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Multiplicative Persistence and Absolute Multiplicative Persistence

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Abstract

We investigate the multiplicative persistence of a number for different bases, including for the bijective representation of bases. We also study the absolute multiplicative persistence, which is the largest multiplicative persistence of a number in any base.

1 Introduction

Sloane [1] introduced the problem of multiplicative persistence of a number, which can be defined as follows: starting with a positive integer, n, multiply the digits d of the number together to provide a new number. Then iterate this procedure on the new number until a single digit is reached. A number in integer base b with k digits may be expressed as follows:

$$n = [d_1 d_2 d_3 \cdots d_k]_b = \sum_{i=1}^{k} d_i b^{k-i}. \tag{1}$$

The function to be iterated can then be written as follows:

$$F(n) = \prod_{i=1}^{k} d_i. \tag{2}$$

The number of times that F can be applied before a single digit is reached is the multiplicative persistence, s. The function, F, is referred to as a Sloane mapping by Faria and Tresser [2], as the function maps one number to another (smaller) one. In base 10, the largest persistence that has been found is $s = 11$ for the number $n = 277777788888899$, which is given in A003001 in the *On-line Encyclopedia of Integer Sequences* [3].

Perez and Styer [4] have considered other bases and found the maximum conjectured persistence in each base 2 through 12. Results from some higher bases will be shown here, as well as some results where the bijective representation of bases is used. Some proofs for the persistence of two-digit numbers are given. The main result of the paper is to introduce a new quantity, the absolute multiplicative persistence. The largest multiplicative persistence of a number in any base is defined as its absolute multiplicative persistence.

2 Two-digit numbers

The smallest number in base b that has persistence of 1 is $[10]_b$, as any number smaller than this must be a single digit already and thus have persistence $s = 0$. The smallest possible number whose digits when multiplied together give a number greater than or equal to b, has a leading digit of 2 and a trailing digit of $\lceil b/2 \rceil$. So the smallest possible number with persistence of 2 is $\lceil 5b/2 \rceil$, and any number smaller than this results in a product of the digits that is less than b. The smallest possible number whose digits when multiplied give $\geq \lceil 5b/2 \rceil$ must have a leading digit of 3 and a trailing digit of $\lceil 5b/6 \rceil$. If the leading digit were still 2, the trailing digit would need to be $\lceil 5b/4 \rceil$, which isn't possible. The smallest values for the subsequent persistence levels are tabulated below for two-digit numbers, down to a persistence level of s. The leading digit increments by 1 as the level of persistence increases, and the trailing digit must always be less than or equal to $b - 1$ (the largest possible digit). Thus $(s! - 1)b/s! \leq b - 1$, i.e., while $b \geq s!$.

Iterations	Leading digit	Trailing digit	n as a fraction
1	1	0	b
2	2	$\left\lceil \frac{b}{2} \right\rceil$	$\left\lceil \frac{5b}{2} \right\rceil$
3	3	$\left\lceil \frac{5b}{6} \right\rceil$	$\left\lceil \frac{23b}{18} \right\rceil$
4	4	$\left\lceil \frac{23b}{24} \right\rceil$	$\left\lceil \frac{119b}{24} \right\rceil$
⋮	⋮	⋮	⋮
s	s	$\left\lceil \frac{(s!-1)b}{s!} \right\rceil$	$\left\lceil \frac{((s+1)!-1)b}{s!} \right\rceil$

Table 1: Number as a function of the base representation and the persistence

Theorem 1. *The smallest two-digit number n represented in base b with multiplicative persistence s is given by*

$$n = \left\lceil \frac{((s+1)!-1)b}{s!} \right\rceil \; \forall b \geq s!. \tag{3}$$

Thus any persistence level can be achieved given an appropriate choice of base.

Proof. For the induction hypothesis assume a number n' in base b that has a persistence of $s-1$, and has the value

$$n' = \left\lceil \frac{(s!-1)b}{(s-1)!} \right\rceil. \tag{4}$$

Consider a larger number, $n = \lceil ((s+1)!-1)b/s! \rceil$ (as given in Equation (3)), that has a leading digit s and trailing digit of $\lceil (s!-1)b/s! \rceil$, and which has a persistence of s. When a Sloane mapping is applied (Equation (2)) to n, we get

$$F(n) = (s)\left(\left\lceil \frac{(s!-1)b}{s!} \right\rceil\right) = s\left\lceil \frac{n'}{s} \right\rceil \geq n'. \tag{5}$$

After one iteration of the Sloane mapping, the number found is the same as (or larger than) the number in the hypothesis, $F(n) \geq n'$. In which case, the multiplicative persistence of n is indeed s, given the multiplicative persistence of $s-1$ assumed for the number n'. There is no requirement for n to be mapped exactly onto n'; $F(n)$ just needs to be no smaller than n'. If n was any smaller, then $F(n) < n'$ which would mean persistence $\neq s$ for n, as n' is the smallest number with persistence $s-1$. For persistence of 1 the smallest number in base b is $[10]_b$ and any smaller number would be a single digit and have persistence of 0. For n, the trailing digit of $\lceil (s!-1)b/s! \rceil$ must always be less than or equal to $b-1$, which is the largest digit, i.e., the theorem holds for all values of $b \geq s!$. □

It is clear that the any level of persistence can be achieved given an appropriate choice of base. The maximum two-digit persistence occurs when $b = s!$, and $n = (b+1)! - 1$. So the number $n = 16! - 1$ represented in base $b = 15!$ has persistence of 15, although it can have a larger persistence in other bases. For example in base 6168, $n = 16! - 1$ is a 4 digit number with a persistence of 27.

