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Complementary Equations with Advanced Subscripts

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Abstract

This paper discusses complementary equations, such as $a_n = b_n + b_{2n}$, in which (a_n) and (b_n) are strictly increasing complementary sequences for which, in addition to the subscript n, at least one subscript in advance of n, such as $n+1$ or $2n$, occurs. Some of the equations are solved with proofs, and others are presented as conjectures or examples based on `Mathematica` programs that appear in the final section of the paper.

1 Introduction

Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Many sequences (a_n) in $\mathbb{N}$ are defined by an equation that tells how to determine each term from preceding terms; e.g., the Fibonacci sequence, (F_n), arises from the equation $F_n = F_{n-1} + F_{n-2}$, starting with $F_1 = 1$ and $F_2 = 1$. Less familiar are sequences generated by a *complementary equation*—that is, an equation that refers to both (a_n) and its complement in $\mathbb{N}$.

In this paper, $a = (a_n)$ denotes an increasing sequence in $\mathbb{N}$ such that the complement in $\mathbb{N}$ of the set $A = \{a_n : n \geq 0\}$ is infinite. We write the complement as B and denote by $b = (b_n)$ the sequence of numbers in B in increasing order. A classic example of a complementary equation is

$$b_n = a_n + n \text{ where } a_1 = 1,$$

with solution given by

$$a_n = \lfloor n\tau \rfloor, \qquad b_n = n + \lfloor n\tau \rfloor,$$

where $\tau = (1+\sqrt{5})/2$, the golden ratio. The sequences (a_n) and (b_n) are the lower and upper Wythoff sequences, A000201 and A001500 in the *On-line Encyclopedia of Integer Sequences* [6]; also see [4] and [5].

2 Advanced subscripts

It may seem surprising that complementary equations with advanced subscripts, such as $a_n = b_n + b_{2n}$, make sense, since $2n$ comes after n. The first theorem will show that, with initial value $b_0 = 1$, this equation and many others make sense and have unique solutions.

Theorem 1. *Suppose that $f : \mathbb{N}^2 \to \mathbb{N}$ satisfies $f(1,1) \geq 2$ and*

$$f(m+1, n) \geq f(m,n) + 1 \text{ and } f(m, n+1) \geq f(m,n) + 1$$

for all $(m,n) \in \mathbb{N}^2$. If $g, h : \mathbb{N}_0 \to \mathbb{N}$ are strictly increasing functions, then the complementary equation

$$a_n = f(b_{g(n)}, b_{h(n)}), \text{ with } b_0 = 1, \tag{1}$$

has a unique solution.

Proof. We assume that a solution $(a, b) = ((a_n), (b_n))$ exists and then confirm inductively that it is uniquely determined; that is, a solution is generated inductively by (1), and it is unique. For $n \geq 0$,

$$\begin{aligned} f(b_{g(n)}, b_{h(n)}) &\geq f(b_{g(n)}, b_{h(n)} - 1) + 1 \\ &\geq f(b_{g(n)}, b_{h(n-1)}) + 1 \\ &\geq f(b_{g(n)} - 1, b_{h(n-1)}) + 2 \\ &\geq f(b_{g(n-1)}, b_{h(n-1)}) + 2, \end{aligned}$$

so that

$$a_n - a_{n-1} \geq 2. \tag{2}$$

Since $b_0 = 1$, we have $a_0 = f(b_0, b_0) = f(1,1) \geq 2$. Then (2) implies $a_1 \geq a_0 + 2$, so that by complementation, the numbers b_i are uniquely determined as the consecutive integers in $[a_0 + 1, a_1 - 1]$, and inductively, the set of b_i are uniquely determined as the consecutive integers in $[a_n + 1, a_{n+1} - 1]$, for all $n \geq 0$. □

Following is a list of some complementary equations to which Theorem 1 applies:

$$\begin{aligned}
a_n &= 2b_n, \\
a_n &= b_n + b_{n+2}, \\
a_n &= 3b_n + 2b_{n+2} - 5, \\
a_n &= 2b_n + b_{2n} + 1, \\
a_n &= b_n^2 + 2b_{n+1}, \\
a_n &= b_{\lfloor n/2 \rfloor} + b_{\lfloor 3n/2 \rfloor}, \\
a_n &= b_n b_{n+1} + 1, \\
a_n &= b_n^2 + b_{n+1}^2.
\end{aligned}$$

