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On the Number of Antichains and Antichain Covers of Labeled Sets

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Abstract

We present a formula for the number of antichains consisting of m subsets of a labeled n-element set, as well as a formula for the number of all m-antichain covers of a labeled n-element set. We also give a simple formula for the number of antichain covers that are composed of sets of the same cardinality.

1 Introduction and preliminaries

In 1897 Dedekind [7] formulated the following question:

Question 1. What is the number of antichains in the power set $\mathcal{P}_n$ of an n-element set (ordered by inclusion)?

Dedekind's problem and the related problem of determining the number of all antichain covers of a labeled n-element set remained unsolved for a long time [20]. These are very difficult—and still baffling—counting problems. Before we formulate these problems precisely, let us define some necessary and fundamental notions.

Definition 2. [21, Def. 1.1.1, p. 2] An *ordered set* (or partially ordered set or poset) is an ordered pair $(P, \leq)$ of a set P and a binary relation $\leq$ contained in $P \times P$, called the order (or the partial order) on P, such that:

1. The relation $\leq$ is reflexive, i.e., $\forall_{p \in P}(p \leq p)$.
2. The relation $\leq$ is antisymmetric, that is,

$$\forall_{p,q \in P}\Big(\big((p \leq q) \wedge (q \leq p)\big) \implies p = q\Big).$$

3. The relation $\leq$ is transitive, i.e.,

$$\forall_{p,q,r \in P}\Big(\big((p \leq q) \wedge (q \leq r)\big) \implies p \leq r\Big).$$

The power set $\mathcal{P}_n$ of an n-element set with the inclusion $\subseteq$ relation is an example of an order set. Another key notion is an antichain defined as follows:

Definition 3. [21, Def. 2.5.1, p. 36] An ordered set P is called an *antichain* iff

$$\forall_{p,q \in P}(p \neq q) \implies (\neg p \leq q \wedge \neg q \leq p).$$

Now we can define the n-th Dedekind number and related numbers.

Definition 4. [21, Def. 2.6.2, p. 39] Let $n \in \mathbb{N}$. We define

1. The *n-th Dedekind number* D_n to be the number of antichains in $\mathcal{P}_n$.
2. T_n to be the number of *antichain covers of a labeled n-element set*, i.e., the number of antichains in $\mathcal{P}(\{1, \ldots, n\})$ such that the union of these antichains is $\{1, \ldots, n\}$.

Since the empty set is an antichain, we obtain straightforwardly: $D_0 = T_0 = 1$, $D_1 = 2$, and $T_1 = 1$. Hitherto only nine initial values of numbers D_n and T_n have been determined. They are listed in the *On-Line Encyclopedia of Integer Sequences* [22] as A000372 and A006126, respectively. Dedekind's problem, in turn, is equivalent to the problem of determining the number of elements in the free bounded distributive lattice on n generators which is the same as the number of monotone Boolean functions in n arguments. T_n is defined primarily in the *On-Line Encyclopedia of Integer Sequences* as the number of hierarchical models on n labeled factors or variables with linear terms forced. Kilibarda and Janovic [11] obtained the antichain cover interpretation of A006126.

An approach to solving Dedekind's problem and related problems is to break the problem into smaller subproblems. There is a simple relation between D_n and T_n.

Proposition 5. *[21, Prop. 2.6.3, p. 40]*

$$D_n = \sum_{k=0}^{n} \binom{n}{k} T_k. \tag{1}$$

Any antichain (cover) in $\mathcal{P}_n$ is a Sperner family. Therefore by Sperner's theorem [23], for every antichain (cover) $C = \{A_1, \ldots, A_m\}$ in $\mathcal{P}_n$ we have $m \leq \binom{n}{\lfloor n/2 \rfloor}$, where $\lfloor x \rfloor := \max\{k \in \mathbb{Z} \colon k \leq x\}$ for any real number $x \in \mathbb{R}$. It follows from the above considerations that if

- D_n^m is the number of all m-element antichains in $\mathcal{P}_n$
- T_n^m is the number of all m-element antichain covers (m-antichain covers) in $\mathcal{P}_n$

then

$$D_n = \sum_{m=0}^{\binom{n}{\lfloor n/2 \rfloor}} D_n^m \quad \text{and} \quad T_n = \sum_{m=0}^{\binom{n}{\lfloor n/2 \rfloor}} T_n^m \tag{2}$$

for all $n \in \mathbb{N}$. The number T_n^2 of all 2-antichain covers of a labeled n-element set coincides with Stirling numbers of second kind $S(n,3)$ as one can see in Tables 1, 3, and 4 as well as in the *On-Line Encyclopedia of Integer Sequences* [22] — the sequence A000392. Then T_n^2 can be easily determined by the following result.

Proposition 6. *[21, Prop. 2.6.6, p. 40]*

$$T_n^2 = \frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} \left(2^k - 1\right), \text{ for all } n \geq 2. \tag{3}$$

Similarly, numbers of 3-antichain and 4-antichain covers of a labeled n-set are listed in the *On-Line Encyclopedia of Integer Sequences* [22] as the sequences A056046 and A056047, respectively. Jovovic and Kilibarda [10] consider these sequences in detail. In particular, they provide simple formulas for these sequences.

In contrast to the problem of determining T_n, Nelsen and Schmidt [19] relatively easily enumerated the chains in the power set of X. Similarly, Macula [18] easily established the number of all (proper) covers of an n-element set as well as the number of all (proper) covers of length m of an n-element set, where $1 \leq m \leq 2^n - 1$. Clarke [6] considered this problem more generally. In his approach, for a given n-element set X (n-set) and a positive integer k, a k-cover of X is a collection of k (not necessarily distinct) subsets of X, whose union is X. A k-cover is minimal if none of its proper subsets covers X. There are four cases to consider: the element of X may be taken to be identical (X is unlabeled) or distinguishable (X is labeled); the order of the sets comprising the k-cover of X may (cover is ordered) or may not be material (cover is disordered). Clarke quoted formulas derived by Hearne and Wagner [9, 24] for the number of (minimal) ordered k-covers of a labeled n-set, the number of ordered k-covers of an unlabeled n-set, and the number of (minimal) disordered k-covers of a labeled n-set. Moreover, Clarke [6] presented and proved a formula for the number of disordered k-covers of an unlabeled n-set.

Kisielewicz [14] proved that

$$D_n = \sum_{k=1}^{2^{2^n}} \prod_{0 \leq i < j \leq 2^n - 1} \left(1 - b_i^k \cdot b_j^k \cdot f(i,j)\right), \tag{4}$$

where $b_i^n = \lfloor n/2^i \rfloor - 2\lfloor n/2^{i+1} \rfloor$ for any $i, n \geq 0$, $\lfloor x \rfloor := \max\{k \in \mathbb{Z}\colon k \leq x\}$ for any real number $x \in \mathbb{R}$,

$$f(i,n) = \begin{cases} 1, & \text{if } i \in S(n); \\ 0, & \text{otherwise;} \end{cases} = \begin{cases} 1, & \text{if } i = 0; \\ \prod_{m=0}^{\log_2(i)} \left(1 - b_m^i + b_m^i b_m^n\right), & \text{for } i \geq 1; \end{cases}$$

and sets $S(n)$ are defined recursively as follows:

$$\begin{aligned} S(0) &= \{0\} \text{ and for } n \geq 0\colon \\ S(2n) &= \{2s\colon s \in S(n)\} \\ S(2n+1) &= S(2n) \cup \{2s+1\colon s \in S(n)\}. \end{aligned}$$

Church [4] computed the first six elements of the sequence A000372, Ward [25] determined D_6, Church [5] calculated D_7, and Wiedemann [26] computed D_8. In recent years, Bakoev, Fidytek et al. [1, 8] tried to improve the computational time. Since obtaining the exact values of Dedekind numbers and related numbers is a very challenging task, many authors tried to determine upper bounds on the Dedekind numbers. In 1969 Kleitman [15] found an upper bound on the logarithm of the Dedekind number. His result was next improved by Kleitman and Markowsky [16] in 1975. In 1981, Korshunov [17] found more sophisticated and accurate estimates. Kahn [12] simplified proofs of these estimates using an ‘entropy’ approach.

Baumann and Strass [2] proved that the number of bipolar Boolean functions in n arguments is equal to

$$b(n) = \sum_{i=0}^{n} 2^i \cdot \binom{n}{i} \cdot T_i$$

and the number of all monotone Boolean functions in n arguments is equal to D_n. A similar result comes from [21, Prop. 2.6.7, p. 41].

Proposition 7. *Let P be a finite ordered set. Then the number of antichains in P is equal to the number of order-preserving maps from P into the two-element chain.*

De Causmaecker and De Wannemacker [3] generalized Dedekind's problem to analysis of the number of antichains in intervals $[\alpha, \beta] = \{\chi \in \mathcal{A}_N\colon \alpha \leq \chi \leq \beta\}$ in a completely distributive lattice $(\mathcal{A}_N, \wedge, \vee)$ for any $\alpha, \beta \in \mathcal{A}_N$, where $\mathcal{A}_N$ is the set of all antichains in $N \subseteq \mathbb{N}$. Moreover if $N = \{1, 2, \ldots, n\}$ then $\mathcal{A}_N$ is denoted by $\mathcal{A}_n$. Antichains $\alpha, \beta \in \mathcal{A}_N$ are partly ordered as

$$\alpha \leq \beta \iff \forall_{A \in \alpha} \exists_{B \in \beta}\colon A \subseteq B.$$

The operators $\wedge$ and $\vee$ are called *meet* and *join*, respectively, and are defined in the following way:

$$\alpha \vee \beta = \max(\alpha \cup \beta), \quad \alpha \wedge \beta = \max(\{A \cap B\colon A \in \alpha, B \in \beta\})$$

The max-operator on an arbitrary set of sets produces an antichain containing only the non dominated sets. For $X \subseteq N$, $X^- = \{X \setminus \{x\}\colon x \in X\}$ and for antichain α denote

$\alpha^- = \bigvee_{X\in\alpha} X^-$ and $\alpha^+ = \bigvee_{X\in 2^N, X^-\leq\alpha}\{X\}$. For each interval $[\alpha,\beta]$ the underlying interval poset of $[\alpha,\beta]$ is defined as follows

$$\mathcal{P}_{[\alpha,\beta]} = (\{X \subseteq N \colon \alpha \vee \{X\} \in]\alpha,\beta]\}, \subseteq).$$

Given an interval $[\alpha,\beta]$, with $\alpha < \beta$ so that $\mathcal{P}_{[\alpha,\beta]} \neq \emptyset$, let l_0 denote the size of the smallest sets in $\mathcal{P}_{[\alpha,\beta]}$ and $l_i = l_0 + i$. For any $l \in \mathbb{N}$ let $\lambda_l = \{X \in \mathcal{P}_{[\alpha,\beta]} \colon |X| = l\}$. For $\chi \subseteq \lambda_l$ define $\chi^+ = \{X \in \lambda_{l+1} \colon X^- \cap \lambda_l \subseteq \chi\}$ and for $l > l_0$, $\chi^- = \{X^- \cap \lambda_{l-1} \colon X \in \chi\}$. De Causmaecker and De Wannemacker [3] proved that with the above notation, for $\alpha \leq \beta \in \mathcal{A}_N$, the size of the interval $[\alpha,\beta]$ is given by

$$|[\alpha,\beta]| = \sum_{\chi_0\subseteq\lambda_{l_0}} \sum_{\chi_2\subseteq\chi_0^{++}} \sum_{\chi_4\subseteq\chi_2^{++}} \cdots 2^{|\chi_0^+|-|\chi_2^-|+|\chi_2^+|-|\chi_4^-|\cdots} \tag{5}$$

$$= \sum_{\chi_1\subseteq\lambda_{l_1}} \sum_{\chi_3\subseteq\chi_1^{++}} \sum_{\chi_5\subseteq\chi_3^{++}} \cdots 2^{|\lambda_{l_0}|-|\chi_1^-|+|\chi_1^+|-|\chi_3^-|\cdots} \tag{6}$$

Let $\mathcal{B}_n \subseteq \mathcal{A}_n$ denote a basic interval of dimension n, i.e.,

$$\mathcal{B}_n = [\{\{1\},\{2\},\ldots,\{n\}\},\{\{1,2,\ldots,n\}\}].$$

Then from the well-known decomposition for the Dedekind numbers $D_n = |\mathcal{A}_n| = \sum_{k=0}^{n} |\mathcal{B}_k|$ and from (1) it follows that $T_n = |\mathcal{B}_n|$ for all natural numbers n. Thus both T_n and D_n can be efficiently computed by (5) or (6) up to $n = 6$.