The persistence level as a function of number in base $b = 720$ is shown in Figure 1. The horizontal lines show $[10]_{720}$, $[100]_{720}$ and $[1000]_{720}$, and these lines represent the boundaries between 2, 3 and 4 digit numbers respectively. The dashed line shows Equation (3) reformulated using the Γ function rather than factorials to get a smooth curve, and as expected the first 6 points lie along this curve.

When $b < s!$, larger persistence levels have a leading digit that increments by more than 1, and the values of n now increase approximately exponentially with persistence, as shown by the first dotted line in Figure 1 that fits persistence levels $s = 6$ to $s = 16$ for the two-digit numbers. The pattern changes again for three-digit numbers, where values of n again grow exponentially with persistence, but at a faster rate than the two-digit numbers. Eventually, (not shown in this figure), the values of n increase their numbers of digits at each step, and n grows at a double exponential rate with s, as will be shown later.

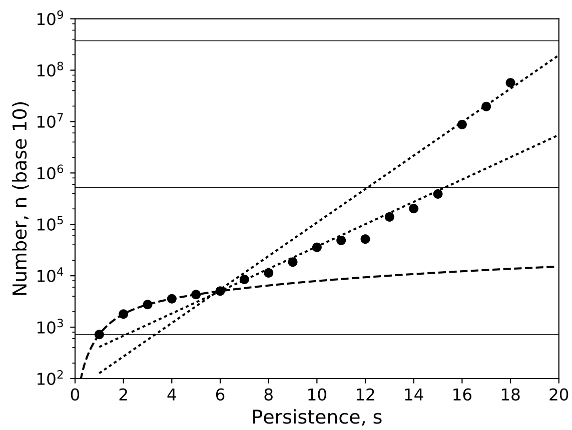

Figure 1: Persistence in base 720

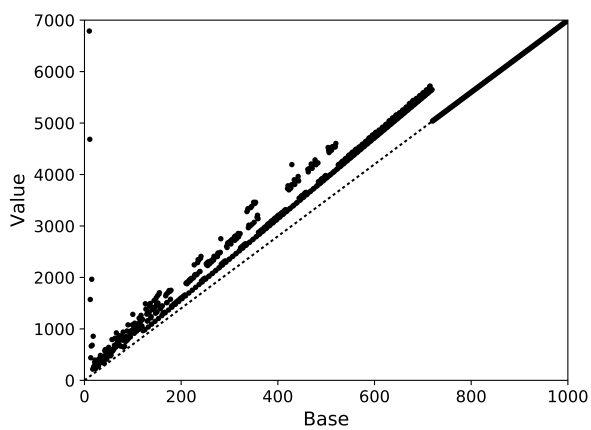

Figure 2: Values for persistence $s = 6$ for different bases

Figure 2 shows the smallest value of the $s = 6$ persistence for different bases from $b = 10$ to $b = 1000$. Kurz A064870 gives this sequence for bases up to base 53. Sequences for other persistence levels are also available in the OEIS for persistence $s = 3 \cdots s = 8$, for sequences A064867, A064868, A064869, A064871, A064872 respectively. In Figure 2 the values decrease very rapidly initially, down to a minimum value of 218 for base $b = 17$. The values subsequently then increase. The minimum value that has a given persistence level is investigated and tabulated in the next section for different persistence levels.

For larger bases, a regular pattern is apparent, with several lines becoming visible. These correspond to the two-digit number cases where the leading digit is a 6, 7, 8 or 9 etc. (when written in the appropriate base). The dotted line shown in Figure 2 fits data points with a leading digit of 6, and the gradient of the line is 5039/720 as would be expected from Equation (3).

3 Absolute persistence

The largest multiplicative persistence of a number in any base is its absolute multiplicative persistence a. For example, the number 218 has an absolute persistence of $a = 6$, which occurs when $b = 17$. There is no choice of base that gives a larger multiplicative persistence for this number. Figure 3 (a) shows the absolute persistences for all the numbers in the range 2 to 1000. From Theorem 1, any persistence level may be found given an appropriate choice of base so absolute persistence is unbounded; thus a increases without limit as n gets large.

Conjecture 2. Any particular absolute persistence level is bounded at top and bottom.

For example, the smallest number with absolute persistence of $a = 1$ is 2 and the largest number with $a = 1$ is conjectured to be 12, as may be seen in Figure 3 (a). Figure 3 (b) shows the corresponding base values that were found to give the absolute persistence for each number. A gap is visible in the scatter of points in Figure 3 (b) which indicates a separation between the two-digit numbers and the three-digit numbers. Much of the time, the largest persistence occurs for a two-digit number, however some of the time, representation in a smaller base with more digits results in the largest persistence. For example, 158 has an absolute persistence of $a = 3$ in base $b = 3$, where it has 5 digits, e.g., $[12212]_3$.

