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On Dedekind Numbers and Two Sequences of Knuth

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Abstract

We consider the sequence whose n^{th} term is the number $F(n)$ of anti-chains in the partially ordered set whose elements are $0, 1, \ldots, n-1$ and the order relation is coordinate-wise on the binary representation of each integer. This sequence is a sort of "background" sequence to its more prominent subsequence of Dedekind numbers, that is, the sequence whose terms are $F(2^k)$. We also consider the sequence of first differences with terms $F(n) - F(n-1)$. We discuss, state, and prove some (recursive) relations between the terms of these three sequences.

1 Introduction

Let P_n denote the partially ordered set (poset) whose elements are $0, 1, \ldots, n-1$ and the order relation is coordinate-wise with respect to the binary representation of the integers in

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P_n. Thus, if $n = 2^k$, then P_n is a Boolean lattice with top element $2^k - 1$ and atoms 2^i for $0 \leq i \leq 2^{k-1}$.

For each n let $F(n)$ be the number of anti-chains in P_n. So $F(2^k)$ is the number of anti-chains in a Boolean lattice with k atoms, that is, $F(2^k)$ is a Dedekind number. The exact values of the Dedekind numbers are known only for $k \leq 8$. The computation for the case $k = 8$ is described in Wiedemann [7]. The sequence of Dedekind numbers is A000372 in the *On-Line Encyclopedia of Integer Sequences* [6]. The many entries in the comments and references for this sequence there attest to the wide interest in it.

The integer sequence with terms $F(n)$ is a background sequence for the sequence of Dedekind numbers. It is A132581 and was authored by D. E. Knuth. The function $\Delta(n) = F(n) - F(n-1)$ gives the terms for the first differences of A132581; we tacitly extend the definition by $\Delta(0) = 0$.

The sequence of Δ values is A132582 and is also authored by Knuth. Another way to define this sequence would be to say that $\Delta(n)$ is the number of anti-chains containing $n-1$ in the partially ordered set P_n.

In this paper we investigate the poset P_n and the two functions F and Δ. Our main interest is in finding expressions for $F(n)$ and for $\Delta(n)$ in terms of F or Δ applied to integers less than n.

In Section 2 we present the basics on posets and lattices that we will use. Section 3 contains lemmas that express the number of anti-chains in an arbitrary finite poset as the sum of the number of anti-chains in certain subposets. We use these lemmas to find two formulas for Dedekind numbers $F(2^k)$ for arbitrary k that involve the function F applied to arguments less than 2^k. In Section 4 we present formulas for $\Delta(n)$ that only involve the F and Δ functions applied to arguments less than n. The final section of the paper is a summary that contains a list of the formulas proved in the paper. As an application, we show how these formulas may be used to determine $\Delta(n)$ in the interval $16 \leq n \leq 32$.

2 Basics

A *semi-ideal* of a poset P is a subset $I \subseteq P$, which is closed downwards, i.e., it satisfies the condition $p \in I, q \leq p \implies q \in I$. A *semi-filter* is a subset F, which is closed upwards, i.e., it satisfies the condition $p \in F, q \geq p \implies q \in F$. An *anti-chain* of P is a set of mutually incomparable elements. For a poset P and an element $p \in P$ let $\mathrm{sub}(p)$ denote the semi-ideal $\{q \in P \mid q \leq p\}$ and $\mathrm{super}(p)$ denote the semi-filter $\{q \in P \mid p \leq q\}$ and let $\mathrm{cone}(p)$ denote the union $\mathrm{sub}(p) \cup \mathrm{super}(p)$ of these. To facilitate notation we will use the term $\mathrm{cocone}(p)$ as an abbreviation of $P \setminus \mathrm{cone}(p)$, as well as $\mathrm{cosub}(p)$ for $P \setminus \mathrm{sub}(p)$ and $\mathrm{cosuper}(p)$ for $P \setminus \mathrm{super}(p)$. If the context of the poset P is necessary we will express this via a prefix $P.$ as in $P.\mathrm{cosub}(p)$ and $P.\mathrm{cosuper}(p)$. This comes in handy in expressions like $P.\mathrm{cosub}(p).\mathrm{cosub}(q)$, which of course is equal to $P \setminus (\mathrm{sub}(p) \cup \mathrm{sub}(q))$. We will also use $\mathrm{cosub}(p, q)$ as a shorthand for this set, as well as $\mathrm{cosuper}(p, q)$ for the poset $P \setminus (\mathrm{super}(p) \cup \mathrm{super}(q))$. For other notation and terminology on partially ordered sets and Boolean lattices we follow the monographs of

Balbes and Dwinger [1] and Birkhoff [3].

It is well known that the set of semi-ideals and the set of semi-filters of a poset P form distributive lattices $\mathcal{I}(P)$ and $\mathcal{F}(P)$ with set intersection and set union. Both these lattices are isomorphic to the lattice of $\mathcal{A}(P)$ of anti-chains of P, where the meet- and join-operations are slightly more complicated. Let $\alpha(P)$ denote the number of elements of $\mathcal{A}(P)$.

We recall from Section 1 the definition of the poset P_n as the set $\{0, \ldots, n-1\}$ with *binary* ordering, i.e., coordinate-wise with respect to the binary representation of the elements. In particular, we note that P_0 is the empty poset. Since the usual ordering of $\mathbb{N}$ is different from the binary ordering we express the latter via $\leq_b$ and $\geq_b$ where necessary. We note for later reference that the natural ordering is an *extension* of the binary ordering in the sense that $p \leq_b q$ implies $p \leq q$, as well as $p \geq_b q$ implies $p \geq q$ for all $p, q \in \mathbb{N}$. To put it differently, we have $P_n.\operatorname{sub}(p) \subseteq [0, p]$ and $P_n.\operatorname{super}(p) \subseteq [p, n-1]$, where we use the handy *interval notation* $[p, q]$ with respect to the natural ordering, i.e., $[p, q] = \{x \mid p \leq x \leq q\}$.

Figure 1 shows the poset P_{16}.

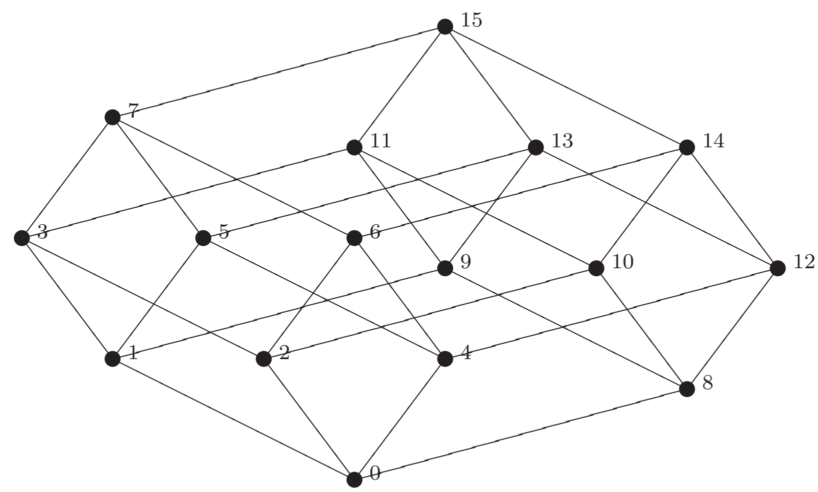

Figure 1: The Boolean poset $B_4 = P_{16}$ with 16 elements.