The method of proof suggests that Theorem 1 could be extended to cover many more types of equations with advanced subscripts, such as these:

$$\begin{aligned}
a_n &= a_{n-1} + b_{n+1}, \\
a_n &= b_n + b_{n+1} + b_{n+2}, \\
a_n &= b_{n+1} + n + 1, \\
a_n &= b_n + b_{2n} + b_{3n} + b_{4n} - 1.
\end{aligned}$$

It appears that for many complementary equations, exact formulas for solutions are elusive and that generating the sequences depends on the mex function (**m**inimal **ex**cludant), which has been described as "unwieldy" [3]. Accordingly, aside from multi-case proofs for certain simple-looking equations (in Sections 2 and 4), we rely on `Mathematica` programs, as shown in Section 7, to generate sequences that reveal both expected and unexpected results, leading to examples and conjectures based on experimentation.

3 The equation $a_n = b_{2n} + b_{4n}$, where $b_0 = 1$

In this section, we consider equations $a_n = b_{hn} + b_{khn}$, where $h \geq 2$. (A different method applies to the case $h = 1$, as in Section 5). First, we consider the special case $a_n = b_{2n} + b_{4n}$, and then we state conjectures for general h and k.

Lemma 2. *Suppose that a sequence β is given by*

$$\beta_n = r(m(n)) + 21\lfloor n/18 \rfloor, \tag{3}$$

where $m(n) = n - 18\lfloor n/18 \rfloor$ and

$$r(m) = \begin{cases} 1, & \text{if } m = 0; \\ m+2, & \text{if } 1 \le m \le 7; \\ m+3, & \text{if } 8 \le m \le 13; \\ m+4, & \text{if } 14 \le m \le 17. \end{cases}$$

Then

$$\beta_{2n+2} - \beta_{2n} = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 3; \\ 2, & \text{if } n \not\equiv 0 \pmod 3) \end{cases}$$

and

$$\beta_{4n+4} - \beta_{4n} = \begin{cases} 4 & \text{if } n \equiv 2 \pmod 3; \\ 5 & \text{if } n \not\equiv 2 \pmod 3. \end{cases}$$

Proof. For $0 \le n \le 17$, we have $m(n) = n$, and values of $r(n)$ as shown in Table 1.

$n = m(n)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$r(n)$	1	3	4	5	6	7	8	9	11	12	13	14	15	16	18	19	20	21

Table 1: Values of $r(n)$

By (3) we have

$$\beta_{2n+2} - \beta_{2n} = r(m(2n+2)) - r(m(2n)) + 21\left(\left\lfloor \frac{n+1}{9} \right\rfloor - \left\lfloor \frac{n}{9} \right\rfloor\right),$$

which leads to cases, first for $\beta_{2n+2} - \beta_{2n}$ (Cases 1.1 and 1.2), then for $\beta_{4n+4} - \beta_{4n}$.

Case 1.1: $\lfloor \frac{n+1}{9} \rfloor - \lfloor \frac{n}{9} \rfloor = 0$. Here, $m(2n+2) - m(2n) = 2$, so that

$$\beta_{2n+2} - \beta_{2n} = r(m(2n+2)) - r(m(2n)),$$

which, as in Table 1, is 3 if $n \equiv 0 \pmod 3$ and 2 otherwise.

Case 1.2: $\lfloor \frac{n+1}{9} \rfloor - \lfloor \frac{n}{9} \rfloor = 1$. Here, n must be of the form $9j-1$, so that $m(2n) = m(18j-2) = 16$ and $m(2n+2) = m(18j) = 0$. Then

$$\beta_{2n+2} - \beta_{2n} = r(0) - r(16) + 21 = 2,$$

as required, since $n \equiv 2 \pmod 3$.