The largest absolute persistence in the range 2 to 100,000 is 18, which occurs for the number 82605 when represented in base 332. The sequence of the least numbers for each absolute persistence level are given in Table 2 and in A330152. This sequence has been extended by Resta [7]. Larger persistences were found in Table 2 by considering just two-digit numbers, so it is possible there may be smaller numbers in existence for the last few tabulated persistences ($a > 27$). The largest conjectured number for each absolute persistence level are given in Table 3, and the number of cases found for each absolute persistence in Table 4. Figure 4 shows a plot of the values from Tables 2 and 3, with the smallest and largest numbers for each absolute persistence level shown by circles and plus symbols respectively.

Conjecture 3. The absolute persistence grows approximately with the log of the value.

For any particular value of n, the expected absolute persistence covers a range of levels with a factor of 2 between the largest and smallest likely values of a, as may be seen from Equations 6 and 7 which describe the dotted lines in Figure 4.

$$a = 2(\log(n_{\text{min}}) - 2) \tag{6}$$

$$a = \log(n_{\text{max}}) - 2 \tag{7}$$

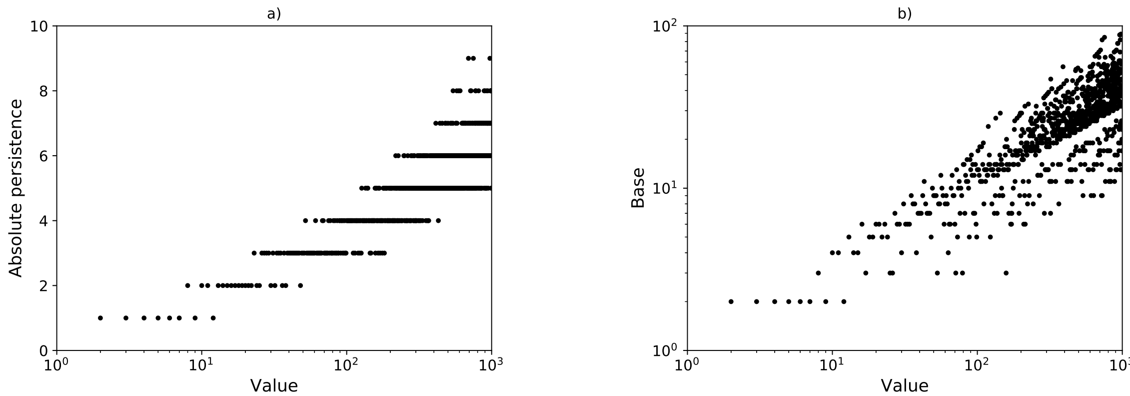

Figure 3: (a) Absolute persistences for values $n = [2, 1000]$; (b) corresponding base to give the absolute persistence.

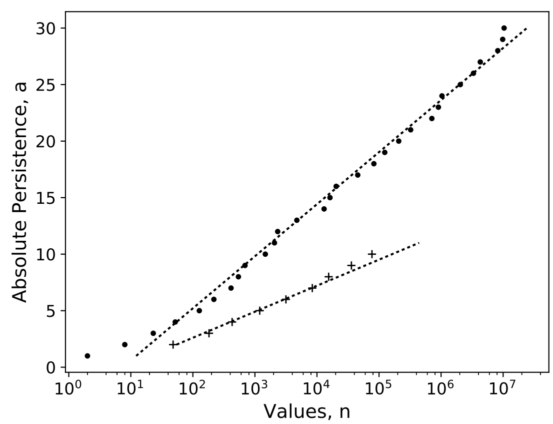

Figure 4: Smallest number $n_{\min}$ (dots), largest number $n_{\max}$ (plus) that has absolute persistence, a. Dotted lines show $a = 2\log(n_{\min}) - 4$ and $a = \log(n_{\max}) - 2$.

Table 4 shows the total number of cases for each absolute persistence level that have been found, and a representative median number with the corresponding base. Where there are an even number of cases, the largest case is excluded for the calculation of the median.

Chaffin [8] has calculated persistence levels in base 10, and with the exception of $s = 1$ the persistence levels are all bounded for the reduced numbers in base 10. A reduced number is the smallest number of the set of numbers that have the same digits excluding the digit 1. For example 58, 85, 15118 all have the same persistence in base 10, but 58 is the reduced number. It can be expected that persistence would also be bounded for reduced numbers in other bases.

Absolute persistence	Number (base 10)	base
0	0	2
1	2	2
2	8	3
3	23	6
4	52	9
5	127	13
6	218	17
7	412	23
8	542	26
9	692	29
10	1471	41
11	2064	53
12	2327	53
13	4739	73
14	13025	123
15	16213	159
16	20388	157
17	45407	251
18	82605	332
19	123706	491
20	207778	587
21	323382	691
22	605338	943
23	905670	1187
24	1033731	1187
25	2041995	1804
26	3325970	2923
27	4282238	2348
28	8200240	3679
29	9840138	3541
30	10364329	3541

Table 2: Smallest number with absolute persistence $a = 1, 2, 3 \cdots$ and the base for which it occurs (A330152). Absolute persistences $a > 27$ may not have the smallest possible numbers, as only two-digit numbers were evaluated.

Absolute persistence	Number (base 10)	base
1	12	2
2	48	5
3	182	11
4	429	15
5	1195	16
6	3160	31
7	8404	41
8	15431	61
9	35893	89
10	76960	97

Table 3: Conjectured largest number with absolute persistence $a = 1, 2, 3 \cdots$ and the base for which it occurs.