If n is of the form $n = 2^k$, then P_n is a Boolean poset, which we also denote by B_k. The atoms of B_k are the elements $2^0,\ 2^1, \ldots, 2^{k-2},\ 2^{k-1}$ and the dual atoms are the elements $2^k - 2,\ 2^k - 3, \ldots, 2^k - 1 - 2^{k-1}$. For this Boolean poset we have a slightly closer connection between the binary and the linear ordering. That is,

$$P_n.\,\mathrm{super}(p) = [p, n-1] \text{ if and only if } p = 2^k - 2^m \text{ for some } m \le k, \tag{1}$$
$$P_n.\,\mathrm{sub}(p) = [0, p] \text{ if and only if } p = 2^m - 1 \text{ for some } m \le k. \tag{2}$$

Both are immediate consequences of the sum formula $2^{m-1} - 1 = \sum_{i=0}^{i=m-1} 2^i$.

Another way to see this is via the binary representation of the elements of B_k as a 0-1–sequence. So for $k = 4$ these elements are

$$\begin{aligned}
2^k - 2^0 &= 15 = 1111,\\
2^k - 2^1 &= 14 = 1110,\\
2^k - 2^2 &= 12 = 1100,\\
2^k - 2^3 &= 8 = 1000,\\
2^k - 2^4 &= 0 = 0000.
\end{aligned}$$

In general, these elements are those, whose binary representation is a sequence with 'leading ones'.

Every automorphism of B_k is determined by its action on the atoms, and therefore the automorpism group of B_k is isomorphic to the symmetric group on k elements. Of particular interest in the following is the *atom inflection* β, which maps the atom 2^0 onto the atom 2^{k-1}, the atom 2^1 onto the atom 2^{k-2}, and so on. Figure 2 shows the order automorphism β of B_4. (Note that in order to avoid excessive arrows we restrict ourselves to indicate the action by an appropriate labelling.) We also denote the isomorphism that exchanges the atoms 2^i and 2^j and keeps all the other atoms fixed, by $\rho_{i,j}$.

Figure 3 shows $\rho_{0,1}$ on B_4.

The *complementation* γ defined by $\gamma(x) = 2^k - 1 - x$ is an order anti-automorphism. Figure 4 shows γ on B_4.

The composition $\delta = \beta\gamma$ is an order anti-automorphism that acts like a *top-down inflection*. Figure 5 shows δ on B_4.

And since γ commutes with all the order isomorphisms, the group of all order automorphisms and order anti-automorphisms of B_k is is isomorphic to the (direct) product of the symmetric group and an inflection group of order 2.

Let D_n denote the lattice $\mathcal{I}(P_n)$. Then for P_8 as shown in Figure 6, the lattice D_8 consists of the 20 semi-ideals

$$\emptyset, \{0\}, \{0,1\}, \{0,2\}, \{0,4\},$$
$$\{0,1,2\}, \{0,1,4\}, \{0,2,4\}, \{0,1,2,3\}, \{0,1,2,4\}, \{0,1,4,5\}, \{0,2,4,6\},$$
$$\{0,1,2,3,4\}, \{0,1,2,4,5\}, \{0,1,2,4,6\}, \{0,1,2,3,4,5\}, \{0,1,2,4,6\}, \{0,1,2,5,6\},$$
$$\{0,1,2,3,4,5,6\}\{0,1,2,3,4,5,6,7\}$$

and has the lattice diagram of Figure 7.

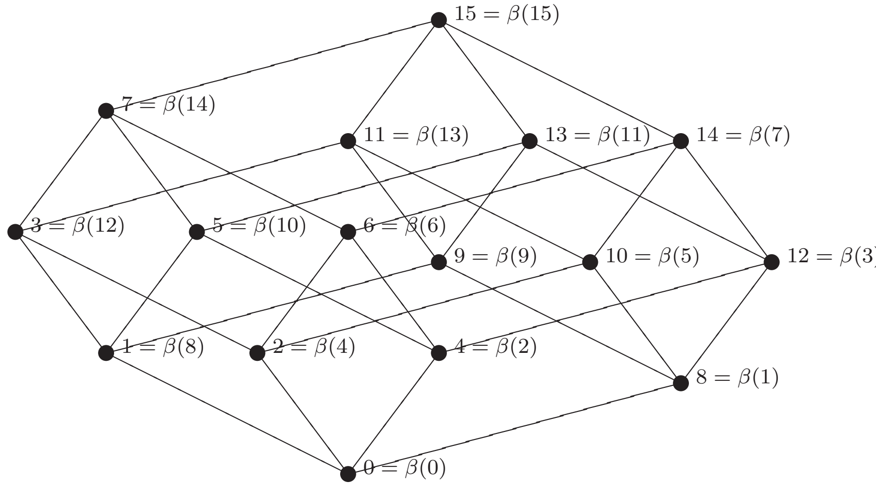

Figure 2: The action of β on $B_4 = P_{16}$.

If we let D_{01} denote the variety of all distributive lattices with 0 and 1, and $\mathrm{FD}_{01}(n)$ denote the free algebra of D_{01} on n generators, then it is well known that $D_{2^n} \cong \mathrm{FD}_{01}(n)$ for all $n \in \mathbb{N}$.

The values of A132581 and its difference sequence A132582 have been computed for the first 212 terms (J. M. Aranda). In order to illustrate the results of the following sections, we list the first 32 terms in Table 1.

3 Formulas for $F(n)$

An old result allowing a (recursive) computation of the number of anti-chains of a poset is taken from Berman and Köhler [2, Thm. 2.1].

Lemma 1. *For every poset P and every $p \in P$, we have $\alpha(P) = \alpha(P \setminus \{p\}) + \alpha(\mathrm{cocone}(p))$.*

Applying this to the poset P_n and the element $n-1$ gives

$$F(n) = F(n-1) + \alpha(\mathrm{cocone}(n-1)). \tag{3}$$

Actually (3) is still the basis for the computation of the known sequence terms. For later use we introduce the notion of a *pivot*. For a poset Q and an element $q \in Q$ we say that the poset Q *pivots* on the element q to produce the two smaller sets $Q \setminus \{q\}$ and $Q.\,\mathrm{cocone}(q)$.