Case 2.1: $\lfloor \frac{2n+2}{9} \rfloor - \lfloor \frac{2n}{9} \rfloor = 0$. Here, $m(4n+4) - m(4n) = 4$, so that

$$\beta_{4n+4} - \beta_{4n} = r(m(4n+4)) - r(m(4n)),$$

which, as in Table 1, is 4 if $n \equiv 2 \pmod 3$ and 5 otherwise.

Case 2.2: $\lfloor \frac{2n+2}{9} \rfloor - \lfloor \frac{2n}{9} \rfloor = 1$. We recognize two exhaustive subcases:

Subcase 2.2.1: $2n = 7 + 9j$ for some j, so that $2n + 2 = 9(j + 1)$. Here, $m(4n) = 14$ and $m(4n + 4) = 0$, whence

$$\beta_{4n+4} - \beta_{4n} = r(0) - r(14) + 21 = 1 - 18 + 21 = 4,$$

as required, since $n \equiv 2 \pmod 3$.

Subcase 2.2.2: $2n = 8 + 9j$ for some j, so that $2n + 2 = 1 + 9(j + 1)$. Then $m(4n) = 16$ and $m(4n + 4) = 2$, whence

$$\beta_{4n+4} - \beta_{4n} = r(3) - r(16) + 21 = 4 - 20 + 21 = 5,$$

as required, since $n \equiv 1 \pmod 3$. □

We are now ready for the main theorem of this section.

Theorem 3. *Let $a = (a_n)$ and $b = (b_n)$ be the strictly increasing complementary sequences determined by the equation*

$$a_n = b_{2n} + b_{4n}, \text{ where } b_0 = 1. \tag{4}$$

Then

$$a_n = \begin{cases} 2 + 21\lfloor n/3 \rfloor, & \text{if } m \equiv 0 \pmod 3; \\ 10 + 21\lfloor n/3 \rfloor, & \text{if } m \equiv 1 \pmod 3; \\ 17 + 21\lfloor n/3 \rfloor, & \text{if } m \equiv 2 \pmod 3. \end{cases} \tag{5}$$

Proof. Since $b_0 = 1$, we have $a_0 = 2$ by (4), and since a and b are strictly increasing and complementary, we have

$$2 < b_1 < b_2 < b_3 < b_4,$$

which implies that $b_i = 2 + i$ for $i = 1, \ldots, 4$. Equation (4) implies $a_1 = b_2 + b_4 = 4 + 6 = 10$, so that $b_i = 3 + i$ for $i = 5, 6, 7$. Then $a_2 = b_4 + b_8 \geq 6 + 10 = 16$, and $a_2 = 6 + 11 = 17$. Thus, (5) is established for $n = 0, 1, 2$. Before assuming (5) as an induction hypothesis, we note that

$$a_3 = b_6 + b_{12} = 8 + 15 = 23$$

implies $b_{13} = 16$ and $b_i = 4 + i$ for $i = 14, \ldots, 17$, so that, in regard to the sequence β_n in Lemma 2, we have $b_n = \beta_n$ for $n = 0, \ldots, 17$. Moreover, since b is the complement of a, we have, for $n = 0, \ldots 17$,

$$b_n = r(m(n)) + 21\lfloor n/18 \rfloor, \tag{6}$$

where $m(n) = n - 18\lfloor n/18 \rfloor$ and

$$r(m) = \begin{cases} 1, & \text{if } m = 0; \\ m + 2, & \text{if } 1 \leq m \leq 7; \\ m + 3, & \text{if } 8 \leq m \leq 13; \\ m + 4, & \text{if } 14 \leq m \leq 17. \end{cases}$$

For use below, we note that (4) implies

$$a_{n+1} = a_n + b_{4n+4} - b_{4n} + b_{2n+2} - b_{2n} \tag{7}$$

for $n \geq 0$. Now, as an induction hypothesis, assume that (5) holds for arbitrary $n \geq 2$. We have three cases, according as $n \equiv 0, 1, 2 \pmod 3$.

Case 1: $n \equiv 0 \pmod 3$. Here, $a_n = 2 + 21\lfloor n/3 \rfloor$, and by Lemma 2, $b_{2n+2} - b_{2n} = 3$ and $b_{4n+4} - b_{4n} = 5$, so that by (7),

$$a_{n+1} = a_n + 8 = 10 + 21\lfloor n/3 \rfloor = 10 + 21\lfloor (n + 1)/3 \rfloor.$$

By complementation, we also have (6) for the numbers b_i required for the following proof for Case 2.

