$

in the right-hand side. Combining the above equations and comparing the coefficients of $\frac{z^n}{n!}$ gives the desired result. □

When $a_1=a_2=0$ in (23),

$$xm^n\sum_{k=0}^{n}\binom{n}{k}w_k\left(\frac{x}{m}\right)w_{n-k}\left(\frac{x}{m}\right)=\widetilde{\mathcal{F}}_{m,m}(n+1;x)+m\widetilde{\mathcal{F}}_{m,m}(n;x).$$

Applying (20) to the right-hand side of this equation gives

$$xm^n\sum_{k=0}^{n}\binom{n}{k}w_k\left(\frac{x}{m}\right)w_{n-k}\left(\frac{x}{m}\right)=\frac{xm^{n+1}w_{n+1}\left(\frac{x}{m}\right)-(-m)^{n+2}}{m+x}+m\frac{xm^nw_n\left(\frac{x}{m}\right)-(-m)^{n+1}}{m+x}$$

which can be simplified into the following identity:

$$\frac{m+x}{m}\sum_{k=0}^{n}\binom{n}{k}w_k\left(\frac{x}{m}\right)w_{n-k}\left(\frac{x}{m}\right)=w_{n+1}\left(\frac{x}{m}\right)+w_n\left(\frac{x}{m}\right). \tag{24}$$

Obviously, this identity boils down to the result obtained by Kargın [10] in (9) when $m=1$.

Theorem 4. *For $x_1\neq x_2$, the following formula holds:*

$$\sum_{k=0}^{n}\binom{n}{k}\widetilde{\mathcal{F}}_{m,a_1}(k;x_1)\widetilde{\mathcal{F}}_{m,a_2}(n-k;x_2)=\frac{x_2\widetilde{\mathcal{F}}_{m,a_1+a_2}(n;x_2)-x_1\widetilde{\mathcal{F}}_{m,a_1+a_2}(n;x_1)}{x_2-x_1}. \tag{25}$$

Proof. Note that we can write

$$\frac{me^{-a_1 z}}{m-x_1(e^{mz}-1)}\cdot\frac{me^{-a_2 z}}{m-x_2(e^{mz}-1)}=\frac{1}{x_2-x_1}\left(\frac{x_2 me^{-(a_1+a_2)z}}{m-x_2(e^{mz}-1)}-\frac{x_1 me^{-(a_1+a_1)z}}{m-x_1(e^{mz}-1)}\right).$$

Following the same method used in the previous theorem leads us to the desired result. □

This theorem contains a formula for the sums of products of noncentral Tanny-Dowling polynomials for different values of x. When $a_1=a_2=0$, (25) reduces to

$$\sum_{k=0}^{n}\binom{n}{k}w_k\left(\frac{x_1}{m}\right)w_{n-k}\left(\frac{x_2}{m}\right)=\frac{x_2 w_n\left(\frac{x_2}{m}\right)-x_1 w_n\left(\frac{x_1}{m}\right)}{x_2-x_1}. \tag{26}$$

It is clear to see that when $m=1$, we recover the sum of products of geometric polynomials in (10).

3 Explicit formulas

In Theorem 1, we obtained an explicit formula that expresses the noncentral Tanny-Dowling polynomials in terms of the geometric polynomials. Now, with $g_n=\frac{1}{a^n}\widetilde{\mathcal{F}}_{m,a}(n;x)$ and $f_j=\left(\frac{m}{a}\right)^j w_j\left(\frac{x}{m}\right)$, the binomial inversion formula

$$f_n=\sum_{j=0}^{n}\binom{n}{j}g_j \Longleftrightarrow g_n=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f_j \tag{27}$$

allows us to express the geometric polynomials $w_n\left(\frac{x}{m}\right)$ in terms of the noncentral Tanny-Dowling polynomials as follows:

$$w_n\left(\frac{x}{m}\right)=\frac{1}{m^n}\sum_{j=0}^{n}\binom{n}{j}a^{n-j}\widetilde{\mathcal{F}}_{m,a}(j;x). \tag{28}$$

In this section, we will derive more explicit formulas for both polynomials.

Using $x-m$ in place of x in (15) gives

$$\begin{aligned}\sum_{n=k}^{\infty}\widetilde{\mathcal{F}}_{m,a}(n;x-m)\frac{z^n}{n!}&=\frac{me^{-(-a-m)(-z)}}{m+x(e^{-mz}-1)}\\&=\sum_{n=k}^{\infty}\widetilde{\mathcal{F}}_{m,-a-m}(n;-x)\frac{(-z)^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\frac{z^n}{n!}$, we get

$$\widetilde{\mathcal{F}}_{m,a}(n;x-m)=(-1)^n\widetilde{\mathcal{F}}_{m,-a-m}(n;-x). \tag{29}$$

Applying (18) to the right-hand side gives

$$\widetilde{\mathcal{F}}_{m,a}(n;x-m)=(-1)^n\left(\frac{(m-x)\widetilde{\mathcal{F}}_{m,-a}(n;-x)-a^n m}{-x}\right).$$

Replacing $-x$ and $-a$ with x and a, respectively, and solving for $\widetilde{\mathcal{F}}_{m,a}(n;x)$ yields

$$\widetilde{\mathcal{F}}_{m,a}(n;x)=\frac{(-1)^n x\widetilde{\mathcal{F}}_{m,-a}(n;-x-m)+(-a)^n m}{m+x}.$$

By (14), we get the next theorem.