Using the arguments of Lemma 1 in a slightly more sophisticated way one can obtain the following alternative from a Campo and Erné [4, Cor. 3.2].

Lemma 2. *For every poset P and every $p \in P$, we have $\alpha(P) = \alpha(\mathrm{cosuper}(p)) + \alpha(\mathrm{cosub}(p))$.*

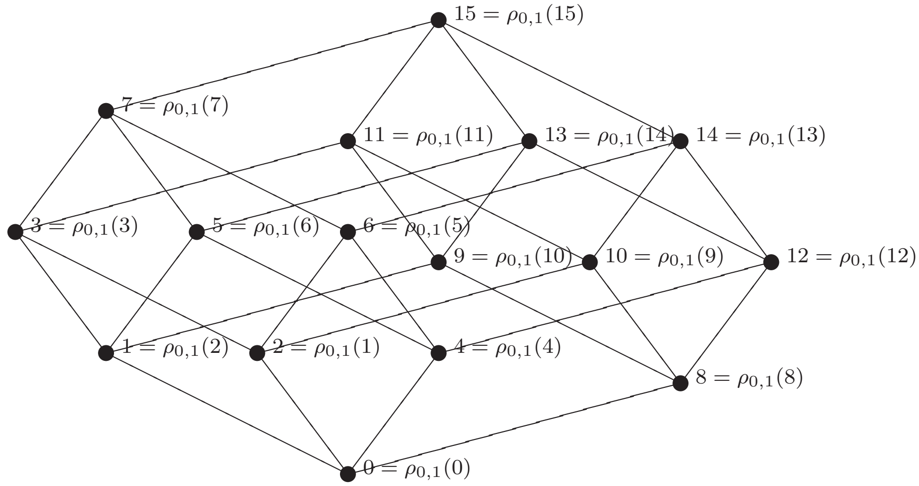

Figure 3: The action of ρ on $B_4 = P_{16}$.

Applying this to the Boolean poset B_n and an atom of the form 2^k we get

Formula 3. *For all $n, k \in \mathbb{N}$ with $k \leq n$, we have $F(2^n) = F(2^n - 2^k) + F(2^n - 2^{n-k})$.*

This gives the following interesting special cases

$$F(2^n) = F(2^n - 1) + 1,$$
$$F(2^n)F(2^n - 2) + F(2^{n-1}),$$
$$F(2^n) = F(2^n - 4) + F(3 \cdot 2^{n-2}).$$

In particular, the case where n is even and $k = \frac{n}{2}$ yields $F(2^n) = 2 \cdot F(2^n - 2^k)$, and this is another proof of the fact that $|\,\mathrm{FD}(n)|$ is even for even n.

Formula 3 was stated without proof by J. M. Aranda in the Formula section of A132581.

Before starting the proof of Formula 3 we begin with some general remarks about the elements $2^n - 2^k$ of B_n. As already mentioned in Section 2 (1),(2), integers of the form $2^n - 2^k$ have the unique and special property that $[2^n - 2^k, 2^n - 1] = \mathrm{super}(2^n - 2^k)$. The elements $2^n - 2^k$ form a maximal chain in B_n that appears on the right side of the diagram of B_n. The complement of $2^n - 2^k$ is $2^k - 1$ and all of these complements appear as a maximal chain on the left side in the diagram of B_n. And it is also easy to see that the anti-automorphism δ maps $2^n - 2^k$ onto $2^n - 2^{n-k}$. As a result, the poset $\mathrm{cone}(2^n - 2^k)$ is dually isomorphic to $\mathrm{cone}(2^n - 2^{n-k})$. In particular, $\mathrm{cone}(2^n - 2^k)$ is isomorphic to the poset B_k placed above the poset B_{n-k} with the bottom element of B_k equal to the top element of B_{n-k}. This can also be expressed as

$$\mathrm{cone}(2^n - 2^k) = [2^n - 2^k, 2^n - 1] \;\cup\; \{i \cdot 2^k \mid i = 0, 1, \ldots, 2^{n-k} - 1\}. \tag{4}$$

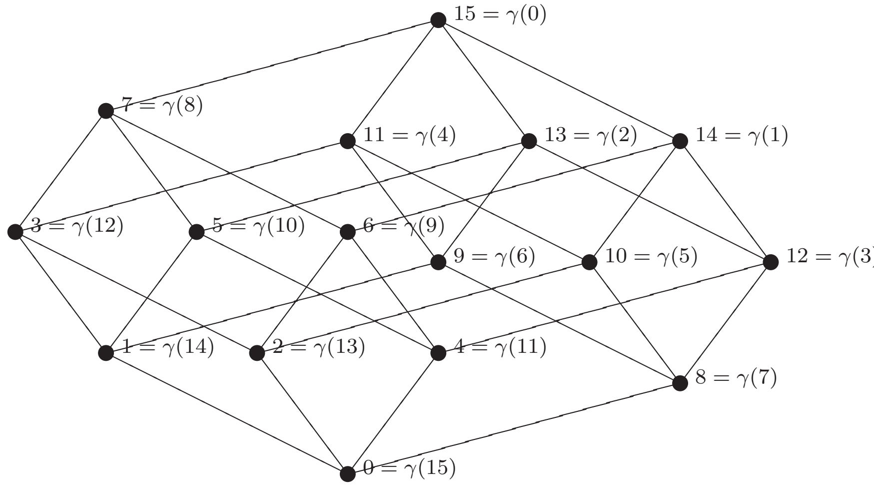

Figure 4: The action of γ on $B_4 = P_{16}$.

In addition

$$B_n.\operatorname{cosuper}(2^n - 2^k) \text{ is isomorphic to } P_{2^n-2^k} \tag{5}$$

$$B_n.\operatorname{cosub}(2^n - 2^k) \text{ is dually isomorphic to } P_{2^n-2^{n-k}}. \tag{6}$$

We also note that (5) and (6) hold for every element of rank $n-k$, i.e., for every element above exactly $n-k$ atoms.

Proof. (of Formula 3) Combining (5), (6), and Lemma 2 gives the result. □

Using the principle of inclusion and exclusion one can easily extend the result of Lemma 2 to a summation of more terms.