Absolute persistence	Total cases	Median (base 10)	base
1	8	5	2
2	20	19	5
3	67	65	6
4	138	172	6
5	280	407	34
6	636	904	13
7	1252	1995	23
8	2626	4254	285
9	5170	8935	145
10	10422	18385	146

Table 4: Number of cases found for absolute persistence $a = 1, 2, 3 \cdots$ and the conjectured median number and the base for which it occurs.

4 Multi-digit numbers

The maximum persistence that has been found in base 10 is $s_{\max} = 11$ for the number $n = 277777788888899$, which has 15 digits. The maximum persistence s_{max} and the number of digits k for which it occurs are shown in Table 5 for other bases up to $b = 24$. This data is also plotted in Figure 5. The prime bases are shown by asterisks, and non-primes by plus symbols. There is a consistent pattern in that the prime bases tend to have a larger maximum persistence than non-primes, and bases with many factors, such as 8, 12, and 24 tend to have smaller maximum persistence. Unpublished results from Kurz [6] show results for some larger bases as well.

b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s_{\max}$	1	3	3	6	5	8	6	7	11	13	7	15	13	11	8
k	1	3	3	22	5	21	9	7	15	51	10	39	27	16	6
b	17	18	19	20	21	22	23	24							
$s_{\max}$	17	10	18	14	14	14	19	9							
k	42	8	59	18	13	20	41	5							

Table 5: Maximum persistence and number of digits for different bases.

Faria and Tresser [2] conjectured that for a multiplication of prime numbers the proportion of a particular digit in the result tends asymptotically to $1/b$. In their example for base $b = 3$ the proportion of zeroes in 2^k tends to one third of the digits when k becomes large enough.

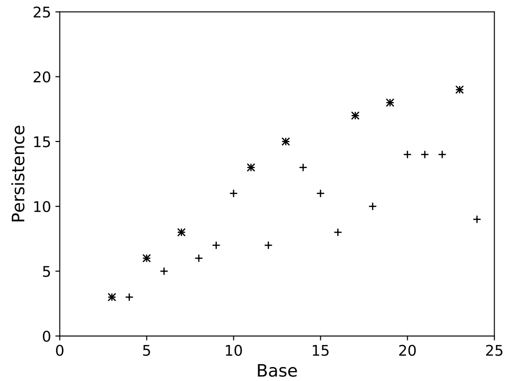

Figure 5: Conjectured maximum persistence for different bases (asterisks show primes).

Base 19 has a maximum persistence of 18, which first occurs for a 59 digit number. The persistence as a function of the smallest number n which has that persistence in base 19, is plotted in Figure 6. For large values of n that have multiple digits, a relationship is plotted in Figure 6 which shows $s = 2\log(\log(n)) + 7.6$. If a persistence of 19 exists in base 19 it is likely to have more than 100 digits, and it was not practical to search for this number. Other bases show the same growth pattern.

Conjecture 4. Persistence grows approximately as $A \log\log n + B$ for all bases as $n \to \infty$ where $A \approx 2$ and B depends on the base used (it increases for larger bases).

Bonuccelli et al. [9] have investigated a somewhat similar problem, the Erdős-Sloane mapping, where all the non-zero digits of a number are multiplied in the mapping function. They proved upper and lower bounds on the persistence, which grows as some factor multiplied by $\log\log n$ as $n \to \infty$.

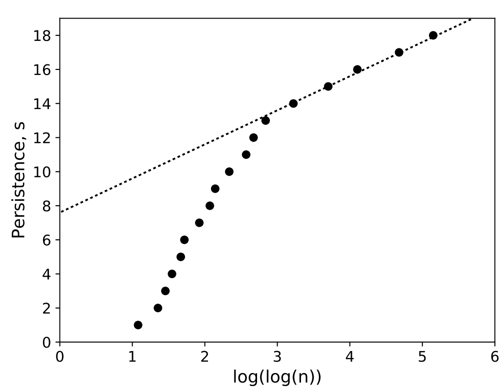

Figure 6: Persistence in base 19, dotted line shows $s = 2\log(\log(n)) + 7.6$.

5 Tables of persistence for different bases

The minimum numbers for each persistence are shown for the bases 3 to 24 in the tables that follow. The numbers are all represented using the convention that lower-case letters in alphabetic order represent digits greater than 9 (e.g. $a = 10, b = 11 \cdots$ etc.), as used for example in hexadecimal notation.

s	$(n)_b$
1	10
2	22
3	222

Table 6: Persistence in base 3

s	$(n)_b$
1	10
2	22
3	333

Table 7: Persistence in base 4

s	$(n)_b$
1	10
2	23
3	233
4	33334
5	444444444444
6	3344444444444444444444

Table 8: Persistence in base 5

s	$(n)_b$
1	10
2	23
3	35
4	444
5	24445

Table 9: Persistence in base 6

s	$(n)_b$
1	10
2	24
3	36
4	245
5	4445
6	44556
7	5555555
8	2222225555555555555666

Table 10: Persistence in base 7

s	$(n)_b$
1	10
2	24
3	37
4	256
5	2777
6	333555577

Table 11: Persistence in base 8

s	$(n)_b$
1	10
2	25
3	38
4	57
5	477
6	45788
7	2577777

Table 12: Persistence in base 9

s	$(n)_b$
1	10
2	25
3	39
4	77
5	679
6	6788
7	68889
8	2677889
9	26888999
10	3778888999
11	277777788888899