Kilibarda [13] presents another generalization of Dedekind's problem. He defines an antichain as a hypergraph satisfying the following property: for every pair of distinct edges, neither one is a subset of the other. Kilibarda [13] enumerates antichains given on an n-set having some of the following properties: being labeled or unlabeled; being ordered or unordered; being a cover or a proper cover; and finally, being a T_0-, T_1- or T_2-hypergraph.

The following part of this article is organized as follows. The second section is devoted to the derivation of the new elegant result — Theorem 11, concerning numbers D_n^m and T_n^m for all $m, n \geq 2$ and $m \leq \binom{n}{\lfloor n/2\rfloor}$. This theorem is preceded by Example 10, which allows easier comprehension of the intuition behind Theorem 11. Proof of Theorem 11 is divided into four Lemmas 14–17 that explicitly and accurately explain the computational process to obtain the formulae (39) and (40) from this theorem.

The third section concerns the number of covers of a labeled n-element set X with antichains consisting of m equinumerous subsets of X.

The last section contains Example 20 demonstrating the use of Theorem 11 for computing the number of all 2 and 3-antichain covers as well as the number of all 2 and 3 element antichains in the power set $\mathcal{P}_n$. In this section, some numerical experiments are also shown. In particular, this section presents:

- values of A016269 — sequence of numbers of monotone Boolean functions of $n-2$ variables with 2 mincuts, that is, sequence of numbers D_n^2 for $n \in \{4,\ldots,15\}$;

- values of sequences A047707 and A051112 of numbers of monotone Boolean functions of n variables with 3 and 4 mincuts, respectively, that is, sequences of numbers D_n^3 and D_n^4 for $n \in \{4, \dots, 15\}$;
- values of sequence A000392 of Stirling numbers of second kind $S(n+1, 3)$, i.e., sequence of numbers T_n^2 for $n \in \{4, \dots, 50\}$;
- values of sequence A056046 of numbers T_n^3 for $n \in \{4, \dots, 50\}$;
- values of sequence A056047 of numbers T_n^4 for $n \in \{4, \dots, 15\}$;
- the numbers of covers of 7-set with antichains consisting of equinumerous sets.

2 Antichains and antichain covers consisting of m subsets of a labeled n-element set

Let $\mathbb{Z}_n$ denote the set $\{0, 1, \dots, n-1\}$, and we let $\mathbb{Z}_n^+$ denote the set $\{1, 2, \dots, n\}$, for all $n \in \mathbb{N}$. Let us denote the cardinality of a set X by $\#X$. Let $\mathcal{A}_n^m$ denote the set of all m-element antichains in the power set $\mathcal{P}_n$, and we let $\mathcal{AC}_n^m$ denote the set of all m-element antichain covers in $\mathcal{P}_n$.

The following lemma will be used later in this article. In particular, the lemma allows us to determine T_n from $D_0, \dots, D_n$ using formula (1).

Lemma 8. *Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $(a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0}$ be sequences such that*

$$b_n = \sum_{k=n_0}^{n} \binom{n}{k} a_k \tag{7}$$

for some $n_0 \in \mathbb{N}_0$ and for all $n \geq n_0$. Then

$$a_n = \sum_{k=0}^{n-n_0} \binom{n}{n-k} \cdot (-1)^k \cdot b_{n-k} \tag{8}$$

for all natural numbers $n \geq n_0$.

This lemma follows from the following more general result: let X be a finite set, and let $f, g \colon 2^X \to \mathbb{N}_0$ be two functions such that

$$\forall_{A \subseteq X} \colon g(A) = \sum_{B \subseteq A} f(B).$$

Then

$$\forall_{A \subseteq X} f(A) = \sum_{B \subseteq A} (-1)^{\#A - \#B} g(B).$$

We obtain Lemma 8 by putting

$$f(A) := \begin{cases} 0, & \text{if } \#A < n_0; \\ a_n, & \text{if } \#A = n \geq n_0; \end{cases}$$
$$g(A) := \begin{cases} 0, & \text{if } \#A < n_0; \\ b_n, & \text{if } \#A = n \geq n_0; \end{cases},$$

but we can also prove this lemma by an inductive procedure.

Proof. Equation (8) is satisfied for $n = n_0$, since by (7), $b_{n_0} = a_{n_0}$. Assume inductively that equation (8) holds for all natural numbers m such that $n_0 \leq m \leq n$ for some natural number $n \geq n_0$. It will be shown that (8) is true for $n+1$, i.e.,

$$a_{n+1} = b_{n+1} - \sum_{k=0}^{n-n_0} \binom{n+1}{n-k} \cdot (-1)^k \cdot b_{n-k} \tag{9}$$

By (7), $b_{n+1} = \sum_{k=n_0}^{n} \binom{n+1}{k} a_k + a_{n+1}$, and hence by the induction hypothesis, we get

$$a_{n+1} = b_{n+1} - \sum_{k=n_0}^{n} \binom{n+1}{k} \sum_{i=0}^{k-n_0} \binom{k}{k-i} \cdot (-1)^i \cdot b_{k-i}.$$

Hence after putting $s := k - i$ we obtain

$$a_{n+1} = b_{n+1} - \sum_{k=n_0}^{n} \binom{n+1}{k} \sum_{s=n_0}^{k} \binom{k}{s} (-1)^{k-s} b_s.$$

After changing the order of summation, we have

$$a_{n+1} = b_{n+1} - \sum_{s=n_0}^{n} \sum_{k=s}^{n} \binom{n+1}{k} \binom{k}{s} (-1)^{k-s} b_s.$$

Thus, after putting $t := n - s$, we obtain

$$a_{n+1} = b_{n+1} - \sum_{t=0}^{n-n_0} \sum_{k=n-t}^{n} \binom{n+1}{k} \binom{k}{n-t} (-1)^{k+t-n} b_{n-t} \tag{10}$$

We shall show that

$$\sum_{k=n-t}^{n} \binom{n+1}{k} \binom{k}{n-t} (-1)^{k-n} = \binom{n+1}{n-t}, \text{ for all } t \in \{0, \ldots, n-n_0\}. \tag{11}$$

Note that

$$\begin{aligned}
&\sum_{k=n-t}^{n} \binom{n+1}{k}\binom{k}{n-t}(-1)^{k-n} \\
&= \sum_{k=n-t}^{n} \frac{(n+1)!}{k!(n+1-k)!} \cdot \frac{k!}{(n-t)!(k+t-n)!}(-1)^{k-n} \\
&= \sum_{k=n-t}^{n} \frac{(n+1)!}{(n-t)!(t+1)!} \cdot \frac{(t+1)!}{(n+1-k)!(k+t-n)!}(-1)^{k-n} \\
&= \sum_{l=0}^{t} \binom{n+1}{n-t}\binom{t+1}{l+1}(-1)^{l} \quad \text{(by setting } l := n-k) \\
&= \binom{n+1}{n-t},
\end{aligned}$$

since $0 = (1-1)^{t+1} = \sum_{l=0}^{t+1} \binom{t+1}{l}(-1)^l$. By (10) and (11), we obtain (9). So by induction, (8) holds for all $n \geq n_0$. □

For small numbers n, formulae (1) and (4), as well as Lemma 8 can be used to compute T_n:

Corollary 9. *We have*

$$T_n = \sum_{k=0}^{n} \binom{n}{n-k}(-1)^k \sum_{l=1}^{2^{2^{n-k}}} \prod_{0 \leq i < j \leq 2^{n-k}-1} \left(1 - b_i^l \cdot b_j^l \cdot f(i,j)\right),$$

where $b_i^n = \lfloor n/2^i \rfloor - 2\lfloor n/2^{i+1} \rfloor$ *for any* $i, n \geq 0$ *and*

$$f(i,n) = \begin{cases} 1, & \text{if } i = 0; \\ \prod_{m=0}^{\log_2(i)} \left(1 - b_m^i + b_m^i b_m^n\right), & \text{for } i \geq 1. \end{cases}$$

The following theorem allows computing the number of all m-element antichains and antichain covers in the power set $\mathcal{P}_n$. Before formulating this theorem the simple example below is presented. It allows us easier comprehension of the intuition behind Theorem 11.

Example 10. We now demonstrate the method for computing T_n^m, as well as D_n^m for $m \in \{2,3\}$ and for all natural numbers $n \geq 3$. Theorem 11 is a generalization of this method for all natural numbers $n \geq 2$ and $2 \leq m \leq \binom{n}{\lfloor n/2 \rfloor}$.

Firstly, we shall show that for all sets $\emptyset \neq A^1, A^2 \subsetneq \mathbb{Z}_n^+$ if

$$\#A^1 \leq \#A^2, \tag{12}$$

then $\{A^1, A^2\} \in \mathcal{A}_n^2$, if and only if A^2 can be decomposed into two subsets A_1^2 and A_2^2 of $\mathbb{Z}_n^+$ satisfying the following conditions:

$$A^2 = A_1^2 \cup A_2^2 \tag{13}$$

$$A_1^2 \subsetneq A^1 \tag{14}$$

$$A_2^2 \subseteq \mathbb{Z}_n^+ \setminus A^1 \tag{15}$$

Assume that $\{A^1, A^2\} \in \mathcal{A}_n^2$ and put $A_1^2 := A^2 \cap A^1$ and $A_2^2 := A^2 \cap (\mathbb{Z}_n^+ \setminus A^1)$. Then (13) and (15) are straightforwardly satisfied. Moreover, $A_1^2 \subseteq A^1$. If A_1^2 were equal to A^1, then A^1 would be included in A^2 and it would contradict the assumption $\{A^1, A^2\} \in \mathcal{A}_n^2$. Thus (14) holds. For the proof of the converse implication assume that A^1, A^2 satisfy conditions (13)–(15). It follows from (14), that there exists $x \in A^1 \setminus A_1^2$. Then by (15), such $x \notin A_2^2$ and by (13), $x \notin A^2$. So $A^1 \not\subseteq A^2$. Similarly, $A^2 \not\subseteq A^1$, since otherwise by (13) and (15), A_2^2 would have to be equal to $\emptyset$, but then $A^2 = A_1^2$ and by (14), $\#A^2 < \#A^1$ — a contradiction to our primary assumption (12).