Case 2: $n \equiv 1 \pmod 3$. Here, $b_{2n+2} - b_{2n} = 2$ and $b_{4n+4} - b_{4n} = 5$, so that

$$a_{n+1} = a_n + 7 = 17 + 21\lfloor n/3 \rfloor = 17 + 21\lfloor (n + 1)/3 \rfloor.$$

By complementation, we also have (6) for the numbers b_i required for the following proof for Case 3.

Case 3: $n \equiv 2 \pmod 3$. Here, $b_{2n+2} - b_{2n} = 2$ and $b_{4n+4} - b_{4n} = 4$, so that

$$a_{n+1} = a_n + 6 = 23 + 21\lfloor n/3 \rfloor = 2 + (21 + 21\lfloor n/3 \rfloor).$$

By complementation, we also have (6) for the numbers b_i required for the next application of the proof for Case 1, so that by the principle of mathematical induction, (5) holds for all $n \geq 0$. □

Corollary 4. *The sequence* (b_n) *in Theorem 3 is identical to the sequence* (β_n) *in Lemma 2.*

Proof. A proof is included in the proof of Theorem 3. □

Corollary 5. *The sequence a in Theorem 3 satisfies the recurrence*

$$a_{n+4} = a_{n+3} + a_{n+1} - a_n.$$

Proof. We shall make repeated use of Corollary 4. Let

$$B_2 = b_{2n+2} - b_{2n}, \quad B_4 = b_{4n+4} - b_{4n}, \quad B_8 = b_{2n+8} - b_{2n+6}, \quad B_{16} = b_{4n+16} - b_{4n+12},$$

so that

$$a_{n+4} - a_{n+3} = B_{16} + B_8 \text{ and } a_{n+1} - a_n = B_4 + B_2.$$

There are three cases:

Case 1: $n \equiv 0 \pmod 3$.

$$\begin{aligned} B_{16} &= v\beta_{4m+4} - \beta_{4m} \text{ for } m = n+3, \text{ so that } m \equiv 0 \pmod 3, \text{ and } B_{16} = 5. \\ B_8 &= \beta_{2m+2} - \beta_{2m} \text{ for } m = n+3, \text{ so that } m \equiv 0 \pmod 3, \text{ and } B_8 = 3. \\ B_4 &= \beta_{2m+4} - \beta_{4m} \text{ for } m = n, \text{ so that } m \equiv 0 \pmod 3, \text{ and } B_4 = 5. \\ B_2 &= \beta_{2m+2} - \beta_{2m} \text{ for } m = n, \text{ so that } m \equiv 0 \pmod 3, \text{ and } B_2 = 3. \end{aligned}$$

Case 2: $n \equiv 1 \pmod 3$.

$$\begin{aligned} B_{16} &= \beta_{4m+4} - \beta_{4m} \text{ for } m = n+3, \text{ so that } m \equiv 1 \pmod 3, \text{ and } B_{16} = 5. \\ B_8 &= \beta_{2m+2} - \beta_{2m} \text{ for } m = n+3, \text{ so that } m \equiv 1 \pmod 3, \text{ and } B_8 = 3. \\ B_4 &= \beta_{2m+4} - \beta_{4m} \text{ for } m = n, \text{ so that } m \equiv 1 \pmod 3, \text{ and } B_4 = 5. \\ B_2 &= \beta_{2m+2} - \beta_{2m} \text{ for } m = n, \text{ so that } m \equiv 1 \pmod 3, \text{ and } B_2 = 3. \end{aligned}$$

Case 3: $n \equiv 2 \pmod 3$.