Table 13: Persistence in base 10

s	$(n)_b$
1	10
2	26
3	3a
4	69
5	269
6	3579
7	26778
8	47788a
9	67899aaa
10	777777778889999
11	7777789999999999999
12	26777777888888889999999aaaaaaaaaa
13	677777777777777777888888899999aaaaaaaaaaaaaaaaaaaa

Table 14: Persistence in base 11

s	$(n)_b$
1	10
2	26
3	3a
4	6b
5	777
6	aab
7	3577777799

Table 15: Persistence in base 12

s	$(n)_b$
1	10
2	27
3	3b
4	5a
5	9a
6	27a
7	8ac
8	35ab
9	9bbb
10	2999bbc
11	28cccccc
12	47788aaabcc
13	355999cccccccccc
14	5555777777aaaaabcccccccc
15	55555555777999999999999999999999ccccccccc

Table 16: Persistence in base 13

s	$(n)_b$
1	10
2	27
3	3c
4	5b
5	99
6	359
7	ccc
8	359ab
9	cdddd
10	3abbdddd
11	488aabcccddd
12	299999999999bbbbcd
13	55599999999999999aaaabbbbbb

Table 17: Persistence in base 14

s	$(n)_b$
1	10
2	28
3	3d
4	5e
5	28c
6	8ae
7	5bbb
8	bbbcc
9	2999bde
10	39bbcccccd
11	2bbbbccccdddddde

Table 18: Persistence in base 15

s	$(n)_b$
1	10
2	28
3	3e
4	5f
5	bb
6	2ab
7	3dde
8	379bdd

Table 19: Persistence in base 16

s	$(n)_b$
1	10
2	29
3	3f
4	5g
5	9f
6	ce
7	3dd
8	9cf
9	2aff
10	55ddf
11	39ddgg
12	degggg
13	6bbbbbeef
14	777bdddddeefg
15	5777bbddddefffffg
16	699999999bbbbbbbbbbbefggggg
17	377779999999bbbbbbbbbbbbffffffffffffffffg

Table 20: Persistence in base 17

s	$(n)_b$
1	10
2	29
3	3f
4	5e
5	8d
6	2bb
7	2ddg
8	aabf
9	8gghh
10	5555aagh

Table 21: Persistence in base 18

s	$(n)_b$
1	10
2	2a
3	3g
4	5f
5	ab
6	dh
7	2bc
8	7bg
9	dii
10	4aah
11	3bgii
12	eefhh
13	adefhh
14	4addddeef
15	9999999bbfhhhi
16	577ddeeefghhhhhhiiiii
17	55bbbbbddddddfhhhhhhhhhhhhhhhhhhhhhhh
18	5555577bbbbbbbdddddddddddddddddddddddddddddddhhhhhhhhhhhhhh

Table 22: Persistence in base 19

s	$(n)_b$
1	10
2	2a
3	3h
4	6d
5	7j
6	di
7	6de
8	cgg
9	2bhi
10	cdgg
11	2degj
12	77bbhj
13	bbbceehhhhh
14	cddddegghhhhhhiiii

Table 23: Persistence in base 20

s	$(n)_b$
1	10
2	2b
3	3i
4	6h
5	ak
6	gh
7	4gi
8	gii
9	5fhj
10	2bgij
11	fhkkk
12	3bdghjk
13	6bbhijjjj
14	3ddgggggggiij

Table 24: Persistence in base 21

s	$(n)_b$
1	10
2	2b
3	3j
4	5h
5	9h
6	hj
7	6gj
8	3dhl
9	cggh
10	6ghkk
11	2dgiiij
12	4deghhjj
13	cijjjjjjjjl
14	77777777777777bbbbbd

Table 25: Persistence in base 22

s	$(n)_b$
1	10
2	2c
3	3k
4	5i
5	7j
6	9k
7	hl
8	55k
9	dgh
10	7lmm
11	hklm
12	chikl
13	ffklll
14	39hijlll
15	6ghiiijkkk
16	5fhhijjjklmmm
17	cddghhhjjjjjjkkl
18	55777bbdddhhhjjjjjjjjjjjjjj
19	9999999ffffffffffffffikkllllllllllllll

Table 26: Persistence in base 23

s	$(n)_b$
1	10
2	2c
3	3k
4	4n
5	7h
6	fm
7	5hm
8	dln
9	9bbdn

Table 27: Persistence in base 24

6 Bijective representation

Bijective representation in base b' uses a digit set such as $\{A, B, C, \ldots, b'\}$ to uniquely represent every non-negative integer. It uses the same positional system as the standard representation, where a digit is a multiple of a power of b' defined by its position in the string, as in Equation (1). In a bijective representation zero can not be represented, and the smallest digit that is possible is 1. Foster [5] gives one of the earliest discussions of bijective number systems. Here the bijective representation uses the convention that upper-case letters in alphabetic order represent the digits greater than 0 ($A = 1, B = 2, \ldots, Z = 26$ etc.). This numeration scheme is commonly used in base 26 for numbering columns in spreadsheets.