Theorem 5. *The noncentral Tanny-Dowling polynomials satisfy the following explicit formula:*

$$\widetilde{\mathcal{F}}_{m,a}(n;x)=x\sum_{k=0}^{n}(-1)^{n+k}k!\widetilde{W}_{m,-a}(n,k)(m+x)^{k-1}+\frac{(-a)^n m}{m+x}. \tag{30}$$

Setting $a=0$ in $\widetilde{W}_{m,a}(n,k)$ allows us to express the noncentral Whitney numbers of the second kind in terms of $\left\{{n \atop k}\right\}$. More precisely, when $a=0$ in [15, Proposition 7], we can see that

$$\widetilde{W}_{m,0}(n,k)=m^{n-k}\left\{{n \atop k}\right\}.$$

Thus, (30) becomes

$$w_n\left(\frac{x}{m}\right)=x\sum_{k=0}^{n}\frac{(-1)^{n+k}k!}{m^k}\left\{{n \atop k}\right\}(m+x)^{k-1} \tag{31}$$

when $a=0$. Furthermore, when $m=1$, we recover the explicit formula in (11). The expression $m^{n-k}\left\{{n \atop k}\right\}$ is actually called translated Whitney numbers of the second kind and is denoted by $\left\{{n \atop k}\right\}^{(m)}$. These numbers satisfy the recurrence relation given by [1, Theorem 8]

$$\left\{{n \atop k}\right\}^{(m)}=\left\{{n-1 \atop k-1}\right\}^{(m)}+mk\left\{{n-1 \atop k}\right\}^{(m)}$$

and the explicit formula [16, Proposition 2]

$$\left\{{n \atop k}\right\}^{(m)}=\frac{1}{m^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(mj)^n.$$

More properties of these numbers can be seen in [14]. With these, we may also write

$$w_n\left(\frac{x}{m}\right)=\frac{x}{m^n}\sum_{k=0}^{n}(-1)^{n+k}k!\left\{{n \atop k}\right\}^{(m)}(m+x)^{k-1}, \tag{32}$$

an explicit formula for the geometric polynomials $w_n\left(\frac{x}{m}\right)$ in terms of the translated Whitney numbers of the second kind.

Now it can be shown that

$$\frac{y^2-1}{2y}\left(\frac{e^{-a(2z)}}{y-e^{mz}}+\frac{e^{-a(2z)}}{y+e^{mz}}\right)=\frac{e^{-a(2z)}}{1-\left(\frac{1}{y^2-1}\right)\left(e^{m(2z)-1}\right)}.$$

Notice that the right-hand side is

$$\begin{aligned}\frac{e^{-a(2z)}}{1-\left(\frac{1}{y^2-1}\right)\left(e^{m(2z)-1}\right)}&=\frac{me^{-a(2z)}}{m-\left(\frac{m}{y^2-1}\right)\left(e^{m(2z)-1}\right)}\\&=\sum_{n=0}^{\infty}2^n\widetilde{\mathcal{F}}_{m,a}\left(n;\frac{m}{y^2-1}\right)\frac{z^n}{n!}.\end{aligned}$$

Also, in the left-hand side, we have

$$\frac{e^{-a(2z)}}{y-e^{mz}}=\frac{1}{y-1}\sum_{n=0}^{\infty}\widetilde{\mathcal{F}}_{m,2a}\left(n;\frac{m}{y-1}\right)\frac{z^n}{n!}$$

and

$$\frac{e^{-a(2z)}}{y+e^{mz}}=\frac{1}{y+1}\sum_{n=0}^{\infty}\widetilde{\mathcal{F}}_{m,2a}\left(n;\frac{-m}{y+1}\right)\frac{z^n}{n!}.$$

Combining these equations and comparing the coefficients of $\frac{z^n}{n!}$ results to

$$2^{n+1}\widetilde{\mathcal{F}}_{m,a}\left(n;\frac{m}{y^2-1}\right)=\frac{y+1}{y}\widetilde{\mathcal{F}}_{m,2a}\left(n;\frac{m}{y-1}\right)+\frac{y-1}{y}\widetilde{\mathcal{F}}_{m,2a}\left(n;\frac{-m}{y+1}\right).$$

Note that if we set $x=\frac{m}{y-1}$, then $y=\frac{m+x}{x}$. Hence, skipping the tedious computations allow us to write

$$(m+2x)\widetilde{\mathcal{F}}_{m,2a}(n;x)=2^{n+1}(m+x)\widetilde{\mathcal{F}}_{m,a}\left(n;\frac{x^2}{m+2x}\right)-m\widetilde{\mathcal{F}}_{m,2a}\left(n;\frac{-mx}{m+2x}\right).$$

The next theorem is obtained by applying (14).