Analogously, for all sets $\emptyset \neq A^1, A^2 \subsetneq \mathbb{Z}_n^+$ if (12) is satisfied, then $\{A^1, A^2\} \in \mathcal{AC}_n^2$ if and only if A^2 can be decomposed into two subsets A_1^2 and A_2^2 of $\mathbb{Z}_n^+$ satisfying conditions (13)–(14). However, instead of condition (15) A_2^2 satisfies the condition:

$$A_2^2 = \mathbb{Z}_n^+ \setminus A^1. \tag{16}$$

If $\emptyset \neq A^1, A^2 \subsetneq \mathbb{Z}_n^+$ and A^1, A^2, A_1^2, A_2^2 satisfy conditions (12) as well as (13)–(15) and we denote $\#A^1$ by a^1, $\#A^2$ by a^2, and $\#A_i^2$ by a_i^2 for $i \in \{1, 2\}$, then we obtain the following bounds:

$$\begin{cases} 1 \;\leq\; a^1 \;\leq\; a^2 \;\leq\; n-1 \\ \quad\;\; 0 \;\leq\; a_1^2 \;\leq\; a^1 - 1 \\ \quad\;\; 0 \;\leq\; a_2^2 \;\leq\; n - a^1 \text{ (or } a_2^2 = n - a^1 \text{ for } \mathcal{AC}_n^2) \end{cases} \tag{17}$$

Then $a_1^2 \geq a^1 - a_2^2 \geq a^1 - (n - a^1) = 2a^1 - n$ and $a_2^2 \geq a^1 - a_1^2 \geq a^1 - (a^1 - 1) = 1$.

If $a^1 < a^2$ then the number of all families of sets $\{A^1, A^2\} \in \mathcal{A}_n^2$, such that A^1, A^2, A_1^2, A_2^2 satisfy conditions (13)–(15) as well as $a^1 = \#A^1$, $a^2 = \#A^2$, $a_1^2 = \#A_1^2$, and $a_2^2 = \#A_2^2$, equals $\binom{n}{a^1}\binom{a^1}{a_1^2}\binom{n-a^1}{a_2^2}$. Otherwise, i.e., if $a^1 = a^2$ then the number of all such families of sets $\{A^1, A^2\} \in \mathcal{A}_n^2$, equals $\binom{n}{a^1}\binom{a^1}{a_1^2}\binom{n-a^1}{a_2^2}/2$, since $\{A^1, A^2\} = \{A^2, A^1\}$. However, if $\{A^1, A^2\} \in \mathcal{AC}_n^2$, then $a_2^2 = n - a^1$. Hence we obtain the following formulas:

$$D_n^2 = \sum_{a^1=1}^{n-1} \; \sum_{a_1^2=\max(0,2a^1-n)}^{a^1-1} \; \sum_{a_2^2=a^1-a_1^2}^{n-a^1} \frac{\binom{n}{a^1}\binom{a^1}{a_1^2}\binom{n-a^1}{a_2^2}}{r(a^1, a^2)} \tag{18}$$

$$T_n^2 = \sum_{a^1=1}^{n-1} \; \sum_{a_1^2=\max(0,2a^1-n)}^{a^1-1} \frac{\binom{n}{a^1}\binom{a^1}{a_1^2}}{r(a^1, a^2)}, \tag{19}$$

where $r(a, b) = \begin{cases} 1, & \text{if } a < b; \\ 2, & \text{otherwise.} \end{cases}$

Similarly, we shall show that for all sets $\emptyset \neq A^1, A^2, A^3 \subsetneq \mathbb{Z}_n^+$ if

$$\#A^1 \leq \#A^2 \leq \#A^3, \tag{20}$$

then $\{A^1, A^2, A^3\} \in \mathcal{A}_n^3$, if and only if A^2 can be decomposed into two sets satisfying conditions (13)–(15) and A^3 can be decomposed into four subsets A_1^3, A_2^3, A_3^3, A_4^3 of $\mathbb{Z}_n^+$ satisfying the following conditions:

$$A^3 = A_1^3 \cup A_2^3 \cup A_3^3 \cup A_4^3 \tag{21}$$
$$A_1^3 \subseteq A_1^2 \tag{22}$$
$$A_2^3 \subseteq A^1 \setminus A_1^2 \tag{23}$$
$$A_3^3 \subseteq A_2^2 \tag{24}$$
$$A_4^3 \subseteq \mathbb{Z}_n^+ \setminus A^1 \setminus A_2^2 \tag{25}$$
$$A_1^3 \cup A_2^3 \subsetneq A^1 \tag{26}$$
$$A_1^3 \cup A_3^3 \subsetneq A^2 \tag{27}$$

Assume that $\{A^1, A^2, A^3\} \in \mathcal{A}_n^3$. Then $\{A^1, A^2\} \in \mathcal{A}_n^2$. From the previous reasoning it follows that A^2 can be decomposed into two sets A_1^2 and A_2^2 satisfying conditions (13)–(15). Put $A_1^3 := A^3 \cap A_1^2$, $A_2^3 := A^3 \cap (A^1 \setminus A_1^2)$, $A_3^3 := A^3 \cap A_2^2$ and $A_4^3 := A^3 \cap (\mathbb{Z}_n^+ \setminus A^1 \setminus A_2^2)$. Then straightforwardly conditions (22)–(25) are satisfied. Moreover, $A_1^3 \cup A_2^3 = A^3 \cap A^1$ and $A_3^3 \cup A_4^3 = A^3 \setminus A^1$. So condition (21) also holds and $A_1^3 \cup A_2^3 \subseteq A^1$. Suppose that $A_1^3 \cup A_2^3 = A^1$, then $A^3 \cap A^1 = A^1$. Hence $A^1 \subseteq A^3$ — a contradiction to the assumption $\{A^1, A^2, A^3\} \in \mathcal{A}_n^3$. Therefore (26) is true. By (13), $A_1^3 \cup A_3^3 = A^3 \cap A^2$. Thus If $A_1^3 \cup A_3^3 = A^2$, then $A^2 \subseteq A^3$ — a contradiction to the assumption $\{A^1, A^2, A^3\} \in \mathcal{A}_n^3$. Therefore condition (27) is also satisfied.

Assume that A^2 can be decomposed into two sets satisfying conditions (13)–(15) and A^3 can be decomposed into four subsets satisfying conditions (21)–(27). We shall prove that $\{A^1, A^2, A^3\} \in \mathcal{A}_n^3$. By the previous reasoning $\{A^1, A^2\} \in \mathcal{A}_n^2$. By the assumption $\#A^1 \leq \#A^2 \leq \#A^3$ it suffices to show that $A^1 \nsubseteq A^3$ and $A^2 \nsubseteq A^3$. By (15),(21), and (24)–(26), $A^3 \cap A^1 = A_1^3 \cup A_2^3 \neq A^1$. So $A^1 \nsubseteq A^3$. Similarly, by (13)–(15), (21)–(25), $A^3 \cap A^2 = A_1^3 \cup A_3^3$. Hence by (27), $A^3 \cap A^2 \neq A^2$. So $A^2 \nsubseteq A^3$.

Analogously, for all sets $\emptyset \neq A^1, A^2, A^3 \subsetneq \mathbb{Z}_n^+$ if condition (20) is satisfied, then

$$\{A^1, A^2, A^3\} \in \mathcal{AC}_n^3$$

if and only if A^2 can be decomposed into two sets satisfying conditions (13)–(15) as well as A^3 can be decomposed into four subsets $A_1^3, A_2^3, A_3^3, A_4^3$ of $\mathbb{Z}_n^+$ satisfying conditions (21)–(27). However, instead of condition (25) A_4^3 satisfies the condition:

$$A_4^3 = \mathbb{Z}_n^+ \setminus A^1 \setminus A_2^2. \tag{28}$$

If A^1, A^2, A^3, A_1^2, A_2^2, A_1^3, A_2^3, A_3^3, A_4^3 satisfy conditions (20), (13)–(15) and (21)–(27), we adopt the notation from the previous paragraph and we denote $\#A_i^3$ by a_i^3 for $i \in \{1, 2, 3, 4\}$, then we obtain the following bounds for a^3 and a^3_i:

$$\begin{cases} 0 & \le & a^3_1 & \le & a^2_1 \\ 0 & \le & a^3_2 & \le & a^1 - a^2_1 \\ 0 & \le & a^3_3 & \le & a^2_2 \\ 0 & \le & a^3_4 & \le & n - a^1 - a^2_2 \text{ (or } a^3_4 = n - a^1 - a^2_2 \text{ for } \mathcal{AC}^3_n) \\ & & a^3_1 + a^3_2 & \le & a^1 - 1 \\ & & a^3_1 + a^3_3 & \le & a^2 - 1 \\ a^2 & \le & a^3 & \le & n-1 \end{cases} \tag{29}$$

If $a^1 < a^2 < a^3$ then the number of all families of sets $\{A^1, A^2, A^3\} \in \mathcal{A}^3_n$, such that A^1, A^2, A^3, A^2_1, A^2_2, A^3_1, A^3_2, A^3_3, A^3_4 satisfy conditions (13)–(15) and (21)–(27), as well as $a^1 = \#A^1$, $a^2 = \#A^2$, $a^2_1 = \#A^2_1$, $a^2_2 = \#A^2_2$, $a^3_i = \#A^3_i$ for $i \in \{1, 2, 3, 4\}$, equals

$$\binom{n}{a^1}\binom{a^1}{a^2_1}\binom{n-a^1}{a^2_2}\binom{a^2_1}{a^3_1}\binom{a^1-a^2_1}{a^3_2}\binom{a^2_2}{a^3_3}\binom{n-a^1-a^2_2}{a^3_4}.$$

If $a^1 < a^2 = a^3$ or $a^1 = a^2 < a^3$ then the number of all such families of sets $\{A^1, A^2, A^3\} \in \mathcal{A}^3_n$, equals

$$\binom{n}{a^1}\binom{a^1}{a^2_1}\binom{n-a^1}{a^2_2}\binom{a^2_1}{a^3_1}\binom{a^1-a^2_1}{a^3_2}\binom{a^2_2}{a^3_3}\binom{n-a^1-a^2_2}{a^3_4}/2.$$

If $a^1 = a^2 = a^3$ then the number of all such families of sets $\{A^1, A^2, A^3\} \in \mathcal{A}^3_n$, equals

$$\binom{n}{a^1}\binom{a^1}{a^2_1}\binom{n-a^1}{a^2_2}\binom{a^2_1}{a^3_1}\binom{a^1-a^2_1}{a^3_2}\binom{a^2_2}{a^3_3}\binom{n-a^1-a^2_2}{a^3_4}/3!.$$