The smallest number in base b' that has persistence of 1 is $[AA]_{b'}$, as any number smaller than this must be a single digit already and thus must have persistence $s = 0$. The smallest possible number whose digits when multiplied together give a number greater than or equal to $b' + 1$ has a leading digit of 2 and a trailing digit of $\lceil (b' + 1)/2 \rceil$. So the smallest possible number with persistence of 2 is $\lceil (5b' + 1)/2 \rceil$, and any number smaller than this results in a product that is less than $b' + 1$. The smallest possible number whose digits when multiplied give $\geq \lceil (5b' + 1)/2 \rceil$ must have a leading digit of 3 and a trailing digit of $\lceil (5b' + 1)/6 \rceil$. The smallest values for the subsequent persistence levels are tabulated below for two-digit numbers, down to a persistence level of s. The leading digit increments by 1 as the level of persistence increases, until $s = b'$, i.e., until the largest two-digit number for that base is reached, e.g., while $b' \geq s$.

Iterations	Leading digit	Trailing digit	n as a fraction
1	1	1	$b' + 1$
2	2	$\left\lceil \frac{b'+1}{2} \right\rceil$	$\left\lceil \frac{5b'+1}{2} \right\rceil$
3	3	$\left\lceil \frac{5b'+1}{6} \right\rceil$	$\left\lceil \frac{23b'+1}{6} \right\rceil$
4	4	$\left\lceil \frac{23b'+1}{24} \right\rceil$	$\left\lceil \frac{119b'+1}{24} \right\rceil$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
s	s	$\left\lceil \frac{(s!-1)b'+1}{s!} \right\rceil$	$\left\lceil \frac{((s+1)!-1)b'+1}{s!} \right\rceil$

Table 28: Number as a function of the bijective representation base b' and the persistence s.

Theorem 5. *The smallest two-digit number n with bijective representation in base b' with multiplicative persistence s is given by*

$$n = \left\lceil \frac{((s+1)! - 1)b' + 1}{s!} \right\rceil \; \forall b' \geq s. \tag{8}$$

Proof. Assume that a number n' in base b' has a persistence of $s-1$, and has the value

$$n' = \left\lceil \frac{(s!-1)b'+1}{(s-1)!} \right\rceil. \tag{9}$$

Consider a larger number, $n = \lceil((s+1)!-1)b'+1/s!\rceil$, that has a leading digit s and trailing digit of $\lceil((s!-1)b'+1)/s!\rceil$, and which has a persistence of s. When a Sloane mapping is applied (Equation (2)) to n,

$$F(n) = (s)\left(\left\lceil \frac{(s!-1)b'+1}{s!} \right\rceil\right) = s\left\lceil \frac{n'}{s} \right\rceil \geq n'. \tag{10}$$

After one iteration of the Sloane mapping, the number found is the same as (or larger than) the hypothesis, $F(n) \geq n'$. In which case, the multiplicative persistence of n is indeed s, given the multiplicative persistence of $s-1$ assumed for the number n'. There is no requirement for n to be mapped exactly onto n'; $F(n)$ just needs to be no smaller than n'. If n was any smaller, then $F(n) < n'$, which would mean a persistence $\neq s$ for n, as n' is the smallest number with persistence $s-1$. For persistence of 1 the smallest number in base b' is $[AA]_{b'}$ as any smaller number would have a single digit and have persistence of 0. The leading digit increments by 1 as the level of persistence increases, until $s = b'$, i.e., the theorem holds for all $b' \geq s$. When $b' = s$ Equation (8) simplifies to $n = s(s+1)$. □

For bijective representation base 1, $s = 1$ is the largest possible persistence, which occurs for $n = [AA]_1$. In bijective representation of base 2, the largest possible persistence occurs for $[BB]_2$ which has persistence 2. This is because all powers of 2 when in bijective representation base 2 contain all A digits except the trailing digit which is B. In general, for two-digit numbers the largest persistence occurs for $n = b'(b'+1)$ in any bijective representation base, where the persistence $s = b'$. For instance in bijective representation base 3, $[CC]_3$ has persistence $s = 3$, and in general any two-digit number that has a maximum trailing digit for that base, has a persistence equal to the leading digit.

Conjecture 6. Persistence is unbounded for bijective representation bases greater than 2.

Persistence using the bijective representation of numbers is somewhat similar to the shifted Sloane persistence problem first introduced by Wagstaff [10], where another fixed integer is added to the digits before multiplying them. The function to be iterated for the shifted Sloane persistence is

$$G(n) = \prod_{i=1}^{k}(d_i + t). \tag{11}$$

A shift of $t = 1$ seems likely to produce behaviour similar to using the bijective representation, as the digit 0 does not occur in the multiplication. Bonuccelli et al. [9] have proved that a double logarithmic dependence of n with s is expected for this shift.