Lemma 4. *For every poset P and every $p, q \in P$, we have $\alpha(P) = \alpha(\operatorname{cosuper}(p,q)) + \alpha(\operatorname{cosub}(p)) + \alpha(\operatorname{cosub}(q)) - \alpha(\operatorname{cosub}(p,q))$.*

Lemma 5. *For every poset P and every $p, q \in P$, we have $\alpha(P) = \alpha(\operatorname{cosub}(p,q)) + \alpha(\operatorname{cosuper}(p)) + \alpha(\operatorname{cosuper}(q)) - \alpha(\operatorname{cosuper}(p,q))$.*

With the help of Lemma 4 and and two conveniently chosen elements of B_n we get

Formula 6. *For $k, n \in \mathbb{N}$ with $1 \leq k < n$, we have $F(2^n) = F(2^n - 2^{n-k} - 2^{n-k-1}) + 2F(2^n - 2^k) - F(2^n - 2^k - 2^{k-1})$.*

E.g., for $n = 5, k = 2$ we have $F(32) = F(20) + 2 \cdot F(28) - F(26)$ and for $n = 5, k = 3$ we have $F(32) = F(26) + 2 \cdot F(24) - F(20)$.

Proof. Let $P = B_n$. We consider the two elements $a_1 = 2^n - 2^{n-k}$ and $a_2 = 2^n - 2^{n-k} - 2^{n-k-1}$ and note that a_2 is the image of a_1 under the automorphism $\rho_{n-k,n-k-1}$, that exchanges the atoms 2^{n-k} and 2^{n-k-1} and keeps the other atoms fixed.

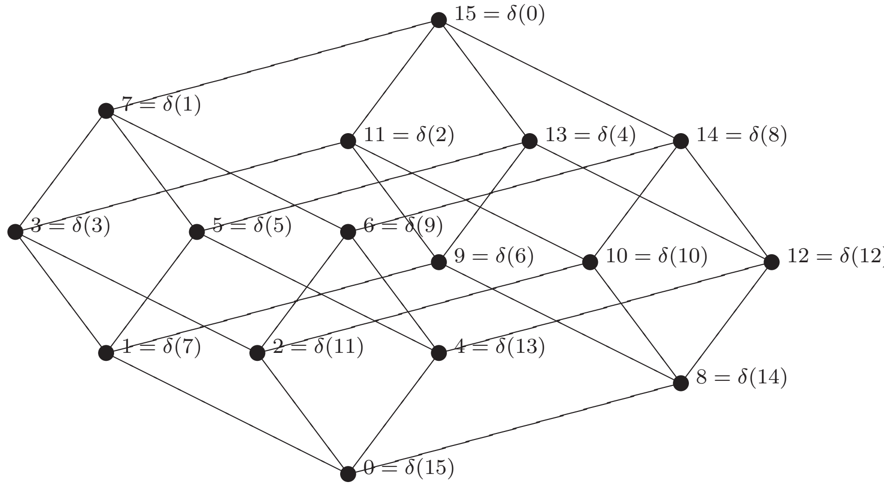

Figure 5: The action of δ on $B_4 = P_{16}$.

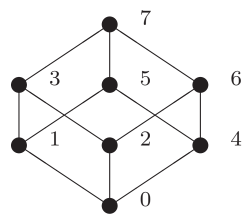

Figure 6: The Boolean poset $B_3 = P_8$ with 8 elements.

By Lemma 4 we have $F(2^n) = \alpha(P.\operatorname{cosuper}(a_1, a_2)) + \alpha(P.\operatorname{cosub}(a_1)) + \alpha(P.\operatorname{cosub}(a_2)) - \alpha(P.\operatorname{cosub}(a_1, a_2))$. We first observe that $P.\operatorname{cosuper}(a_1, a_2) = P_{a_2}$, which accounts for the first term in the formula.

Moreover $\alpha(P.\operatorname{cosub}(a_1)) = F(2^n - 2^k)$ by (5). And as a_1 is mapped onto a_2 by the automorphism $\rho_{n-k,n-k-1}$, we have the same result for $P.\operatorname{cosub}(a_2)$. This accounts for the second term in the formula.

To account for the subtraction of the last term we use the anti-automorphism δ as in the derivation of (4). Let $b_i = \delta(a_1)$ for $i = 1, 2$. Then $P.\operatorname{cosub}(a_1, a_2)$ is dually isomorphic to $P.\operatorname{cosuper}(b_1, b_2)$ and by the argument of the first part this shows that $\alpha(\operatorname{cosub}(a_1, a_2) = \alpha(P.\operatorname{cosuper}(b_1, b_2)) = F(b_2)$. $\square$

In addition, it would be possible to formulate an analog to Lemma 4 for three elements. There was, however, no immediate application of that to gain another recursion formula involving F-terms.

Other recursions are possible, but they are more conveniently formulated using the terms of the Δ-sequence—we will discuss that in the following section.

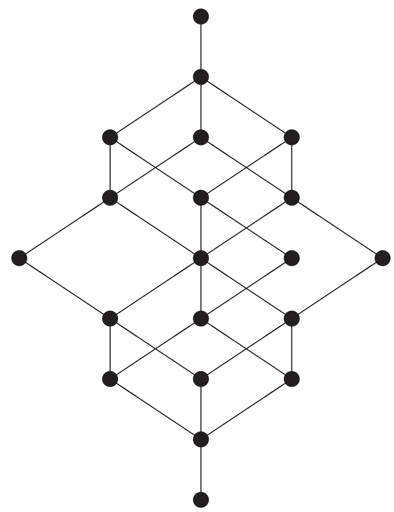

Figure 7: The lattice D_8.

4 Formulas for $\Delta(n)$

Let $d(n)$ denote the poset $P_n.\operatorname{cosub}(n-1)$. As $n-1$ is a maximal element of P_n we have $P_n.\operatorname{cosuper}(n-1) = P_n \setminus \{n-1\}$ and $P_n.\operatorname{cosub}(n-1) = P_n.\operatorname{cocone}(n-1)$ and therefore $\Delta(n) = \alpha(d(n))$ by Lemma 1 and Lemma 2.

We wish to find formulas for $\Delta(n)$ that involve the functions Δ and F applied to arguments that are less than n.

Formula 7. *For all $n, k \in \mathbb{N}$ with $0 \le k \le n$, we have $\Delta(2^n - 2^k + 1) = \Delta(2^n - 2^{n-k} + 1)$.*

E.g., for $n = 5, k = 2$ we have $\Delta(29) = \Delta(25)$.