$$\begin{aligned} B_{16} &= \beta_{4m+4} - \beta_{4m} \text{ for } m = n+3, \text{ so that } m \equiv 2 \pmod 3, \text{ and } B_{16} = 4. \\ B_8 &= \beta_{2m+2} - \beta_{2m} \text{ for } m = n+3, \text{ so that } m \equiv 2 \pmod 3, \text{ and } B_8 = 2. \\ B_4 &= \beta_{2m+4} - \beta_{4m} \text{ for } m = n, \text{ so that } m \equiv 2 \pmod 3, \text{ and } B_4 = 4. \\ B_2 &= \beta_{2m+2} - \beta_{2m} \text{ for } m = n, \text{ so that } m \equiv 2 \pmod 3, \text{ and } B_2 = 2. \end{aligned}$$

In all the cases,

$$B_{16} + B_8 = B_4 + B_2,$$

which is equivalent to

$$a_{n+4} - a_{n+3} = a_{n+1} - a_n.$$

□

Next, we give a conjecture based on Program 7.1, which was used to generated Table 2.

Conjecture 6. Let $a = (a_n)$ and $b = (b_n)$ be the strictly increasing complementary sequences determined by the equation

$$a_n = b_{hn} + b_{khn},$$

where $h \geq 2$, $k \geq 2$, and $b_0 = 1$. Then a is a linear recurrence sequence given by

$$a_n = a_{n-1} + a_{n-k-1} - a_{n-k-2}$$

with initial terms

$$a_0 = 2, \quad a_1 = h(k+1) + 4, \quad a_2 = 2h(k+1) + 5, \ldots, \quad a_k = kh(k+1) + k + 3.$$

The sequence a consists of the positive integers congruent modulo $(k+1)(2k+h+1)$ to the numbers in $\{a_0, a_1, \ldots, a_k\}$, and the sequence b satisfies the linear recurrence

$$b_n = b_{n-1} + b_{n-m} - b_{n-m-1}, \text{ where } m = (k+2)hk + h - 5.$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$a_n = b_{2n} + b_{4n}$	2	10	17	23	31	38	44	52	59	65	73	80	86	94
$a_n = b_{2n} + b_{6n}$	2	12	21	30	38	48	57	66	74	84	93	102	110	120
$a_n = b_{2n} + b_{8n}$	2	14	25	36	47	57	69	80	91	102	112	124	135	146
$a_n = b_{3n} + b_{6n}$	2	13	23	32	43	53	62	73	83	92	103	113	122	133
$a_n = b_{3n} + b_{9n}$	2	16	29	42	54	68	81	94	106	120	133	146	158	172
$a_n = b_{4n} + b_{8n}$	2	16	29	41	55	68	80	94	107	119	133	146	158	172

Table 2: First fourteen terms, $a_n = b_{hn} + b_{khn}, b_0 = 1$

In the next section we extend our earlier attention to the special case $a_n = b_{2n} + b_{4n}$ by adding a nonzero constant to the right-hand side.

4 The equation $a_n = b_{2n} + b_{4n} + c$, where $b_0 = 1$

Theorem 3 gives a precise solution of $a_n = b_{2n} + b_{4n}$, with $b_0 = 1$, and Corollary 5 shows that (a_n) is more "regular" than might be thought if (5) were the only representations. We shall see, however, that this sort of regularity does not extend to equations of the form $a_n = b_{2n} + b_{4n} + c$ when $c \geq 1$.

Theorem 7. *Let $a = (a_n)$ and $b = (b_n)$ be the strictly increasing complementary sequences determine by the equation*

$$a_n = b_{2n} + b_{4n} + 1, \text{ where } b_0 = 1.$$

Then

$$a_n = 7n + \begin{cases} 3, & \text{if } n \equiv 0 \pmod 3; \\ 3 \text{ or } 4, & \text{if } n \not\equiv 0 \pmod 3. \end{cases} \tag{8}$$

Proof. We have [2] inductively

$$b_{6n+3} \geq 7n+5 \text{ and } b_{6n+1} \leq 7n+2,$$

so that

$$t + \lfloor (t+3)/6 \rfloor + 1 \leq b_t \leq t + \lceil (t-1)/6 \rceil + 1 \tag{9}$$

for $t = 6k+3, \ldots, 6k+7$. Thus, $b_t = t + \lceil (t-1)/6) \rceil + 1$ except for

$$7k+3 \leq b_{6k+2} \leq 7k+4.$$

Consequently, $7n+3 \leq a_n \leq 7n+4$, as desired. □

To illustrate the "irregularity" of (8) when $n \not\equiv 0 \pmod 3$, let $r = (r_n)$ be the increasing sequence of numbers n for which $a_n = 7n+4$:

$$r = (1, 5, 7, 10, 13, 14, 17, 19, 23, 25, 28, 32, 34, 37, 41, 44, 46, 47, 49, 50, \ldots). \tag{10}$$