Only the first dozen persistences are shown for the bijective representation bases 3 to 12 in the tables that follow. Figure 7 shows how the size of the number grows with persistence

s	$(n)_{b'}$
1	AA
2	BB
3	CC
4	BBC
5	CCC
6	BBCCCC
7	CCCCCCC
8	CCCCCCCCC
9	BBBBBBBBBBBBBBCCCC
10	BBBBBBBCCCCCCCCCCCCCC
11	BBBBBBBBBBBBBBBBBBBBBBBBBBCCCCCCCCCCCCC
12	BBCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

Table 29: Persistence in bijective representation base 3

s	$(n)_{b'}$
1	AA
2	BC
3	CD
4	DD
5	CCD
6	DDD
7	CCCCD
8	DDDDD
9	BCCCCCCCCCC
10	DDDDDDDDDDDD
11	BCCCCCCCCCCCCCCCCCCCCCCCCC
12	BCCCCCCCCCCCCCCCCDDDDDDDDDDDDDDDDDDDDDDDD

Table 30: Persistence in bijective representation base 4

s	$(n)_{b'}$
1	AA
2	BC
3	CE
4	DE
5	EE
6	BCE
7	EEE
8	CCCCE
9	CDDDDEE
10	DDDDEEEE
11	BCCCDDDEEEE
12	BCCCCCDDDDDDDDDE

Table 31: Persistence in bijective base 5

s	$(n)_{b'}$
1	AA
2	BD
3	CF
4	DF
5	EF
6	FF
7	DDF
8	DDEE
9	DFFFF
10	CCCFFFF
11	DDDEEEEFF
12	EFFFFFFFFFF

Table 32: Persistence in bijective base 6

s	$(n)_{b'}$
1	AA
2	BD
3	CF
4	DG
5	EG
6	FG
7	GG
8	BDG
9	FFG
10	GGG
11	EGGG
12	BDGGG

Table 33: Persistence in bijective base 7

s	$(n)_{b'}$
1	AA
2	BE
3	CG
4	DH
5	EH
6	FH
7	GH
8	HH
9	BFF
10	DGH
11	EHH
12	GGH

Table 34: Persistence in bijective base 8

s	$(n)_{b'}$
1	AA
2	BE
3	CI
4	DI
5	EI
6	FI
7	GI
8	HI
9	II
10	BEI
11	DFI
12	DII

Table 35: Persistence in bijective base 9

s	$(n)_{b'}$
1	AA
2	BF
3	CI
4	DJ
5	EJ
6	FJ
7	GJ
8	HJ
9	IJ
10	JJ
11	CGJ
12	GII

Table 36: Persistence in bijective base 10

s	$(n)_{b'}$
1	AA
2	BF
3	CJ
4	DK
5	EK
6	FK
7	GK
8	HK
9	IK
10	JK
11	KK
12	BFK

Table 37: Persistence in bijective base 11

s	$(n)_{b'}$
1	AA
2	BG
3	CK
4	DL
5	EL
6	FL
7	GL
8	HL
9	IL
10	JL
11	KL
12	LL

Table 38: Persistence in bijective base 12

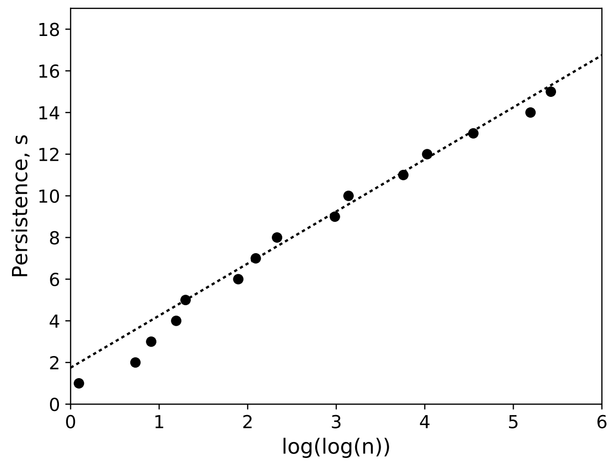

Figure 7: Persistence in bijective representation of base 3, dotted line shows $s = 2.5\log(\log(n)) + 1.75$.

for bijective representation base 3. There is a double logarithmic dependence of n with s, just as was found for the conventional bases, and as shown in Figure 6 for base 19.

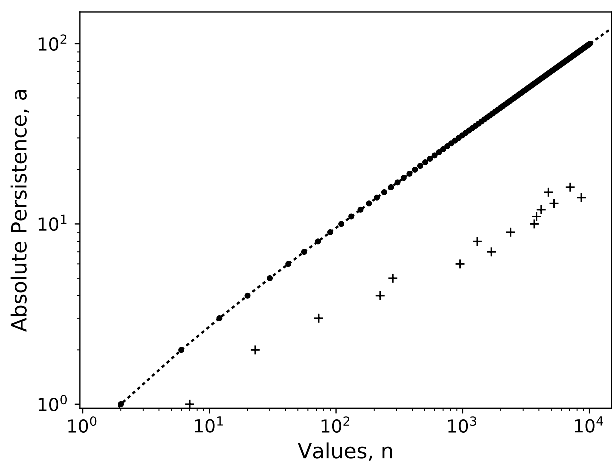

Figure 8: Smallest number $n_{\min}$ (dots), largest number $n_{\max}$ (plus) that has absolute persistence a, for bijective representation of numbers. Dotted line shows $n = a(a+1)$.

7 Absolute persistence using bijective representation

It has already been shown that for any bijective representation base the largest two-digit persistence occurs in all cases when $n = b'(b'+1)$ and the persistence $s = b'$. The least number for each absolute persistence level follows this pattern, as shown in Table 39 and plotted in Figure 8.

Conjecture 7. Absolute persistence is unbounded using bijective representation, and the smallest number of any absolute persistence level a is given by

$$n = a(a+1), \tag{12}$$

when n is represented in base $b' = a$.