Theorem 6. *The noncentral Tanny-Dowling polynomials satisfy the following explicit formula:*

$$\widetilde{\mathcal{F}}_{m,2a}(n;x)=\sum_{k=0}^{n}k!x^k\left(\frac{2^{n+1}(m+x)x^k\widetilde{W}_{m,a}(n,k)+(-m)^{k+1}\widetilde{W}_{m,2a}(n,k)}{(m+2x)^{k+1}}\right).\tag{33}$$

Since it is already known that $\widetilde{W}_{m,0}(n,k) = {n \brace k}^{(m)}$, then the right-hand side can be expressed in terms of the translated Whitney numbers of the second kind when $a = 0$. That is,

$$w_n\left(\frac{x}{m}\right) = \frac{1}{m^n}\sum_{k=0}^{n} k!x^k {n \brace k}^{(m)} \left(\frac{2^{n+1}(m+x)x^k + (-m)^{k+1}}{(m+2x)^{k+1}}\right). \tag{34}$$

Lastly, we recover the explicit formula in (12) when $m = 1$.

Finally, we will conclude this paper by mentioning an explicit formula for $\widetilde{\mathcal{F}}_{m,a}(n;x)$ established in [15, Theorem 19] that is given by

$$\widetilde{\mathcal{F}}_{m,a}(n;x) = \frac{m}{m+x}\sum_{k=0}^{\infty}\left(\frac{x}{m+x}\right)^k (mk-a)^n. \tag{35}$$

This explicit formula entails interesting particular cases. For instance, when $a = 0$,

$$w_n\left(\frac{x}{m}\right) = \frac{m}{m+x}\sum_{k=0}^{\infty}\left(\frac{x}{m+x}\right)^k k^n. \tag{36}$$

When $m = 1$ and then $x = 1$, we get formulas for the ordinary geometric polynomials and numbers. That is,

$$w_n(x) = \frac{1}{x+1}\sum_{k=0}^{\infty}\left(\frac{x}{x+1}\right)^k k^n \tag{37}$$

and

$$w_n = \sum_{k=0}^{\infty}\frac{k^n}{2^{k+1}}. \tag{38}$$

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References

[1] H. Belbachir and I. Bousbaa, Translated Whitney and r-Whitney numbers: a combinatorial approach, *J. Integer Sequences* **16** (2013), Article 13.8.6.

[2] M. Benoumhani, On Whitney numbers of Dowling lattices, *Discrete Math.* **159** (1996), 13–33.

[3] M. Benoumhani, On some numbers related to Whitney numbers of Dowling lattices, *Adv. Appl. Math.* **19** (1997), 106–116.

[4] K. N. Boyadzhiev, A Series transformation formula and related polynomials, *Int. J. Math. Math. Sci.* **23** (2005), 3849–3866.

[5] K. N. Boyadzhiev, and A. Dil, Geometric polynomials: properties and applications to series with zeta values, *Anal. Math.* **42** (2016), 203–224.

[6] A. Z. Broder, The r-Stirling numbers, *Discrete Math.* **49** (1984), 241–259.

[7] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Co., 1974.

[8] R. B. Corcino, The (r, β)-Stirling numbers, *The Mindanao Forum*, **14** (1999), 91–99.

[9] A. Dil and V. Kurt, Investigating geometric and exponential polynomials with Euler-Seidel matrices, *J. Integer Sequences* **14** (2011), Article 11.4.6.

[10] L. Kargın, Some formulae for products of geometric polynomials with applications, *J. Integer Sequences* **20** (2017), Article 17.4.4.

[11] L. Kargın and B. Çekim, Higher order generalized geometric polynomials, *Turkish J. Math.*, **42** (2018), 887–903.

[12] L. Kargın and R. B. Corcino, Generalization of Mellin derivative and its applications, *Integral Transforms Spec. Funct.* **27** (2016), 620–631.

[13] M. Koutras, Non-central Stirling numbers and some applications, *Discrete Math.* **42** (1982), 73–89.

[14] M. M. Mangontarum and A. M. Dibagulun, On the translated Whitney numbers and their combinatorial properties, *British J. Appl. Sci. Technology* **11** (2015), 1–15.

[15] M. M. Mangontarum, O. I. Cauntongan, and A. P. M.-Ringia, The noncentral version of the Whitney numbers: a comprehensive study, *Int. J. Math. Math. Sci.* **2016** (2016).

[16] M. M. Mangontarum, A. P. M.-Ringia, and N. S. Abdulcarim, The translated Dowling polynomials and numbers, *International Scholarly Research Notices* **2014**, Article ID 678408, 8 pages, (2014).

[17] I. Mező, A new formula for the Bernoulli polynomials, *Results Math.* **58** (2010), 329–335.

[18] J. Stirling, *Methodus Differentialissme Tractus de Summatione et Interpolatione Serierum Infinitarum*, London, 1730.

[19] S. M. Tanny, On some numbers related to the Bell numbers, *Canad. Math. Bull.* **17** (1975), 733–738.

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