Proof. In order to prove the formula it suffices to show that the poset $d(2^n - 2^k + 1)$ is dually isomorphic to $d(2^n + 2^{n-k} + 1)$. By definition, we have

$$d(2^n - 2^k + 1) = P_{2^n - 2^k + 1}.\operatorname{cocone}(2^n - 2^k).$$

And by (5) we have $B_n = [0, 2^n - 2^k] \ \cup\ \operatorname{super}(2^n - 2^k)$. Therefore

$$B_n.\operatorname{cocone}(2^n - 2^k) = P_{2^n - 2^k + 1}.\operatorname{cocone}(2^n - 2^k) = d(2^n - 2^k + 1).$$

And as $\delta(2^n - 2^k) = 2^n - 2^{n-k}$, we get

$$\delta(d(2^n - 2^k + 1)) = \delta(B_n.\operatorname{cocone}(2^n - 2^k)) = B_n.\operatorname{cocone}(2^n - 2^{n-k}) = d(2^n - 2^{n-k} + 1).$$

□

n	$F(n)$	$\Delta(n)$	n	$F(n)$	$\Delta(n)$
1	2	1	17	335	167
2	3	1	18	483	148
3	5	2	19	765	282
4	6	1	20	849	84
5	11	5	21	1466	617
6	14	3	22	1681	215
7	19	5	23	1988	307
8	20	1	24	2008	20
9	39	19	25	3700	1692
10	53	14	26	4414	714
11	78	25	27	5489	1075
12	84	6	28	5573	84
13	134	50	29	7265	1692
14	148	14	30	7413	148
15	167	19	31	7580	167
16	168	1	32	7581	1

Table 1: $F(n)$ and $\Delta(n)$.

The fact that $B_n.\operatorname{cocone}(2^n - 2^k) = d(2^n - 2^k + 1)$ allows for a description of the poset $d(2^n - 2^k + 1)$ as follows. The minimal elements of this poset are the k atoms $2^0, 2^1, \ldots, 2^{k-1}$ of B_n. The maximal elements of this poset are the $n - k$ co-atoms

$$(2^n - 1) - 2^{n-1}, (2^n - 1) - 2^{n-2}, \ldots, 2^k$$

of B_n. An element b of B_n is in $d(2^n - 2^k + 1)$ if and only if there exist elements a and c such that $a \leq b \leq c$ with a one of the k atoms in this list of minimal elements and c one of the $n - k$ co-atoms in this list of maximal elements. Despite this description, we do not have a direct formula for the number of anti-chains in this poset.

We do have, however, a sum formula, which can be proven along the same lines as Formula 6.

Formula 8. *For all $k, n \in \mathbb{N}$ with $1 \leq k \leq n - 1$, we have*

$$\Delta(2^n - 2^k + 1) = \Delta(2^n - 2^k - 2^{k-1} + 1) + \Delta(2^n - 2^{n-k} - 2^{n-k-1} + 1).$$

E.g., for $n = 5, k = 2$ we have $\Delta(29) = \Delta(27) + \Delta(21)$ and, combining this with Formula 7, $\Delta(27) = \Delta(25) - \Delta(21)$.

Proof. Let $a = 2^n - 2^k$ and $P = P_{a+1}$, Then $d(a + 1) = P.\operatorname{cosub}(a)$. Let

$$x = \rho_{k-1,k}(a) = 2^n - 2^k - 2^{k-1}.$$

Then $P.\operatorname{super}(x) = \{i \mid x \leq i < a\}$, since x is covered by the $k-1$ elements $x+2^0, \ldots, x+2^{k-2}$. As a result, $P.\operatorname{cosuper}(x) = P \setminus \{i \mid x \leq i < a\}$ and that is easily seen to be isomorphic to P_{x+1} via the isomorphism $\rho_{k-1,k}$.

We form $d_1 = d(a+1).\operatorname{cosub}(x)$ and $d_2 = d(a+1).\operatorname{cosuper}(x)$. By Lemma 2 we have $\Delta(a+1) = \alpha(d_1) + \alpha(d_2)$. So it remains to show that $\alpha(d_2) = \alpha(d(2^n - 2^k - 2^{k-1} + 1))$ and $\alpha(d_1) = \alpha(d(2^n - 2^{n-k} - 2^{n-k-1} + 1))$.

For d_2 this is rather immediate, since $d_2 = P.\operatorname{cosuper}(x).\operatorname{cosub}(a)$ and that is, by the remark above, isomorphic to $P_{x+1}.\operatorname{cosub}(x)$, which is $d(x+1)$.

For d_1 this is not as obvious, but we can make use of the dual isomorphism between $d(a+1)$ and $d(2^n + 2^{n-k} + 1)$ of the proof of Formula 7. In fact, $\delta(a) = 2^n - 2^{n-k}$ and $\delta(x) = 2^n - 2^{n-k} - 2^{n-k-1}$. This implies that d_1 is dually isomorphic to

$$d(2^n + 2^{n-k} + 1).\operatorname{cosuper}(2^n - 2^{n-k} - 2^{n-k-1}).$$

And this finally is, by the same reasoning as for d_2 above, isomorphic to $d(2^n - 2^{n-k} - 2^{n-k-1} + 1)$. $\square$

Applying Formula 8 for specific combinations of n, k gives some interesting connections. For even n, say $n = 2q$, we get

$$\Delta(2^{2q} - 2^q + 1) = 2 \cdot \Delta(2^{2q} - 2^q - 2^{q-1} + 1),$$

e.g., $\Delta(13) = 2 \cdot \Delta(11)$ or $\Delta(57) = 2 \cdot \Delta(53)$.

An additional curiosity here is that $\Delta(13) = 50$ and 50 is the largest Δ value among all $\Delta(i)$ for $i \leq 2^4$ and $\Delta(57) = 2453690$ is the largest Δ value among all $\Delta(i)$ for $1 \leq i \leq 2^6$. This suggests the general result for $n = 2q$ that $\Delta(2^{2q} - 2^q + 1)$ is the largest value of $\Delta(i)$ for $i \leq n$.

If we continue to speculate in this way for n odd, say, $n = 2q - 1$ and if we look for the value of $\Delta(i)$ that is largest among all $i \leq 2^n$, then

for $n = 2 \cdot 2 - 1 = 3$ the largest value is $5 = \Delta(5) = \Delta(7)$,

for $n = 2 \cdot 3 - 1 = 5$ the largest value is $1692 = \Delta(25) = \Delta(29)$,

for $n = 2 \cdot 4 - 1 = 7$ the largest value is $510955171111 = \Delta(113) = \Delta(121)$.

This suggests that in general the largest $\Delta(i)$ value for $i \leq n = 2q - 1$ is obtained by $\Delta(2^{2q-1} - 2^{q-1} + 1)$ and by $\Delta(2^{2q-1} - 2^q + 1)$. Proving this may be quite difficult.

We also note that from Table 1 we see that the equalities

$$\begin{aligned} \Delta(2^5 - 3) &= 6\Delta(2^4 + 3) \quad \text{and} \\ \Delta(2^4 - 3) &= 2 \cdot \Delta(2^3 + 3) \end{aligned}$$

hold. But $\Delta(2^6 - 3) = 580655 = 5 \cdot 116131$, and 116131 is prime and is not a Δ or an F value.

Here are some other recursion formulas.