Equation 12 is shown by the dotted line in Figure 8. The largest number of any absolute persistence level grows at a slower rate of approximately half the rate of the smallest number (plus symbols compared to the circles), as was found in Figure 4 for the conventional representation of bases. In this case, the variability of the largest numbers (plus symbols) is quite large and a line fit has not been plotted in Figure 8.

The absolute persistences for all the numbers in the range 2 to 1000 are plotted in Figure 9. Conjecture 2 was applicable to all bases with the conventional representation, with the absolute persistence being unbounded, but individual persistence levels bounded at top and bottom. This appears to be equally applicable when using the bijective representation of bases as well.

Conjecture 8. Any particular absolute persistence level is bounded at top and bottom when using bijective representation.

For example, the smallest number with absolute persistence of $a = 1$ is 2 and the largest is 7, as may be seen in Figure 9. The largest conjectured number for each persistence level is tabulated in Table 40, and the number of cases of each absolute persistence found is shown in Table 41. A representative median number is also shown in Table 41. Where there are an even number of cases, the largest case is excluded for the calculation of the median.

Figure 10 shows a maximum absolute persistence of $a = 100$ which occurs when $n = 10100$ represented in $b' = 100$. There is a sharp cut-off in the figure where $a \approx 10100/b'$. Large values of n, e.g., $9900 < n < 10100$ that can be represented in the form $n = a.b'$, lie along this boundary. As more values of n are added to the plot, a increases with b'.

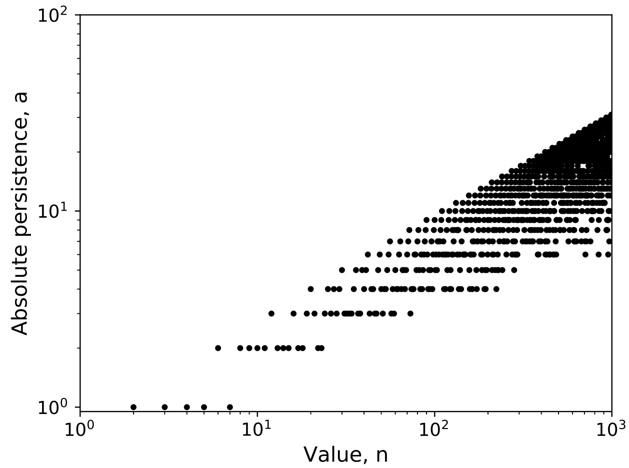

Figure 9: Absolute persistences for values $n = [2, 1000]$ using bijective representation.

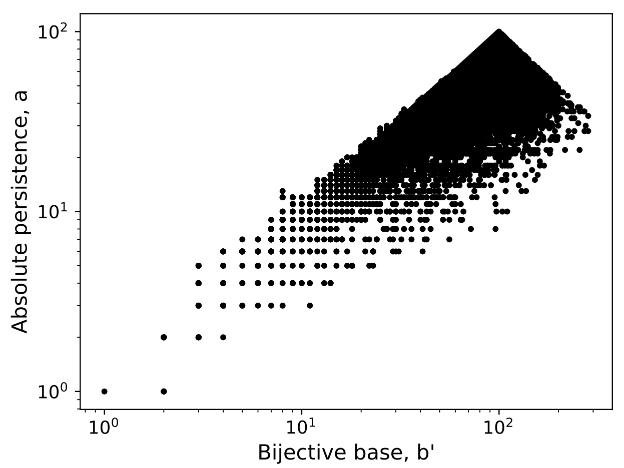

Figure 10: Absolute persistence a for $n \leq 10100$ and bijective representation of base b' for which it occurs.

Absolute persistence	Number (base 10)	Bijective base
0	1	1
1	2	1
2	6	2
3	12	3
4	20	4
5	30	5
6	42	6
7	56	7
8	72	8
9	90	9
10	1471	10
11	132	11
12	156	12
13	182	13
14	210	14
15	240	15
16	272	16
17	306	17
18	342	18
19	380	19
20	420	20
21	462	21
22	506	22
23	552	23
24	600	24
25	650	25
26	702	26
27	756	27
28	812	28
29	870	29
30	930	30

Table 39: Smallest number with absolute persistence $a = 1, 2, 3 \cdots$ and the bijective representation of base for which it occurs.

Absolute persistence	Number (base 10)	Bijective base
1	7	2
2	23	3
3	73	5
4	223	3
5	281	3
6	953	23
7	1684	7
8	1303	12
9	2389	8
10	3673	110
11	3831	21
12	4166	14
13	5261	130
14	8641	24
15	4751	16

Table 40: Conjectured largest number with absolute persistence $a = 1, 2, 3 \cdots$ and the bijective representaion of base for which it occurs.

Absolute persistence	Total cases	Median (base 10)	Bijective base
1	5	4	2
2	12	13	2
3	20	33	3
4	36	75	11
5	31	101	6
6	53	159	15
7	53	277	9
8	50	292	8
9	68	343	7
10	70	449	24
11	76	623	12
12	115	657	9
13	97	853	14
14	89	887	33
15	106	1027	40

Table 41: Number of cases found for absolute persistence $a = 1, 2, 3 \cdots$ and the conjectured median number and the bijective representation of base for which it occurs.

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This work was inspired by the Numberphile video by Haran and Parker, “What’s special about 277777788888899?” [11].

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