However, if $\{A^1, A^2, A^3\} \in \mathcal{AC}^3_n$, then $a^3_4 = n - a^1 - a^2_2$. Hence we obtain the following formulae:

$$D^3_n = \sum_{a^1=1}^{n-1} \sum_{a^2_1=\max(0,2a^1-n)}^{a^1-1} \sum_{a^2_2=a^1-a^2_1}^{n-a^1} \sum_{(a^3_1,a^3_2,a^3_3,a^3_4) \text{ satisfying conditions (29)}} \frac{\binom{n}{a^1}\binom{a^1}{a^2_1}\binom{n-a^1}{a^2_2}\binom{a^2_1}{a^3_1}\binom{a^1-a^2_1}{a^3_2}\binom{a^2_2}{a^3_3}\binom{n-a^1-a^2_2}{a^3_4}}{r(a^1,a^2,a^3)} \tag{30}$$

$$T^3_n = \sum_{a^1=1}^{n-1} \sum_{a^2_1=\max(0,2a^1-n)}^{a^1-1} \sum_{(a^3_1,a^3_2,a^3_3,a^3_4) \text{ satisfying conditions (29)}} \frac{\binom{n}{a^1}\binom{a^1}{a^2_1}\binom{n-a^1}{a^2_2}\binom{a^2_1}{a^3_1}\binom{a^1-a^2_1}{a^3_2}\binom{a^2_2}{a^3_3}}{r(a^1,a^2,a^3)}, \tag{31}$$

where $r(a,b,c) = \begin{cases} 1, & \text{if } a < b < c; \\ 2, & \text{if } a < b = c \text{ or } a = b < c;. \\ 3!, & \text{if } a = b = c. \end{cases}$

Theorem 11. *Let*

$$I_i^m := \bigcup_{k=0}^{2^{i-1}-1} \left[2k \cdot 2^{m-1-i} + 1; (2k+1) \cdot 2^{m-1-i}\right] \cap \mathbb{N} \tag{32}$$

for $m \geq 2$ *and* $i \in \mathbb{Z}_{m-1}^+$. *For a natural number* n *and sequence of sequences of natural numbers* $(\underline{\boldsymbol{a}}^1, \underline{\boldsymbol{a}}^2, \ldots)$ *such that* $\underline{\boldsymbol{a}}^m = (a_1^m, \ldots, a_{2^{m-1}}^m) \in \mathbb{N}_0^{2^{m-1}}$ *for* $m \in \mathbb{N}$ *define:*

$$a^m = \sum_{k=1}^{2^{m-1}} a_k^m \tag{33}$$

$$b_1^1 := n \tag{34}$$

$$b_k^m := \begin{cases} a_{(k+1)/2}^{m-1}, & \text{if } 2 \nmid k \\ b_{k/2}^{m-1} - a_{k/2}^{m-1}, & \text{if } 2 \mid k \end{cases}, \text{for } m \geq 2 \text{ and } k \in \mathbb{Z}_{2^{m-1}}^+ \tag{35}$$

$$S_m^n(\underline{\boldsymbol{a}}^1, \ldots, \underline{\boldsymbol{a}}^{m-1}) := \left\{ \begin{array}{ll} \underline{\boldsymbol{a}}^m \in \mathbb{N}_0^{2^{m-1}} : 0 \leq a_k^m \leq b_k^m, & \text{for } k \in \mathbb{Z}_{2^{m-1}}^+ \& \\ a^{m-1} \leq a^m \leq n-1 & \& \\ \sum_{k \in I_i^m} a_k^m \leq a^i - 1, & \text{for } i \in \mathbb{Z}_{m-1}^+ \end{array} \right\}, \tag{36}$$

for $m \geq 2$

$$\overline{S_m^n}(\underline{\boldsymbol{a}}^1, \ldots, \underline{\boldsymbol{a}}^{m-1}) := \left\{\underline{\boldsymbol{a}}^m \in S_m^n(\underline{\boldsymbol{a}}^1, \ldots, \underline{\boldsymbol{a}}^{m-1}) \colon a_{2^{m-1}}^m = b_{2^{m-1}}^m\right\}, \text{ for } m \geq 2 \tag{37}$$

For a non-decreasing sequence of natural numbers $(a_1, \ldots, a_m)$, *natural numbers* $m_1, \ldots, m_k$, *and* $M_1, \ldots, M_k$ *such that:*

1. $m_1 + m_2 + \cdots + m_k = m$
2. $M_1 = 0$
3. $M_i := m_1 + \cdots + m_{i-1}$ *for* $2 \leq i \leq k$
4. $\forall_{i \in \mathbb{Z}_k^+} \forall_{j \in \mathbb{Z}_{m_i}^+} a_{M_i+j} = a_{M_i+1}$
5. $\forall_{i \in \{2,\ldots,k\}} a_{M_i} < a_{M_i+1}$.

define function

$$r(a_1, \ldots, a_m) := \prod_{i=1}^{k} m_i!. \tag{38}$$

With the above notation, for all natural numbers $m, n \geq 2$ *if* $m \leq \binom{n}{\lfloor n/2 \rfloor}$ *then:*

$$D_n^m = \sum_{a^1=1}^{n-1} \sum_{\underline{\boldsymbol{a}}^2 \in S_2^n(a^1)} \cdots \sum_{\underline{\boldsymbol{a}}^{m-1} \in S_{m-1}^n(\underline{\boldsymbol{c}}^{m-2})} \sum_{\underline{\boldsymbol{a}}^m \in S_m^n(\underline{\boldsymbol{c}}^{m-1})} \prod_{i=1}^{m} \prod_{j=1}^{2^{i-1}} \frac{\binom{b_j^i}{a_j^i}}{r(a^1, \ldots, a^m)} \tag{39}$$

$$T_n^m = \sum_{a^1=1}^{n-1} \sum_{\underline{\boldsymbol{a}}^2 \in S_2^n(a^1)} \cdots \sum_{\underline{\boldsymbol{a}}^{m-1} \in S_{m-1}^n(\underline{\boldsymbol{c}}^{m-2})} \sum_{\underline{\boldsymbol{a}}^m \in \overline{S_m^n}(\underline{\boldsymbol{c}}^{m-1})} \prod_{i=1}^{m} \prod_{j=1}^{2^{i-1}} \frac{\binom{b_j^i}{a_j^i}}{r(a^1, \ldots, a^m)} \tag{40}$$

where $\underline{\boldsymbol{c}}^k = \left(\underline{\boldsymbol{a}}^1, \underline{\boldsymbol{a}}^2, \dots, \underline{\boldsymbol{a}}^k\right)$.

Remark 12. The set I_i^m defined in (32) for $m \geq 2$ and $i \in \mathbb{Z}_{m-1}^+$ is intuitively the set of indices of sets belonging to the partition $\{A_1^m, \dots, A_{2^{m-1}}^m\}$ of set A^m, which are included in A^i. Properties of these partitions are described by (13)–(15) and (21)–(27) in Example 10 for A^2 and A^3, respectively, as well as in Lemma 17 occurring later in this article. So the condition $\sum_{k\in I_i^m} a_k^m \leq a^i - 1$ from (36) can be translated in the context of these partitions as $\bigcup_{k\in I_i^m} A_k^m \subsetneq A^i$ and it prevents $A^i \subseteq A^m$ for all $i \in \mathbb{Z}_{m-1}^+$. Thus this condition guarantees that sets $A^1, \dots, A^m$, consisting of these partitions form an antichain.

The following proposition gives an explicit formula for expressions defined recursively by (34) and (35).

Proposition 13. *Let* $m, n \in \mathbb{N}$, $a_1^0 := n$ *and* b_k^m *be defined by* (34) *and* (35). *Then*

$$b_k^m = a_{(r+1)/2}^{m-l-1} - \sum_{i=1}^{l} a_{2^{l-i}r}^{m-i}. \tag{41}$$

for all $r \in \mathbb{N}$ *and* $l \in \mathbb{Z}_m$ *such that* $2 \nmid r$, $k = 2^l \cdot r$, *and* $k \leq 2^{m-1}$.

Proof. The proof is by induction on m. For the base step of the induction, assume that $m = 1$. Then $k = 1 = 2^0$ and by (34), $b_1^1 = n = a_1^0 = a_{(r+1)/2}^{m-l-1}$ for $l = 0$ and $r = 1$.

Assume that (41) holds for some $m \in \mathbb{N}$. We shall show that (41) holds for $m+1$. We shall consider two cases. Firstly, let us assume that $2 \nmid k$. Then by (35), $b_k^{m+1} = a_{(k+1)/2}^m = a_{(r+1)/2}^{m+1-l-1}$ for $l = 0$ and $r = k$. Therefore in this case (41) is true.

Assume that $k = 2^l \cdot r$, where $0 < l \leq m$ and $2 \nmid r$. Then by (35),

$$b_k^{m+1} = b_{2^{l-1}\cdot r}^m - a_{2^{l-1}\cdot r}^m. \tag{42}$$

By the induction hypothesis,

$$b_{2^{l-1}\cdot r}^m = a_{(r+1)/2}^{m-(l-1)-1} - \sum_{i=1}^{l-1} a_{2^{l-1-i}r}^{m-i}. \tag{43}$$

Therefore by (42) and (43),

$$\begin{aligned} b_k^{m+1} &= a_{(r+1)/2}^{m-(l-1)-1} - \sum_{i=1}^{l-1} a_{2^{l-1-i}r}^{m-i} - a_{2^{l-1}\cdot r}^m \\ &= a_{(r+1)/2}^{m+1-l-1} - \sum_{i=1}^{l} a_{2^{l-i}r}^{m+1-i}. \end{aligned}$$

So (41) holds for $m+1$, and by an inductive procedure, (41) is true for all $m \in \mathbb{N}$. $\square$

Proof of Theorem 11.

Lemma 14. *Let I_i^m be defined by (32). Then*

$$\bigcup_{i \in \mathbb{Z}_{m-1}^+} I_i^m = \mathbb{Z}_{2^{m-1}}^+ \setminus \left\{2^{m-1}\right\} \tag{44}$$

for all natural numbers $m \geq 2$.