Formula 9. *For all $k, n \in \mathbb{N}$ with $0 \le k \le n-1$, we have $\Delta(2^n - 2^k) = F(2^{n-1} - 2^k)$.*

Proof. Let $P = P_{2^n}$ and $a = 2^n - 2^k$. Then by (5) $P.\operatorname{cosuper}(a) = P_a$ and therefore $d(a) = P.\operatorname{cosuper}(a).\operatorname{cosub}(a-1)$. We now observe that $a - 1 = 2^n - 1 - 2^k$ is a dual atom of P and, as noted right after (6), this implies that the poset Q with

$$Q = P.\operatorname{cosub}(a-1)$$

is dually isomorphic to $P_{2^{n-1}}$. If ϕ is this dual isomorphism then

$$\phi(Q.\operatorname{cosuper}(a)) = P_{2^{n-1}}.\operatorname{cosuper}(\phi(a))$$

and $\phi(a)$ has rank k in $P_{2^{n-1}}$. As a result, $\phi(Q.\operatorname{cosuper}(a))$ is isomorphic to $P_{2^{n-1}-2^k}$, which proves Formula 9. $\square$

Formula 10. *For all $n, k \in \mathbb{N}$ with $0 \le k \le n-1$, we have $\Delta(2^n + 2^k) = F(2^n - 2^k)$.*

Proof. $\Delta(2^n + 2^k) = \alpha(P_{2^n+2^k}.\operatorname{cocone}(2^n + 2^k - 1))$. The element $2^n + 2^k - 1$ is the join in the lattice B_{n+1} of the atom 2^n and $2^k - 1$. So $2^k - 1$ is covered by $2^n + 2^k - 1$. Thus, the semi-ideal $\operatorname{sub}(2^n + 2^k - 1)$ is the union of the interval $[0, 2^k - 1]$ and the interval $[2^n, 2^n + 2^k - 1]$. Therefore, the poset $P_{2^n+2^k}.\operatorname{cocone}(2^n + 2^k - 1)$ consists of $P_{2^n} \setminus P_{2^k}$, which is dually isomorphic to $P_{2^n-2^k}$. $\square$

We next find formulas for $\Delta(2^n + i)$ for $i \le 7$ that only involve the functions Δ and F applied to arguments less than $2^n + i$. By virtue of Formula 10 we need only consider $i = 3, 5, 6, 7$. The proofs will involve frequent use of the pivot notion introduced right after Lemma 1.

Let Q denote $P_{2^n+i}.\operatorname{cocone}(2^n + i - 1)$. Then $\Delta(2^n + i) = \alpha(Q)$ and for any $p \in Q$ by Lemma 1 we have $\alpha(Q)$ is the sum $\alpha(Q \setminus \{p\}) + \alpha(Q.\operatorname{cocone}(p))$. If both summands are known then we are done. If not then choose a poset for which the α value is not known and apply Lemma 1. Continue in this way until all the α values are known and add up these values. This will be a formula for $\Delta(2^n + i)$.

In the context in which we are working a poset S has the value of $\alpha(S)$ known if S or its dual is isomorphic to a P_k for $k < 2^n + i$ or to a poset whose α value was determined at an earlier stage in the proof.

An important fact to consider in choosing the element on which a poset is to pivot is that for any $j \le i$ the poset $[0, 2^n - 1] \cap \operatorname{cocone}(2^n + j)$ is dually isomorphic to P_{2^n-j}, which is a poset with a known α value. We use the heuristic of choosing pivot elements of the form $2^n + j$ wherever possible so as to obtain posets that are either entirely in $[0, 2^n - 1]$ that are dually isomorphic to P_r for $r \le 2^n$ or are isomorphic to a P_r for $r = 2^n + j$ for which $\alpha(P_r)$ has already been determined. This method suffices in our proofs for $i = 3, 6$, and 7. For determining $\alpha(Q)$ for $Q \subseteq [0, 2^n - 1]$ that are not isomorphic or dually isomorphic to a P_k we look for a poset Q' for which $\alpha(Q')$ had been determined by the use of Lemma 1 earlier in the proof. This is the case for our proof of the formula for $\Delta(2^n + 5)$.

Formula 11. *For every $n \in \mathbb{N}$ with $n \geq 2$, we have $\Delta(2^n + 3) = F(2^n - 2) + F(2^n - 3)$.*

Proof. The poset $Q = P_{2^n+3}.\operatorname{cocone}(2^n+2)$ consists of $[0, 2^n+2] \setminus \{0, 2^1, 2^n, 2^n+2\}$. Pivoting it with $p = 2^n + 1$ yields $Q \setminus \{p\} = P_{2^n} \setminus \{0, 1\}$ and $Q.\operatorname{cocone}(p) = P_{2^n} \setminus \{0, 1, 2\}$. And these posets are dually isomorphic to P_{2^n-2} and P_{2^n-3} as desired. □

Formula 12. *For every $n \in \mathbb{N}$ with $n \geq 1$, we have*

$$\Delta(2^n + 5) = 2 \cdot \Delta(2^n + 3) + \Delta(2^n + 4) - \Delta(2^{n-1} + 3) - \Delta(2^{n-1} + 4).$$

E.g., $\Delta(21) = 2 \cdot \Delta(19) + \Delta(20) - \Delta(11) - \Delta(12) = 2 \cdot 282 + 84 - 25 - 6 = 617$.

Proof. Let $Q = d(2^n + 5)$. Then by definition

$$Q = P_{2^n+5}.\operatorname{cocone}(2^n + 4) = P_{2^n+5} \setminus \{0, 4, 2^n, 2^n + 4\}.$$

First pivot Q on the element $2^n + 3$. Then we get the pivot components Q_1 and Q_2 with $Q_1 = Q \setminus \{2^n + 3\}$ and $Q_2 = Q.\operatorname{cocone}(2^n + 3) = [5, 2^n - 1]$, which is easily seen to be dually isomorphic to P_{2^n-5}. This pivot of Q gives

$$\alpha(Q) = \alpha(Q_1) + \alpha(Q_2) \tag{7}$$
$$\alpha(Q_2) = F(2^n - 5). \tag{8}$$

Another pivoting of Q_1 at $2^n + 2$ leads to the components $Q_{11} = Q_1 \setminus \{2^n + 2\}$ and $Q_{12} = Q_1.\operatorname{cocone}(2^n + 2)$. Now Q_{11} is easily seen to be isomorphic to $d(2^n + 3)$ via the isomorphism $\rho_{1,2}$ of B_n that interchanges the atoms 2^1 and 2^2. This accounts for one of the occurrences of $\Delta(2^n + 3)$ in the formula.

$$\alpha(Q_1) = \alpha(Q_{11}) + \alpha(Q_{12}) \tag{9}$$
$$\alpha(Q_{11}) = \alpha(d(2^n + 3))) \tag{10}$$

To obtain the other parts, we show that they arise as pivot components of known posets. In fact, if we pivot Q_{11} at 2, we get $Q_{111} = Q_{11} \setminus \{2\}$ and $Q_{112} = Q_{11}.\operatorname{cocone}(2)$. We see that $Q_{111} = Q_{12}$ and Q_{112} is mapped onto $d(2^{n-1} + 3)$ by the automorphism η of P_{n+1} that fixes the atom 2^0 and cycles the other atoms via $2^n \mapsto \cdots \mapsto 2^2 \mapsto 2^1 \mapsto 2^n$. Figure 8 shows η on B_4.