Conjecture 8. The sequence r is not linearly recurrent, and

$$\{a_n - a_{n-1} : n \geq 1\} = \{1, 2, 3, 4, 5\}.$$

The five differences shown in 10 all occur as n ranges from 1 to 28. No other difference occurs as n ranges up to 10000. Similar mysteries are found for other choices of c. Initial terms of (a_n) are shown in Table 3, which was used to construct Table 3. Looking ahead, it appears that the sequences (a_n) are much more "regular" for $c \in \{2, 3, 5\}$, than for some other values of c; see Conjectures 14-16.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$a_n = b_{2n} + b_{4n} + 0$	2	10	17	23	31	38	44	52	59	65	73	80	86	94
$a_n = b_{2n} + b_{4n} + 1$	3	11	17	24	31	39	45	53	59	66	74	80	87	95
$a_n = b_{2n} + b_{4n} + 2$	4	11	18	25	32	39	46	53	60	67	74	81	88	95
$a_n = b_{2n} + b_{4n} + 3$	5	12	19	26	33	40	47	54	61	68	75	82	89	96
$a_n = b_{2n} + b_{4n} + 4$	6	12	19	27	34	41	48	54	62	69	75	82	90	96
$a_n = b_{2n} + b_{4n} + 5$	7	13	20	28	34	41	49	55	62	70	76	83	91	97
$a_n = b_{2n} + b_{4n} + 6$	8	14	21	28	35	42	50	56	63	71	77	84	92	98
$a_n = b_{2n} + b_{4n} + 7$	9	15	22	28	36	43	50	57	64	70	78	85	92	99
$a_n = b_{2n} + b_{4n} + 8$	10	16	22	29	36	44	50	58	65	71	79	86	92	100

Table 3: $a_n = b_{2n} + b_{4n} + c$, for $c = 0, 1, \ldots, 8$

With one exception, the data in Table 3, generated by Program 7.2, show nondecreasing columns. It turns out that this exception is indicative of many others. In order to describe them, we introduce some notation: for each $c \geq 1$ and the equations

$$a_n = b_{2n} + b_{4n} + c \text{ and } a_n^* = b_{2n}^* + b_{4n}^* + c + 1, \text{where } b_0 = b_0^* = 1, \tag{11}$$

throughout this section let

$$\begin{aligned} N &= \text{an upper bound for } n\text{, depending on context;}\\ m(c) &= \min\{a_n^* - a_n\} \text{ for } 0 \le n \le N = 5000;\\ M(c) &= \max\{a_n^* - a_n\} \text{ for } 0 \le n \le N = 5000;\\ T(c,i) &= \text{sequence of indices } n \text{ for which } a_n^* - a_n = i;\\ \Delta(c,i) &= \text{set of distinct differences in } T(c,i). \end{aligned}$$

The choice of N varies as an experimental upper bound when sampling various (presumably) unbounded sequences.

Note that the sets $T(c,i)$ for $i = m(c), \ldots, M(c)$ partition $\mathbb{N}_0$. We shall refer to m, M, T, and Δ in the following examples, which show various sorts of unwieldiness that appear when comparing the sequences (a_n) and (a_n^*) given by (11).

Example 9. For $c = 0$, we have $m(c) = 0, M(c) = 1$, and

$$\begin{aligned} T(0,0) &= (2,4,8,11,16,20,\ldots); \quad &\Delta(0,0) = \{1,2,3,4,5,7,8,9,10,12,13\};\\ T(0,1) &= (0,1,3,5,6,7,9,\ldots); \quad &\Delta(0,1) = \{1,2,3\}. \end{aligned}$$

In Example 9, $\Delta(0,0)$ consists of 11 integers in the interval $[1,13]$, with 6 and 11 missing. Table 4, generated by Program 7.3, shows the frequency of each difference $a_n^* - a_n$, for n up to $N = 10000$.