Proof. Fix any natural number $m \geq 2$ and $x \in \mathbb{Z}_{2^{m-1}}^+ \setminus \{2^{m-1}\}$. Let

$$i := m - 1 - \lfloor \log_2(2^{m-1} - x) \rfloor \tag{45}$$

and $k := 2^{i-1} - 1$. Note that $i \in \mathbb{Z}_{m-1}^+$. We shall show that

$$k \cdot 2^{m-i} < x \leq k \cdot 2^{m-i} + 2^{m-i-1} :$$

Note that

$$k \cdot 2^{m-i} = 2^{m-1} - 2 \cdot 2^{\lfloor \log_2(2^{m-1}-x) \rfloor}, \tag{46}$$

and

$$2^{m-1} - 2 \cdot 2^{\lfloor \log_2(2^{m-1}-x) \rfloor} < 2^{m-1} - 2 \cdot 2^{\log_2(2^{m-1}-x)-1} = x \tag{47}$$

$$2^{m-1} - 2 \cdot 2^{\lfloor \log_2(2^{m-1}-x) \rfloor} \geq 2^{m-1} - 2 \cdot 2^{\log_2(2^{m-1}-x)} = 2x - 2^{m-1} \tag{48}$$

Thus by (46) and (47) we have $k \cdot 2^{m-i} < x$ and

$$\begin{aligned} x &\leq 2^{m-1} - 2^{\lfloor \log_2(2^{m-1}-x) \rfloor}, \text{by (48)} \\ &= k \cdot 2^{m-i} + 2^{\lfloor \log_2(2^{m-1}-x) \rfloor}, \text{by (46)} \\ &= k \cdot 2^{m-i} + 2^{m-i-1}, \text{by (45)} \end{aligned}$$

Therefore $x \in I_i^m$. □

Let $m, n \geq 2$ be natural numbers and

$$\begin{aligned} \mathfrak{A}_m^n := \Bigg\{ &\left\{ \left(A^1, A^2, \ldots, A^m\right), \left\{ \left(A_1^k, \ldots, A_{2^{k-1}}^k, B_1^k, \ldots, B_{2^{k-1}}^k\right) : k \in \mathbb{Z}_m^+ \right\} \right\} : \\ &\forall_{k \in \mathbb{Z}_m^+} \left(\emptyset \neq A^k \subsetneq \mathbb{Z}_n^+ \quad \wedge \quad \#A^{k-1} \leq \#A^k \quad \wedge \quad \forall_{i \in \mathbb{Z}_{2^{k-1}}^+} : \left(A_i^k \subseteq B_i^k \subseteq \mathbb{Z}_n^+\right) \right) \Bigg\}, \end{aligned} \tag{49}$$

where $A^0 := \emptyset$.

Lemma 15. *Let $m, n \geq 2$ be natural numbers and $\mathcal{A} \in \mathfrak{A}_m^n$. If $\mathcal{A}$ satisfies condition*

$$B_i^k = \begin{cases} A_{(i+1)/2}^{k-1}, & \text{if } 2 \nmid i; \\ B_{i/2}^{k-1} \setminus A_{i/2}^{k-1}, & \text{otherwise}; \end{cases} \tag{50}$$

for all $k \in \{2, \ldots, m\}$ and $i \in \mathbb{Z}_{2^{k-1}}^+$, where $B_1^1 = \mathbb{Z}_n^+$ and $A_1^1 = A^1$. Then for all $k \in \{2, \ldots, m\}$:

$$\bigcup_{i \in \mathbb{Z}_{2^{k-1}}^+} B_i^k = \mathbb{Z}_n^+ \tag{51}$$

$$B_i^k \cap B_j^k = \emptyset, \text{ for all distinct } i, j \in \mathbb{Z}_{2^{k-1}}^+ \tag{52}$$

Proof. The proof is by induction on k. For $k = 2$ by (50), $B_1^2 = A_1^1 = A^1$ and $B_2^2 = B_1^1 \setminus A_1^1 = \mathbb{Z}_n^+ \setminus A^1$. Thus (51) and (52) hold.

Assume that (51) and (52) are satisfied for some natural number $2 \leq k < m$. We shall show that (51) holds for $k+1$: By (50) and the induction hypothesis we have

$$\begin{aligned} \bigcup_{i \in \mathbb{Z}_{2^k}^+} B_i^{k+1} &= \bigcup_{i \in \mathbb{Z}_{2^{k-1}}^+} \left(B_{2i-1}^{k+1} \cup B_{2i}^{k+1}\right) \\ &= \bigcup_{i \in \mathbb{Z}_{2^{k-1}}^+} \left(A_i^k \cup \left(B_i^k \setminus A_i^k\right)\right) = \bigcup_{i \in \mathbb{Z}_{2^{k-1}}^+} B_i^k = \mathbb{Z}_n^+. \end{aligned}$$

We shall show that (52) holds for $k+1$: Assume that $i, j \in \mathbb{Z}_{2^k}^+$ and $i \neq j$.

1. If $2 \nmid i$ and $2 \nmid j$ then by (50) and finally by the induction hypothesis,

$$B_i^{k+1} \cap B_j^{k+1} = A_{(i+1)/2}^k \cap A_{(j+1)/2}^k \subseteq B_{(i+1)/2}^k \cap B_{(j+1)/2}^k = \emptyset.$$

2. If $2 \nmid i$ and $2 \mid j$ then

   - if $(i+1)/2 = j/2$ then by (50), we have

$$B_i^{k+1} \cap B_j^{k+1} = A_{j/2}^k \cap (B_{j/2}^k \setminus A_{j/2}^k) = \emptyset;$$

   - if $(i+1)/2 \neq j/2$, then by (50) and the induction hypothesis we have

$$B_i^{k+1} \cap B_j^{k+1} = A_{(i+1)/2}^k \cap (B_{j/2}^k \setminus A_{j/2}^k) \subseteq B_{(i+1)/2}^k \cap B_{j/2}^k = \emptyset.$$

3. If $2 \mid i$ and $2 \mid j$ then by (50) and the induction hypothesis we have

$$B_i^{k+1} \cap B_j^{k+1} = (B_{i/2}^k \setminus A_{i/2}^k) \cap (B_{j/2}^k \setminus A_{j/2}^k) \subseteq B_{i/2}^k \cap B_{j/2}^k = \emptyset.$$

□

Lemma 16. *Let $m, n \geq 2$ be natural numbers and $\mathcal{A} \in \mathfrak{A}^n_m$. If $\mathcal{A}$ satisfies condition (50) and the following condition:*

$$A^k = \bigcup_{i=1}^{2^{k-1}} A^k_i \tag{53}$$

for all $k \in \{2, \dots, m\}$. Then

$$\bigcup_{i \in I^k_j} B^k_i = A^j, \text{ for all } j \in \mathbb{Z}^+_{k-1} \tag{54}$$

$$B^k_{2^{k-1}} = \mathbb{Z}^+_n \setminus \bigcup_{i=1}^{k-1} A^i \tag{55}$$

for all natural numbers $2 \leq k \leq m$.

Proof. The proof of (54) is by induction on k. For $k = 2$, by (32), $I^2_1 = \{1\}$. Hence $\bigcup_{i \in I^2_1} B^2_i = B^2_1 = A^1$. Therefore (54) holds.

Assume that(54) is satisfied for some natural number k such that $2 \leq k < m$. Firstly we shall prove that $\bigcup_{i \in I^{k+1}_k} B^{k+1}_i = A^k$: By (32), $I^{k+1}_k = \{2i + 1 \colon i \in \mathbb{Z}_{2^{k-1}}\}$. Thus, by (50) and (53),

$$\bigcup_{i \in I^{k+1}_k} B^{k+1}_i = \bigcup_{i \in \mathbb{Z}_{2^{k-1}}} B^{k+1}_{2i+1} = \bigcup_{i \in \mathbb{Z}^+_{2^{k-1}}} A^k_i = A^k$$

Fix any $j \in \{1, \dots, k-1\}$. Then by (32), I^{k+1}_j is a union of 2^{j-1} 2^{k-j}-element sets, furthermore $\{(i+1)/2 \colon i \in I^{k+1}_j \quad \& \quad 2 \nmid i\} = I^k_j$. Therefore by (50) and the induction hypothesis,

$$\begin{aligned}
\bigcup_{i \in I^{k+1}_j} B^{k+1}_i &= \bigcup_{\substack{i \in I^{k+1}_j \\ 2 \nmid i}} \left(B^{k+1}_i \cup B^{k+1}_{i+1}\right) \\
&= \bigcup_{\substack{i \in I^{k+1}_j \\ 2 \nmid i}} \left(A^k_{(i+1)/2} \cup \left(B^k_{(i+1)/2} \setminus A^k_{(i+1)/2}\right)\right) = \bigcup_{i \in I^k_j} B^k_i = A^j.
\end{aligned}$$

To prove (55), firstly we utilize (51) and (52) from Lemma 15, from which it follows that

$$B^k_{2^{k-1}} = \mathbb{Z}^+_n \setminus \bigcup_{i=1}^{2^{k-1}-1} B^k_i.$$

By (44) from Lemma 14, and next by (54) from Lemma 16 we have

$$\begin{aligned} B^k_{2^{k-1}} &= \mathbb{Z}^+_n \setminus \bigcup_{i\in\bigcup_{j\in\mathbb{Z}^+_{k-1}} I^k_j} B^k_i \\ &= \mathbb{Z}^+_n \setminus \bigcup_{j\in\mathbb{Z}^+_{k-1}} \bigcup_{i\in I^k_j} B^k_i \\ &= \mathbb{Z}^+_n \setminus \bigcup_{j\in\mathbb{Z}^+_{k-1}} A^j \end{aligned}$$

□

Lemma 17. *Let $m, n \geq 2$ be natural numbers, $m \leq \binom{n}{\lfloor n/2 \rfloor}$, $A^k \subseteq \mathbb{Z}^+_n$ for all $k \in \mathbb{Z}^+_m$, and $1 \leq \#A^1 \leq \ldots \leq \#A^m \leq n-1$, then*

1. *$\mathcal{A} = \{A^1, \ldots, A^m\} \in \mathcal{A}^m_n$ if and only if for each $k \in \{2, \ldots, m\}$ there exist sets $A^k_1, \ldots, A^k_{2^{k-1}}$ and $B^k_1, \ldots, B^k_{2^{k-1}}$ such that conditions (50), (53), and*

$$A^k_i \subseteq B^k_i, \text{ for all } i \in \mathbb{Z}^+_{2^{k-1}} \tag{56}$$

$$\bigcup_{i\in I^k_j} A^k_i \subsetneq A^j, \text{ for all } j \in \mathbb{Z}^+_{k-1} \tag{57}$$

   are satisfied for all $k \in \{2, \ldots, m\}$.

2. *$\mathcal{A} = \{A^1, \ldots, A^m\} \in \mathcal{AC}^m_n$ if and only if for each $k \in \{2, \ldots, m\}$ there exist sets $A^k_1, \ldots, A^k_{2^{k-1}}$ and $B^k_1, \ldots, B^k_{2^{k-1}}$ such that conditions (50), (53), (56), (57) are satisfied for all $k \in \{2, \ldots, m\}$ and $A^m_{2^{m-1}} = B^m_{2^{m-1}}$.*

Proof. The proof is by induction on m. Firstly we shall prove this fact for $m = 2$. Assume that $\{A^1, A^2\} \in \mathcal{A}^2_n$. Define $A^1_1 := A^1$, $B^1_1 := \mathbb{Z}^+_n$, $B^2_1 := A^1$, $B^2_2 := \mathbb{Z}^+_n \setminus A^1$, $A^2_1 := A^1 \cap A^2$, and $A^2_2 := A^2 \setminus A^1$. Then conditions (50), (53), and (56) are straightforwardly satisfied. Moreover $I^2_1 = \{1\}$. Hence $\bigcup_{i\in I^2_1} A^2_i = A^2_1 \subsetneq A^1$, since $A^1 \not\subseteq A^2$. Thus condition (57) is also true. If additionally $\{A^1, A^2\} \in \mathcal{AC}^2_n$, then $\mathbb{Z}^+_n \setminus A^1 \subseteq A^2$. Hence $A^2_2 = B^2_2$.