This gives

$$\alpha(Q_{11}) = \alpha(d(2^n + 3)) = \alpha(Q_{12}) + \alpha(d(2^{n-1} + 3)). \tag{11}$$

Finally we consider $d(2^n + 4) = \{4, \ldots, 2^n - 1\}$ and pivot it on 4. Obviously $d(2^n + 4) \setminus \{4\} = Q_2$. And $d(2^n + 4).\operatorname{cocone}(4)$ is mapped onto $d(2^{n-1} + 4)$ by the automorphism $\rho_{2,n-1}$ that exchanges the atoms 2^2 and 2^{n-1}. This gives

$$\alpha(d(2^n + 4)) = \alpha(d(2^{n-1} + 4)) + \alpha(Q_2). \tag{12}$$

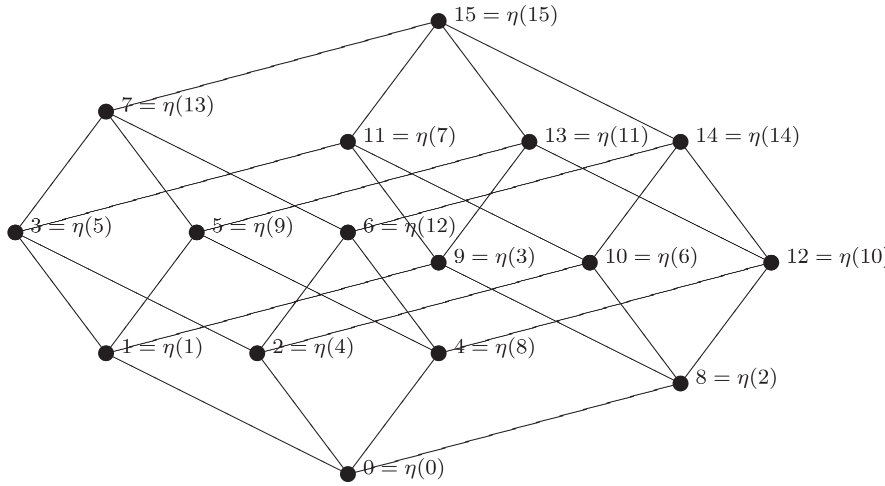

Figure 8: The action of η on $B_4 = P_{16}$.

Combining all these intermediate results we end up with

$$\alpha(Q_2) = \alpha(d(2^n + 4)) - \alpha(d(2^{n-1} + 4)) \text{ by (12)} \tag{13}$$
$$\alpha(Q_{12}) = \alpha(d(2^n + 3)) - \alpha(d(2^{n-1} + 3)) \text{ by (11)} \tag{14}$$
$$\alpha(Q_1) = 2\alpha(d(2^n + 3)) - \alpha(d(2^{n-1} + 3)) \text{ by (9), (10) and (14)} \tag{15}$$
$$\alpha(Q) = 2\alpha(d(2^n + 3)) + \alpha(d(2^n + 4)) - \alpha(d(2^{n-1} + 4) - \alpha(d(2^{n-1} + 3)) \text{ by (7), (13) and (15),} \tag{16}$$

and this is Formula 12. □

Formula 13. *For every $n \in \mathbb{N}$ with $n \geq 3$, we have*

$$\Delta(2^n + 2^2 + 2^1) = \Delta(2^n + 6) = F(2^n - 4) + F(2^n - 5) + F(2^n - 6).$$

E.g., $\Delta(22) = F(12) + F(11) + F(10) = 53 + 78 + 84 = 215$.

Proof. By definition the poset $d(2^n + 2^2 + 2^1)$ is $P_{2^n+2^2+2^1}.\,\text{cocone}(2^n + 2^2 + 2^0)$, which is $P_{2^n+2^2} \setminus \{0, 1, 4, 5, 2^n, 2^n + 1\}$. Let P denote $d(2^n + 2^2 + 2^1)$. First pivot P on the element $2^n + 2 + 1$. We have $P.\,\text{cocone}(2^n + 2 + 1) = P_{2^n} \setminus \{0, 1, 2, 3, 4, 5\}$, which is dually isomorphic to P_{2^n-6}.

Let Q denote $P \setminus \{2^n + 2 + 1\}$. We pivot Q on the element $2^n + 2$. We have

$$Q.\,\text{cocone}(2^n + 2) = P_{2^n} \setminus \{0, 1, 2, 4, 5\},$$

which is dually isomorphic to P_{2^n-5}. We also have $Q \setminus \{2^n + 2\} = P_{2^n} \setminus \{0, 1, 4, 5\}$, which is dually isomorphic to P_{2^n-4}. Thus, $\Delta(2^n + 2^2 + 2^1)$ is the sum of $F(2^n - 6), F(2^n - 5)$, and $F(2^n - 4)$. □

Formula 14. *For every $n \in \mathbb{N}$ with $n \geq 3$, we have*

$$\Delta(2^n + 2^2 + 2^1 + 2^0) = \Delta(2^n + 7) = F(2^n - 7) + F(2^n - 6) + \Delta(2^n + 6).$$

E.g., $\Delta(23) = F(9) + F(10) + \Delta(22) = 39 + 53 + 215 = 307$.

Note that this formula for a Δ value involves both F values and a Δ value.

Proof. Let P be the poset

$$\begin{aligned} P &= d(2^n + 2^2 + 2^1 + 2^0) = d(2^n + 7) = P_{2^n+7}.\operatorname{cocone}(2^n + 6) \\ &= (P_{2^n} \setminus \{0, 2, 4, 6\}) \cup \{2^n + 1, 2^n + 3, 2^n + 5\}. \end{aligned}$$

We pivot P on the element $2^n + 5$.

Then $P.\operatorname{cocone}(2^n+5) = (P_{2^n} \setminus \{0, 1, 2, 4, 5, 6\}) \cup \{2^n+3\}$. We pivot $P.\operatorname{cocone}(2^n+5)$ on the element $2^n + 3$ to get the posets $P_{2^n} \setminus \{0, 1, 2, 4, 5, 6\}$ and $P_{2^n} \setminus \{0, 1, 2, 3, 4, 5, 6\}$. These are dually isomorphic to P_{2^n-6} and P_{2^n-7}, respectively.