1	2	3	4	5	6	7	8	9	10	11	12	13
316	581	737	461	188	0	45	129	129	23	0	8	53

Table 4: Frequencies of differences in $\Delta(0,0), n = 1, 2, \ldots, N = 10000$

Example 10. For $c = 1$, we have $m(c) = 0, M(c) = 1$, and

$$\begin{aligned} T(1,0) &= (1,5,7,10,13,14,17,19\ldots),\\ \Delta(1,0) &= \{1,2,3,4,5\};\\ T(1,1) &= (0,2,3,4,6,8,9,11,12\ldots),\\ \Delta(1,1) &= \{1,2,3\}. \end{aligned}$$

Example 11. For $c = 2$, we have $m(c) = M(c) = 1$, so that $T(2,1) = \mathbb{N}_0$ and $\Delta(2,1) = \{1\}$.

Regarding Example 11, we pose this question: *is 2 the only value of c for which $m(c) = M(c)$?* (Program 7.3, with $c = 2$ in the first instance and $c = 3$ in the second, confirms that $a_n^* - a_n = 1$ for n up to $N = 10000$).

In connection with Table 4, the difference $a_n^* - a_n$ is never 6 or 11 for $N = 10000$, and is 12 relatively rarely; indeed, the least n for which $a_n^* - a_n = 12$ is $n = 1325$.

Example 12. For $c = 6$, we have $m(c) = -1, M(c) = 1$, and

$$\begin{aligned} T(6,-1) &= (9,18,24,27,36,45,48\ldots), \\ \Delta(6,-1) &= \{3,6,9\}; \\ T(6,0) &= (3,6,12,31,33,39,51\ldots), \\ \Delta(6,0) & \text{ (see note below)}; \\ T(6,1) &= (0,1,2,4,5,7,8,10,\ldots), \\ \Delta(6,1) &= \{1,2\}. \end{aligned}$$

It appears that $\Delta(6,0)$ is a much larger set, for large n, than all other sampled $\Delta(c,i)$. Indeed, it seems possible that if n is not restricted to an upper bound, then $\Delta(6,0)$ is infinite, as suggested by the following data, in which "newcomers" are bolded:

$$\begin{aligned} &\text{For } n = 1,\ldots,N = 6000, \\ &T(6,0) = \{3,6,9,12,18,24,27,36,39,42,48,51,54,57,66,75,81,93,99, \\ &\quad 111,117,120,123,156,162,198,237,279,354,360,480,531,660\}. \\ &\text{For } n = 1,\ldots,N = 12000, \\ &T(6,0) = \{3,6,9,12,18,24,27,36,39,42,48,51,54,57,66,75,81,93,99, \\ &\quad 111,117,120,123,156,162,\mathbf{171},\mathbf{180},198,237,279,354,360,480, \\ &\quad 531,660,\mathbf{711},\mathbf{849},\mathbf{1065}\}. \\ &\text{For } n = 1,\ldots,N = 16000, \\ &T(6,0) = \{3,6,9,12,18,24,27,36,39,42,48,51,54,57,66,75,81,93,99, \\ &\quad 111,117,120,123,156,162,171,\mathbf{174},180,198,\mathbf{225},237,\mathbf{270},279, \\ &\quad 354,360,480,531,660,711,849,\mathbf{1062},1065,\mathbf{1092}\}. \end{aligned}$$

Example 13. For $c = 9$, we have $m(c) = -1, M(c) = 2$, and

$$\begin{aligned} T(9,-1) &= (6), \Delta(9,-1) = \text{(empty)}; \\ T(9,0) &= (8,10,11,13,16,20,\ldots), \Delta(9,0) = \{1,2,3,4,6,7\}; \\ T(9,1) &= (0,1,2,3,4,5,6,7,9,\ldots), \Delta(9,1) = \{1,2,3\}; \\ T(9,2) &= (20), \Delta(9,2) = \text{(empty)}. \end{aligned}$$

In Example 13, $T(9,-1) = (6)$ comes from $a_6^* - a_6 = 44 - 45 = -1$, and $T(9,2) = (2,0)$ comes from $a_{20}^* - a_{20} = 144 - 142 = 2$. Otherwise, for $0 \le n \le N = 5000$, we have $0 \le a_n^* - a_n \le 1$.