Now assume that conditions (50), (53), (56), (57) are satisfied. It follows from (57) that $A^2_1 \subsetneq A^1$, and by (56) and (50), $A^2_2 \subseteq \mathbb{Z}^+_n \setminus A^1$. Therefore by (53), $A^1 \not\subseteq A^2$. Moreover, $\#A^1 \leq \#A^2$. Hence $A^2 \not\subseteq A^1$, since otherwise $A^2 = A^1$ which contradicts $A^1 \not\subseteq A^2$. So $\{A^1, A^2\} \in \mathcal{A}^2_n$. If $A^2_2 = B^2_2$, then by (53) and (50), $A^1 \cup A^2 = A^1 \cup A^2_2 = A^1 \cup B^2_2 = \mathbb{Z}^+_n$. Thus $\{A^1, A^2\} \in \mathcal{AC}^2_n$ and Lemma 17 is true for $m = 2$.

Assume that Lemma 17 is true for some natural number m such that $2 \leq m < \binom{n}{\lfloor n/2 \rfloor}$. We shall show that it holds for $m+1$. Assume $\{A^1, \ldots, A^{m+1}\} \in \mathcal{A}^{m+1}_n$. Then $\{A^1, \ldots, A^m\} \in \mathcal{A}^m_n$ and by the induction hypothesis, for each $k \in \{2, \ldots, m\}$ there exist sets $A^k_1, \ldots, A^k_{2^{k-1}}$ and $B^k_1, \ldots, B^k_{2^{k-1}}$ such that conditions (50), (53), (56), (57) are satisfied for all $k \in \{2 \ldots, m\}$.

Let $A_i^{m+1} := A^{m+1} \cap B_i^{m+1}$, where B_i^{m+1} defined by (50) exist by the induction hypothesis, for all $i \in \mathbb{Z}^+_{2^m}$. Then conditions (50) and (56) are also straightforwardly satisfied for $k = m+1$. By (51) from Lemma 15,

$$\bigcup_{i=1}^{2^m} A_i^{m+1} = A^{m+1} \cap \bigcup_{i=1}^{2^m} B_i^{m+1} = A^{m+1};$$

thus (53) is also satisfied for $m+1$. Therefore by (54) from Lemma 16, we have $\bigcup_{i \in I_j^{m+1}} B_i^{m+1} = A^j$ for all $j \in \mathbb{Z}^+_m$. So

$$\bigcup_{i \in I_j^{m+1}} A_i^{m+1} = A^{m+1} \cap \bigcup_{i \in I_j^{m+1}} B_i^{m+1} = A^{m+1} \cap A^j \subsetneq A^j,$$

since otherwise $A^j \subseteq A^{m+1}$ which would contradict with assumption $\{A^1, \ldots, A^{m+1}\} \in \mathcal{A}_n^{m+1}$. It follows from the above reasoning that (57) is satisfied for all $k \in \{2, \ldots, m+1\}$.

Assume now that $\{A^1, \ldots, A^{m+1}\} \in \mathcal{AC}_n^{m+1}$. Then by the previous part of the proof, conditions (50), (53), (56), (57) are satisfied for all $k \in \{2, \ldots, m+1\}$. We shall prove that $A_{2^m}^{m+1} = B_{2^m}^{m+1}$. By (56), $A_{2^m}^{m+1} \subseteq B_{2^m}^{m+1}$. Suppose that $A_{2^m}^{m+1} \subsetneq B_{2^m}^{m+1}$. Then by (55), $A_{2^m}^{m+1} \subsetneq \mathbb{Z}^+_n \setminus \bigcup_{i=1}^m A^i$, and hence

$$A_{2^m}^{m+1} \cup \bigcup_{i=1}^{m} A^i \subsetneq \mathbb{Z}^+_n. \tag{58}$$

On the other hand, by (57), $\bigcup_{j \in \mathbb{Z}^+_m} \bigcup_{i \in I_j^{m+1}} A_i^{m+1} \subseteq \bigcup_{j \in \mathbb{Z}^+_m} A^j$. So by Lemma 14 and (53), $A^{m+1} \setminus A_{2^m}^{m+1} \subseteq \bigcup_{j \in \mathbb{Z}^+_m} A^j$. Therefore

$$\bigcup_{j \in \mathbb{Z}^+_{m+1}} A^j = \bigcup_{j \in \mathbb{Z}^+_m} A^j \cup A_{2^m}^{m+1}. \tag{59}$$

We see that (58) and (59) contradict assumption $\{A^1, \ldots, A^{m+1}\} \in \mathcal{AC}_n^{m+1}$.

Now assume that conditions (50), (53), (56), (57) are satisfied for all $k \in \{2, \ldots, m+1\}$. We shall show that $\{A^1, \ldots, A^{m+1}\} \in \mathcal{A}_n^{m+1}$. By assumption $\#A^i \leq \#A^{i+1}$ for all $i \in \mathbb{Z}^+_m$ and the induction hypothesis, it suffices to prove that $A^j \nsubseteq A^{m+1}$ for all $j \in \mathbb{Z}^+_m$. Fix any $j \in \mathbb{Z}^+_m$. Then: by (54), $A^{m+1} \cap A^j = A^{m+1} \cap \bigcup_{i \in I_j^{m+1}} B_i^{m+1}$; by (53), $A^{m+1} \cap \bigcup_{i \in I_j^{m+1}} B_i^{m+1} = \bigcup_{i \in \mathbb{Z}^+_{2^m}} A_i^{m+1} \cap \bigcup_{i \in I_j^{m+1}} B_i^{m+1}$; by Lemma 14, (51), (52), and (56), $\bigcup_{i \in \mathbb{Z}^+_{2^m}} A_i^{m+1} \cap \bigcup_{i \in I_j^{m+1}} B_i^{m+1} = \bigcup_{i \in I_j^{m+1}} A_i^{m+1}$; hence by (57), $A^{m+1} \cap A^j \subsetneq A^j$. Therefore $A^j \nsubseteq A^{m+1}$.

If additionally $A_{2^m}^{m+1} = B_{2^m}^{m+1}$, then by (55), we have $A_{2^m}^{m+1} \cup \bigcup_{i \in \mathbb{Z}^+_m} A^i = \mathbb{Z}^+_n$. So by (53), $\bigcup_{i \in \mathbb{Z}^+_{m+1}} A^i = \mathbb{Z}^+_n$. Thus $\{A^1, \ldots, A^{m+1}\} \in \mathcal{AC}_n^{m+1}$. □

Using Lemma 17 we can determine the number of all antichains as well as the number of all antichain covers of an n-element set X which consist of m subsets of X. Let

$$\mathfrak{C}_m^n := \left\{(A^1,\dots,A^m)\colon \exists_{\mathcal{A}\in\mathfrak{A}_m^n}\colon (A^1,\dots,A^m)\in\mathcal{A} \;\&\; \mathcal{A} \text{ satisfies conditions (50), (53), (57)}\right\}$$

and

$$\mathfrak{D}_m^n := \left\{(A^1,\dots,A^m)\in\mathfrak{C}_m^n\colon A_{2^{m-1}}^m = B_{2^{m-1}}^m\right\}$$

By Lemma 17, if $\#A^1 \le \ldots \le \#A^m$, then $\{A^1,\dots,A^m\}\in\mathcal{A}_n^m$ iff $(A^1,\dots,A^m)\in\mathfrak{C}_m^n$ and $\{A^1,\dots,A^m\}\in\mathcal{AC}_n^m$ iff $(A^1,\dots,A^m)\in\mathfrak{D}_m^n$.

If $(A^1,\dots,A^m)\in\mathfrak{C}_m^n$, then A^1 can be chosen in $\binom{b_1^1}{a^1}$ ways, where $b_1^1 = n$, $a^1 = \#A^1$, and a^1 can differ from 1 to $n-1$. Generally, it follows from Lemma 17, that for fixed sets $A^1,\dots,A^k$ of cardinality $a^1,\dots,a^k$, respectively, set A_i^{k+1} can be chosen in $\binom{b_i^{k+1}}{a_i^{k+1}}$ ways, for all $i\in\mathbb{Z}_{2^k}^+$, where b_i^{k+1} are defined by (35), $\left(a_1^{k+1},\dots,a_{2^k}^{k+1}\right)\in S_m^n\left(\underline{a}^1,\dots,\underline{a}^k\right)$, and S_m^n is defined by (36). Moreover if $(A^1,\dots,A^m)\in\mathfrak{D}_m^n$, then by Lemma 17, $A_{2^{m-1}}^m = B_{2^{m-1}}^m$. So it can be chosen in one way. Furthermore, note that for any set $\{A^1,\dots,A^m\}\in\mathcal{A}_n^m$ if $a_i = \#A^i$ for all $i\in\mathbb{Z}_m^+$ and

$$\underbrace{a_1=\cdots=a_{m_1}}_{m_1} < \underbrace{a_{m_1+1}=\cdots=a_{m_1+m_2}}_{m_2} < \cdots < \underbrace{a_{M_k+1}=\cdots=a_{M_k+m_k}}_{m_k}$$

where $M_i := m_1+\cdots+m_{i-1}$ for all $1<i\le k$, then the number of all $(B^1,\dots,B^m)\in\mathfrak{C}_m^n$ such that $\{B^1,\dots,B^m\}=\{A^1,\dots,A^m\}$ is equal to $\prod_{i=1}^k m_i!$. Therefore (39) and (40) are true. □

3 Covers of labeled sets with antichains consisting of equinumerous sets

Let X be an n-element set and numbers $k,m\in\mathbb{N}$ be such that $\frac{n}{k}\le m\le\binom{n}{k}$. We let $\mathcal{AC}^{m,k}(X)$ denote the set of all m-antichain covers $\{A_1,\dots,A_m\}$ in the power set $\mathcal{P}(X)$ such that $\#A_i = k$ for all $i\in\mathbb{Z}_m^+$. If $X=\mathbb{Z}_n^+$, then $\mathcal{AC}^{m,k}(X)$ is denoted by $\mathcal{AC}_n^{m,k}$ and $T_n^{m,k} = \#\mathcal{AC}_n^{m,k}$.

Note that the assumption $\frac{n}{k}\le m\le\binom{n}{k}$ is a necessary condition for the correctness of this notion, since for any n-element set X and $\{A_1,\dots,A_m\}\in\mathcal{AC}^{m,k}(X)$:

$$n = \#X = \#\left(\bigcup_{i=1}^m A_i\right) \le \sum_{i=1}^m \#A_i = m\cdot k$$

and $A_1,\dots,A_m$ are distinct k-element subsets of X whose total number equals $\binom{n}{k}$.

Lemma 18. *Let X be an n-element set, $m,k\in\mathbb{N}$ be such that $m\le\binom{n}{k}$,*

$$C_k^m(X) := \left\{\{A_1,\dots,A_m\}\colon A_i\subseteq X \text{ and } \#A_i = k \text{ for all } i\in\{1,\dots,m\}\right\},$$

$n_0 := \min\left\{n' \in \mathbb{N} \colon \binom{n'}{k} \geq m\right\}$. *Then:*

$$C_k^m(X) = \bigcup_{Y \subseteq X \colon n_0 \leq \#Y \leq \min(n, m\cdot k)} \mathcal{AC}^{m,k}(Y) \tag{60}$$

$$\#C_k^m(X) = \binom{\binom{n}{k}}{m} \tag{61}$$

$$\sum_{l=n_0}^{\min(n,m\cdot k)} \binom{n}{l} \cdot T_l^{m,k} = \binom{\binom{n}{k}}{m}. \tag{62}$$

Proof.