We also have $P_{2^n+5} = (P_{2^n} \setminus \{0, 2, 4, 6\}) \cup \{2^n + 1, 2^n + 3\}$. But then

$$d(2^n + 6) = P_{2^n+6}.\operatorname{cocone}(2^n + 5) = (P_{2^n} \setminus \{0, 1, 4, 5\}) \cup \{2^n + 2, 2^n + 3\},$$

which is isomorphic to the poset P_{2^n+5} by the automorphism $\rho_{0,1}$ of the Boolean lattice $B_{2^{m+1}}$ that interchanges the atoms 2^0 and 2^1 and leaves all the other atoms fixed. Hence the number of anti-chains in the poset P_{2^n+5} is equal to $\Delta(2^n + 6)$. Thus $\Delta(2^n + 7) = \Delta(2^n + 6) + F(2^n - 6) + F(2^n - 7)$. □

The proof of our final formula does not involve pivots but does make use of the inclusion-exclusion argument of Lemma 5.

Formula 15. *For every $n \in \mathbb{N}$ with $n \geq 1$, we have*

$$\Delta(2^n + 2^{n-1} + 1) = F(2^n + 2^{n-1} - 1) - 2 \cdot F(2^n - 2).$$

E.g., $\Delta(25) = F(23) - 2 \cdot F(14)$.

Proof. Let $a = 2^n + 2^{n-1}$, and let Q be the poset P_a. Then $Q = \{0, \ldots, a - 1\}$ and it is easily seen that $d(a + 1) = Q \setminus \{0, 2^{n-1}, 2^n\} = Q.\operatorname{cosub}(2^{n-1}, 2^n)$.

From Lemma 5 we infer that

$$\begin{aligned} \alpha(Q) = \alpha(Q.\operatorname{cosub}(2^{n-1}, 2^n)) + \alpha(Q.\operatorname{cosuper}(2^{n-1}) +& \\ \alpha(Q.\operatorname{cosuper}(2^n)) - \alpha(Q.\operatorname{cosuper}(2^{n-1}, 2^n)).& \end{aligned}$$

Again it is easily seen that $Q.\operatorname{cosuper}(2^n)$ is equal to P_{2^n} and via the isomorphism $\rho_{n-1,n}$ of P_{n+1} we see that $Q.\operatorname{cosuper}(2^{n-1})$ is isomorphic to P_{2^n}.

Similarly we infer that $Q.\operatorname{cosuper}(2^{n-1}, 2^n)$ is equal to $P_{2^{n-1}}$.

Combining all these observations we see that

$$F(a) = \Delta(a+1) + 2 \cdot F(2^n) - F(2^{n-1}). \tag{17}$$

Now by the definition of Δ, we have

$$F(a) = F(a-1) + \Delta(a) \tag{18}$$

and by Formula 10

$$\Delta(a) = F(2^n - 2^{n-1}) = F(2^{n-1}). \tag{19}$$

Substituting first (18) and then (19) into (17) we get

$$\Delta(a+1) = F(a-1) + 2 \cdot F(2^{n-1}) - 2 \cdot F(2^n). \tag{20}$$

By Formula 3 we have

$$F(2^n) = F(2^n - 2^1) + F(2^n - 2^{n-1}) = F(2^n - 2) + F(2^{n-1}) \tag{21}$$

Finally substituting (21) into (20) we arrive at

$$\Delta(a+1) = F(a-1) - 2 \cdot F(2^n - 2), \tag{22}$$

which is Formula 15. □

5 Summary

These are the formulas obtained so far.

For $k, n \in \mathbb{N}$ with $0 < k < n$

Formula 3: $F(2^n) = F(2^n - 2^k) + F(2^n - 2^{n-k})$,

Formula 6: $F(2^n) = F(2^n - 2^{n-k} - 2^{n-k-1}) + 2F(2^n - 2^k) - F(2^n - 2^k - 2^{k-1})$,

Formula 7: $\Delta(2^n - 2^k + 1) = \Delta(2^n - 2^{n-k} + 1)$,

Formula 8: $\Delta(2^n - 2^k + 1) = \Delta(2^n - 2^k - 2^{k-1} + 1) + \Delta(2^n - 2^{n-k} - 2^{n-k-1} + 1)$,

Formula 9: $\Delta(2^n - 2^k) = F(2^{n-1} - 2^k)$,

Formula 10: $\Delta(2^n + 2^k) = F(2^n - 2^k)$,

Formula 11: $\Delta(2^n + 2^1 + 2^0) = F(2^n - 2) + F(2^n - 3)$,

Formula 12: $\Delta(2^n + 2^2 + 2^0) = 2 \cdot \Delta(2^n + 3) + \Delta(2^n + 4) - \Delta(2^{n-1} + 3) - \Delta(2^{n-1} + 4)$,

Formula 13: $\Delta(2^n + 2^2 + 2^1) = F(2^n - 4) + F(2^n - 5) + F(2^n - 6)$, Formula 14: $\Delta(2^n + 2^2 + 2^1 + 2^0) = F(2^n - 7) + F(2^n - 6) + \Delta(2^n + 6)$,

Formula 15: $\Delta(2^n + 2^{n-1} + 1) = F(2^n + 2^{n-1} - 1) - 2 \cdot F(2^n - 2)$.

An application of these formulas gives numerical values for the values of Δ for $n = 16, \ldots, 32$—with the exception of $\Delta(26)$, where we leave it as an open question whether the formula given there is a special case of a more general one. Thus we have

$\Delta(16) = 1$ by Formula 3,

$\Delta(17) = F(15) = 167$ by Formula 10,

$\Delta(18) = F(14) = 148$ by Formula 10,

$\Delta(19) = F(13) + F(14) = 282$ by Formula 11,

$\Delta(20) = F(12) = 84$ by Formula 10,

$\Delta(21) = 2 \cdot \Delta(19) + \Delta(20) - \Delta(11) - \Delta(12) = 617$ by Formula 12,

$\Delta(22) = F(12) + F(11) + F(10) = 215$ by Formula 10,

$\Delta(23) = F(9) + F(10) + \Delta(22) = 307$ by Formula 14,

$\Delta(24) = F(8) = 20$ by Formula 10,

$\Delta(25) = F(23) - 2 \cdot F(14) = 1692$ by Formula 15,

$\Delta(26) = F(18) + F(13) + F(11) + F(7) = 714$,

$\Delta(27) = \Delta(25) - \Delta(21) = 1075$ by Formulas 7 and 8,

$\Delta(28) = F(12) = 84$ by Formula 9,

$\Delta(29) = \Delta(25) = 1692$ by Formula 7,

$\Delta(30) = F(14) = 148$ by Formula 9,

$\Delta(31) = F(15) = 167$ by Formula 9,

$\Delta(32) = 1$ by Formula 3.

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