1. Fix any $\{A_1, \ldots, A_m\} \in C_k^m(X)$. Let $Y := \bigcup_{i=1}^m A_i$. Then $\{A_1, \ldots, A_m\} \in \mathcal{AC}^{m,k}(Y)$, $Y \subseteq X$; moreover, we have, $n' := \#Y \leq \min(n, m \cdot k)$ and $\binom{n'}{k} \geq m$. Thus $n' \geq n_0$. The converse inclusion is obvious.

2. The number of all k-element subsets of an n-element set equals $\binom{n}{k}$. Thus $\#C_k^m(X)$ as the number of all m-element subsets of the family of all k-element subsets of an n-element set equals $\binom{\binom{n}{k}}{m}$.

3. It follows straightforwardly from (60) and (61), since

$$\#\mathcal{AC}^{m,k}(Y) = \#\mathcal{AC}^{m,k}(Y') = T_l^{m,k}$$

for all $Y, Y' \subseteq X$ such that $n_0 \leq l := \#Y = \#Y' \leq \min(n, m \cdot k)$ and the number of all l-element subsets of X equals $\binom{n}{l}$.

□

Theorem 19. *Assume that* $m, k \in \mathbb{N}$ *and* $n_0 := \min\left\{n' \in \mathbb{N} \colon \binom{n'}{k} \geq m\right\}$. *Then*

$$T_n^{m,k} = \sum_{t=0}^{n-n_0} \binom{n}{n-t} \cdot \binom{\binom{n-t}{k}}{m} \cdot (-1)^t \tag{63}$$

for all $n \in \{n_0, \ldots, m \cdot k\}$

Proof. Fix $m, k \in \mathbb{N}$ and put $n_0 := \min\left\{n' \in \mathbb{N} \colon \binom{n'}{k} \geq m\right\}$. Let $a_n := T_n^{m,k}$ and $b_n := \binom{\binom{n}{k}}{m}$. Then by (62) from Lemma 18, $b_n := \sum_{k=n_0}^{n} \binom{n}{k} a_k$. Thus assumptions of Lemma 8 are satisfied. Therefore by the same lemma, (63) holds. □

4 Numerical results

Example 20. Let us return to Example 10 and compute D_n^m for all $n \geq 3$ and $m \in \{2, 3\}$ using Theorem 11 and Proposition 13. By formulas (39) and (40),

$$D_n^2 = \sum_{a^1=1}^{n-1} \sum_{\underline{\mathbf{a}}^2 \in S_2^n(a^1)} \frac{\binom{b_1^1}{a_1^1}\binom{b_1^2}{a_1^2}\binom{b_2^2}{a_2^2}}{r(a^1, a^2)}$$

$$T_n^2 = \sum_{a^1=1}^{n-1} \sum_{\underline{\mathbf{a}}^2 \in \overline{S_2^n}(a^1)} \frac{\binom{b_1^1}{a_1^1}\binom{b_1^2}{a_1^2}\binom{b_2^2}{a_2^2}}{r(a^1, a^2)}$$

$$D_n^3 = \sum_{a^1=1}^{n-1} \sum_{\underline{\mathbf{a}}^2 \in S_2^n(a^1)} \sum_{\underline{\mathbf{a}}^3 \in S_3^n(a^1, \underline{\mathbf{a}}^2)} \frac{\binom{b_1^1}{a_1^1}\binom{b_1^2}{a_1^2}\binom{b_2^2}{a_2^2}\binom{b_1^3}{a_1^3}\binom{b_2^3}{a_2^3}\binom{b_3^3}{a_3^3}\binom{b_4^3}{a_4^3}}{r(a^1, a^2, a^3)}$$

$$T_n^3 = \sum_{a^1=1}^{n-1} \sum_{\underline{\mathbf{a}}^2 \in S_2^n(a^1)} \sum_{\underline{\mathbf{a}}^3 \in \overline{S_3^n}(a^1, \underline{\mathbf{a}}^2)} \frac{\binom{b_1^1}{a_1^1}\binom{b_1^2}{a_1^2}\binom{b_2^2}{a_2^2}\binom{b_1^3}{a_1^3}\binom{b_2^3}{a_2^3}\binom{b_3^3}{a_3^3}\binom{b_4^3}{a_4^3}}{r(a^1, a^2, a^3)}$$

By (41), $b_1^1 = a_1^0 = n$, $b_1^2 = a_1^1 = a^1$, $b_2^2 = a_1^0 - a_1^1 = n - a^1$, $b_1^3 = a_1^2$, $b_2^3 = a_1^1 - a_1^2 = a^1 - a_1^2$, $b_3^3 = a_2^2$, and $b_4^3 = a_1^0 - a_2^2 - a_1^1 = n - a^1 - a_2^2$. Note that by (32), $I_1^2 = \{1\}$, $I_1^3 = \{1, 2\}$, and $I_2^3 = \{1, 3\}$ So by (33) and (36),

$$\begin{aligned}
S_2^n(a^1) &= \{(a_1^2, a_2^2) \colon 0 \leq a_1^2 \leq a^1 - 1 \quad \& \quad 0 \leq a_2^2 \leq n - a^1 \quad \& \quad a^1 \leq a^2 \leq n - 1\} \\
\overline{S_2^n}(a^1) &= \{(a_1^2, a_2^2) \colon 0 \leq a_1^2 \leq a^1 - 1 \quad \& \quad a^1 \leq a^2 \leq n - 1 \quad \& \quad a_2^2 = n - a^1\} \\
S_3^n(a^1, \underline{\mathbf{a}}^2) &= \{(a_1^3, a_2^3, a_3^3, a_4^3) \colon 0 \leq a_1^3 \leq a_1^2 \quad \& \quad 0 \leq a_2^3 \leq a^1 - a_1^2 \quad \& \quad 0 \leq a_3^3 \leq a_2^2 \quad \& \\
&\quad 0 \leq a_4^3 \leq n - a^1 - a_2^2 \quad \& \quad a^2 \leq a^3 \leq n - 1 \quad \& \\
&\quad a_1^3 + a_2^3 \leq a^1 - 1 \quad \& \quad a_1^3 + a_3^3 \leq a^2 - 1\} \\
\overline{S_3^n}(a^1, \underline{\mathbf{a}}^2) &= \{(a_1^3, a_2^3, a_3^3, a_4^3) \colon 0 \leq a_1^3 \leq a_1^2 \quad \& \quad 0 \leq a_2^3 \leq a^1 - a_1^2 \quad \& \quad 0 \leq a_3^3 \leq a_2^2 \quad \& \\
&\quad 0 \leq a_4^3 = n - a^1 - a_2^2 \quad \& \quad a^2 \leq a^3 \leq n - 1 \quad \& \\
&\quad a_1^3 + a_2^3 \leq a^1 - 1 \quad \& \quad a_1^3 + a_3^3 \leq a^2 - 1\}
\end{aligned}$$

It follows from the system of inequalities in $S_2^n(a^1)$, that: $\max(0, 2a^1 - n) \leq a_1^2 \leq a^1 - 1$ and

$$1 \leq a^1 - a_1^2 \leq a_2^2 \leq n - a^1 \leq n - 1 - a_1^2.$$

From the system of inequalities and equality from $\overline{S_2^n}(a^1)$ we obtain the same bound for a_1^2 as above. Hence we obtain the same formulae (18) and (19) as in Example 10, as follows:

$$D_n^2 = \sum_{a^1=1}^{n-1} \sum_{a_1^2=\max(0,2a^1-n)}^{a^1-1} \sum_{a_2^2=a^1-a_1^2}^{n-a^1} \frac{\binom{n}{a_1^1}\binom{a^1}{a_1^2}\binom{n-a^1}{a_2^2}}{r(a^1,a^2)}$$

$$T_n^2 = \sum_{a^1=1}^{n-1} \sum_{a_1^2=\max(0,2a^1-n)}^{a^1-1} \frac{\binom{n}{a_1^1}\binom{a^1}{a_1^2}}{r(a^1,a^2)}.$$

Note that the systems of inequalities in $S_3^n(a^1,\underline{\mathbf{a}}^2)$ and $\overline{S_3^n}(a^1,\underline{\mathbf{a}}^2)$ are the same as in (29) from Example 10. Solving these systems we obtain the following formulas:

$$D_n^3 = \sum_{a^1=1}^{n-1} \sum_{a_1^2=\max(0,2a^1-n)}^{a^1-1} \sum_{a_2^2=a^1-a_1^2}^{\min(n-a^1,n-2)} \sum_{a_1^3=\max(0,a^2+a_1^2-n)}^{\min(a_1^2,n-a_2^2-2)}$$
$$\sum_{a_2^3=\max(0,a^1+a_2^2-n+1,a^1+a^2-n-a_1^3)}^{\min(a^1-a_1^2,a^1-1-a_1^3)} \sum_{a_3^3=\max(0,a^2+a_2^2-n,a^1+a^2+a_2^2-n-a_1^3-a_2^3)}^{\min(a_2^2,a^2-1-a_1^3)}$$
$$\sum_{a_4^3=\max(0,a^2-a_1^3-a_2^3-a_3^3)}^{\min(n-a^1-a_2^2,n-1-a_1^3-a_2^3-a_3^3)} \frac{\binom{n}{a_1^1}\binom{a^1}{a_1^2}\binom{n-a^1}{a_2^2}\binom{a_1^2}{a_1^3}\binom{a^1-a_1^2}{a_2^3}\binom{a_2^2}{a_3^3}\binom{n-a^1-a_2^2}{a_4^3}}{r(a^1,a^2,a^3)}$$

$$T_n^3 = \sum_{a^1=1}^{n-1} \sum_{a_1^2=\max(0,2a^1-n)}^{a^1-1} \sum_{a_2^2=a^1-a_1^2}^{\min(n-a^1,n-2)} \sum_{a_1^3=\max(0,a^2+a_1^2-n)}^{\min(a_1^2,n-a_2^2-2)} \sum_{a_2^3=\max(0,a^1+a_2^2-n+1,a^1+a^2-n-a_1^3)}^{\min(a^1-a_1^2,a^1-1-a_1^3)}$$
$$\sum_{a_3^3=\max(0,a^2+a_2^2-n,a^1+a^2+a_2^2-n-a_1^3-a_2^3)}^{\min(a_2^2,a^2-1-a_1^3)} \frac{\binom{n}{a_1^1}\binom{a^1}{a_1^2}\binom{n-a^1}{a_2^2}\binom{a_1^2}{a_1^3}\binom{a^1-a_1^2}{a_2^3}\binom{a_2^2}{a_3^3}}{r(a^1,a^2,a^3)}.$$

Table 1 gives the number of all summands in the sums (39) denoted by $N_A^m(n)$, the number of all summands in the sums (40) denoted by $N_{AC}^m(n)$, as well as values of D_n^m and T_n^m for $n \in \{4,5\}$ and $m \in \left\{2,\ldots,\binom{n}{\lfloor n/2\rfloor}-1\right\}$. Trivial cases when $m \in \left\{0,1,\binom{n}{\lfloor n/2\rfloor}\right\}$ are omitted in the table. They are considered in the following remark.

Remark 21. For all natural numbers $n \in \mathbb{N}_0$:

1. $D_n^0 = T_n^0 = 1$;
2. $D_n^1 = 2^n$ and $T_n^1 = 1$;
3. if $m = \binom{n}{\lfloor n/2\rfloor}$, then $D_n^m = T_n^m = \begin{cases} 1, & \text{if } 2 \mid n; \\ 2, & \text{if } 2 \nmid n; \end{cases}$

4. if $m = \binom{n}{\lfloor n/2 \rfloor} - 1$, then $T_n^m = \begin{cases} \binom{n}{\lfloor n/2 \rfloor}, & \text{if } 2 \mid n; \\ 2\binom{n}{\lfloor n/2 \rfloor}, & \text{if } 2 \nmid n; \end{cases}$

5. if $\binom{n-1}{\lfloor (n-1)/2 \rfloor} < m \leq \binom{n}{\lfloor n/2 \rfloor}$, then $T_n^m = D_n^m$;

n	m	$N_A^m(n)$	$N_{AC}^m(n)$	D_n^m	T_n^m
4	2	7	4	55	25
	3	11	9	64	56
	4	19	19	25	25
	5	30	30	6	6
5	2	13	6	285	90
	3	40	29	1090	790
	4	164	145	2020	1895
	5	760	730	2146	2116
	6	3180	3150	1380	1375
	7	11148	11148	490	490
	8	31104	31104	115	115
	9	60480	60480	20	20

Table 1: The number of all summands in the sums (39) — $N_A^m(n)$, (40) — $N_{AC}^m(n)$, D_n^m, and T_n^m for $n \in \{4, 5\}$ and $m \in \left\{2, \ldots, \binom{n}{\lfloor n/2 \rfloor} - 1\right\}$.

Tables 2 and 3 depict the number of all summands in the sums (39) and (40) — $N_A^m(n)$ and $N_{AC}^m(n)$, respectively, as well as values of D_n^m and T_n^m, respectively, for $n \in \{6, \ldots, 15\}$ and $m \in \{2, 3, 4\}$.

n	$N_A^2(n)$	D_n^2	$N_A^3(n)$	D_n^3	$N_A^4(n)$	D_n^4
6	22	1351	113	14000	913	82115
7	34	6069	272	153762	3889	2401910
8	50	26335	585	1533504	13850	58089465
9	70	111645	1154	14356610	43157	1245331920
10	95	465751	2129	128722000	121243	24625121455
11	125	1921029	3718	1119607522	313162	460316430970
12	161	7859215	6208	9528462944	754557	8266174350005
13	203	31964205	9976	79817940930	1714126	144171200793620
14	252	129442951	15520	660876543600	3702041	2461016066613195
15	308	522538389	23470	5424917141282	7650964	41343340015862430

Table 2: The number of all summands in the sums (39) — $N_A^m(n)$ and values of D_n^m for $n \in \{6, \ldots, 15\}$ and $m \in \{2, 3, 4\}$.

n	$N^2_{AC}(n)$	T^2_n	$N^3_{AC}(n)$	T^3_n	$N^4_{AC}(n)$	T^4_n
6	9	301	73	8380	749	70370
7	12	966	159	76482	2976	1868650
8	16	3025	313	638736	9961	41062035
9	20	9330	569	5043950	29307	802349205
10	25	28501	975	38390660	78086	14514339340
11	30	86526	1589	285007162	191919	249104207000
12	36	261625	2490	2079779416	441395	4120588431245
13	42	788970	3768	14995363110	959569	66392465654515
14	49	2375101	5544	107204473740	1987915	1049608974433110
15	56	7141686	7950	761823557042	3948923	16365222591176550

Table 3: The number of all summands in the sums (40) — $N^m_{AC}(n)$ and values of T^m_n for $n \in \{6,\dots,15\}$ and $m \in \{2,3,4\}$.

Table 4 presents values of T^m_n for $n \in \{16,\dots,50\}$ and $m \in \{2,3\}$.

Table 5 presents values of $T^{m,k}_n$ for $n = 7$, $m \in \{2,\dots,35\}$, and all possible k, i.e., $k \geq \frac{7}{m}$ and $\binom{7}{k} \geq m$.

Methods enabling the computation of T^m_n, D^m_n for all $n \in \mathbb{N}$ and $1 \leq m \leq \binom{n}{\lfloor n/2 \rfloor}$, as well as a method allowing the determination of $T^{m,k}_n$, for all $m,n,k \in \mathbb{N}$ such that $\frac{n}{k} \leq m \leq \binom{n}{k}$ were programmed in R. These methods were used to obtain results from Tables 1–3, 5, and 6. Methods permitting the determination of T^m_n for $m \in \{2,3,4\}$ and all natural numbers $n \geq 4$ were programmed in Maple, and were used to obtain the results in Table 4. All methods written in R were implemented on a PC with Intel Core i7-8750H 2.20-GHz CPU and 16 GB of RAM. However, methods written in Maple were implemented on a PC with Intel Core i5-3230M 2.60-GHz CPU and 6 GB of RAM.

Table 6 presents the running time of the method $\mathrm{Ind}(n,m,\mathrm{ind}(n,m-1))$ determining the list $\mathrm{ind}(n,m)$ of all possible sequences $\underline{a}^1,\dots,\underline{a}^m$ for $m \in \{2,3,4\}$ and $n \in \{12,13,14,15\}$ such that $\underline{a}^k \in S^n_k(\underline{a}^1,\dots,\underline{a}^{k-1})$, where $S^n_k(\underline{a}^1,\dots,\underline{a}^{k-1})$ are defined by (36), as well as the method $\mathrm{AC}(n,m,\mathrm{ind}(n,m))$ computing D^m_n and T^m_n according to (39) and (40) for $m \in \{2,3,4\}$, $n \in \{12,13,14,15\}$, and the list of sequences $\mathrm{ind}(n,m)$. The list $\mathrm{ind_c}(n,m)$ of all possible sequences $\underline{a}^1,\dots,\underline{a}^m$ suitable for (40) can be easily and quickly obtained from the list $\mathrm{ind}(n,m)$ of all possible sequences $\underline{a}^1,\dots,\underline{a}^m$ suitable for (39). Therefore the running time of the procedure convToCl, converting one list into another, is omitted.

It can be seen in Table 6 that methods Ind and AC, based on Theorem 11, allow us to effectively compute D^m_n and T^m_n for small $m \in \{2,3\}$ and relatively large n.

n	T_n^2	T_n^3
16	21457825	5390550296096
17	64439010	38026057186270
18	193448101	267656481977620
19	580606446	1881017836414122
20	1742343625	13204444871932776
21	5228079450	92618543463601430
22	15686335501	649270263511862300
23	47063200806	4549607376865786402
24	141197991025	31870882201493713456
25	423610750290	223214539710301456590
26	1270865805301	1563094445025734127780
27	3812664524766	10944627831536630201882
28	11438127792025	76627241923504206742136
29	34314651811530	536464983051964328959750
30	34314651811530	3755626545565209968614060
31	308834550658326	26291245004929410021308562
32	926505799458625	184048014423958633597002816
33	2779521693343170	1288382611752502969073278910
34	8338573669964101	9018910891539127690443801140
35	25015738189761486	63133539508515093365692674442
36	75047248929022825	441940593783822075229938159496
37	225141815506545210	3093613246149811622777248142070
38	675425583958589101	21655438185531666666398473609020
39	2026277026753674246	151588794667806792127575720587522
40	6078831630016836625	1061125199746752166787452564356176
41	18236495989562137650	7427894584494498851907618171437230
42	54709490167709668501	51995353026424486474819809382197700
43	164128474901175516606	363967925874293633608399150016855802
44	492385433499619572025	2547777754624695453699489227563016856
45	1477156318091044760490	17834455650128183338493646918462812390
46	4431468989457506370301	124841246390602318150989020114676767180
47	13294407038741263288566	873889008936455232899697038492373607282
48	39883221256961278221025	6117224483581236975434924744272427645536
49	119649664052358811373730	42820578490258331781970654161998901275550
50	358948992720026387542501	299744084957994372050034044992597868357460

Table 4: Values of T_n^m for $n \in \{16, \ldots, 50\}$ and $m \in \{2, 3\}$.

m	k	$T_7^{m,k}$	m	k	$T_7^{m,k}$	m	k	$T_7^{m,k}$	m	k	$T_7^{m,k}$
2	4	70	8	2	159390	15	2	54257	22	3	1476337800
	5	105		3	22654975		3	3247834632		4	1476337800
	6	21		4	23490775		4	3247943153	23	3	834451800
3	3	945		5	203490		5	54264		4	834451800
	4	3570	9	2	259105	16	2	20349	24	3	417225900
	5	1190		3	69431950		3	4059895035		4	417225900
	6	35		4	70572425		4	4059928950	25	3	183579396
4	2	315		5	293930		5	20349		4	183579396
	3	22820	10	2	331716	17	2	5985	26	3	70607460
	4	42910		3	182286125		3	4537559670		4	70607460
	5	5880		4	183558375		4	4537567650	27	3	23535820
	6	35		5	352716		5	5985		4	23535820
5	2	4410	11	2	343161	18	2	1330	28	3	6724520
	3	221396		3	416050180		3	4537566320		4	6724520
	4	303632		4	417216345		4	4537567650	29	3	1623160
	5	20307		5	352716		5	1330		4	1623160
	6	21	12	2	290745	19	2	210	30	3	324632
6	2	23604		3	833570010		3	4059928810		4	324632
	3	1356250		4	834448615		4	4059928950	31	3	52360
	4	1588125		5	293930		5	210		4	52360
	5	54257	13	2	202755	20	2	21	32	3	6545
	6	7		3	1475795160		3	3247943153		4	6545
7	1	1		4	1476337065		4	3247943160	33	3	595
	2	73755		5	203490		5	21		4	595
	3	6184400	14	2	116175	21	2	1	34	3	35
	4	6679475		3	2319688080		3	2319959400		4	35
	5	116280		4	2319959295		4	2319959400	35	3	1
	6	1		5	116280		5	1		4	1

Table 5: Values of $T_n^{m,k}$ for $n = 7$, $m \in \{2, \dots, 35\}$, and all possible k.

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		Computational time (s)		
n	m	$\mathrm{Ind}(n, m, \mathrm{ind}(n, m-1))$	$\mathrm{AC}(n, m, \mathrm{ind}(n, m))$	$\mathrm{AC}(n, m, \mathrm{ind_c}(n, m))$
12	2	0.01	0.01	0
	3	0.19	0.05	0.01
	4	205.98	12.89	7.76
13	2	0	0	0
	3	0.24	0.11	0.03
	4	536.09	29.55	17.03
14	2	0	0	0
	3	0.32	0.18	0.06
	4	1668.4	65.19	34.6
15	2	0	0	0
	3	0.47	0.25	0.079
	4	5217.23	136.77	73.77

Table 6: The running time of methods $\mathrm{Ind}(n, m)$, $\mathrm{AC}(n, m, A)$ for $m \in \{2, 3, 4\}$, $n \in \{12, 13, 14, 15\}$, and $A \in \{\mathrm{ind}(n, m), \mathrm{ind_c}(n, m)\}$.

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