For $0 \le n \le N = 5000$, and perhaps for all $n > N = 5000$,

$$\begin{aligned}
(m(c), M(c)) &= (0,1) \text{ for } c = 0,1,3,4,5,7,8,10,11,12,13,15,16,17,24,25,\\
&\qquad 27,28,32,33,36,48,50,56,57;\\
(m(c), M(c)) &= (-1,1) \text{ for } c = 6,14,18,19,20,21,23,26,34,37,39,40,44,\\
&\qquad 45,47,55,58;\\
(m(c), M(c)) &= (-1,2) \text{ for } c = 9,29,30,31,35,38,41,42,43,49,51,52,53,\\
&\qquad 54,59,60;\\
(m(c), M(c)) &= (-1,3) \text{ for } c = 22,46.
\end{aligned}$$

The next three conjectures, based on $N = 10000$, suggest that $a_n^* - a_n$ is "regular" for $c = 2, 3, 5$, (whereas, it appears, the regularity does not extend to $c = 4$ or $c = 6$).

Conjecture 14. Suppose that $c \ge 0$, and let (a_n) and $(\hat{a}_n)$ be solutions of

$$a_n = b_{2n} + b_{4n} \text{ and } \hat{a}_n = \hat{b}_{2n} + \hat{b}_{4n} + 2,$$

respectively. Then

$$\hat{a}_n - a_n = \begin{cases} 2, & \text{if } n \equiv 0 \pmod 3; \\ 1, & \text{if } n \not\equiv 0 \pmod 3). \end{cases}$$

Conjecture 15. Suppose that $c \ge 0$, and let (a_n) and $(\hat{a}_n)$ be solutions of

$$a_n = b_{2n} + b_{4n} \text{ and } \hat{a}_n = \hat{b}_{2n} + \hat{b}_{4n} + 3,$$

respectively. Then

$$\hat{a}_n - a_n = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 3; \\ 2, & \text{if } n \not\equiv 0 \pmod 3). \end{cases}$$

Conjecture 16. Suppose that $c \ge 0$, and let (a_n) and $(\hat{a}_n)$ be solutions of

$$a_n = b_{2n} + b_{4n} \text{ and } \hat{a}_n = \hat{b}_{2n} + \hat{b}_{4n} + 5,$$

respectively. Then

$$\hat{a}_n - a_n = \begin{cases} 5, & \text{if } n \equiv 0 \pmod 3; \\ 3, & \text{if } n \not\equiv 0 \pmod 3). \end{cases}$$

5 The equation $a_n = b_n + b_{2n}$

In contrast to results in Section 3, we find less "regularity" in the present section.

Theorem 17. *The solution of the complementary equation* $a_n = b_n + b_{2n}$*, with* $b_0 = 1$*, satisfies the following conditions for all* $n \geq 0$ *and* $i \geq 0$*:*

$$\begin{aligned} 4n+2 \leq a_n &\leq 4n+3, \\ b_{3i} &= 4i+1, \\ 4i+2 \leq b_{3i+1} &\leq 4i+3, \\ b_{3i+2} &= 4i+4. \end{aligned}$$

Proof. We paraphrase a proof given in MathOverflow [1]. Clearly the inequality

$$4n+2 \leq a_n \leq 4n+3 \tag{12}$$

holds for $n \in \{0,1,2\}$. We assume (12) as an induction hypothesis for $n \geq 2$. For convenience the phrase "is in a" will refer to terms of the sequence a, and likewise for b. We have three cases.

Case 1: $n = 3i$. By the induction hypothesis,

$$4i-2 \leq a_{i-1} \leq 4i-1, \tag{13}$$

so that the number of terms a_m that are $\leq 4i-1$ is i. (Specifically, these i numbers are $a_0, a_1, \ldots, a_{i-1}$.) Since every integer in $[1, 4i-1]$ is in a or b, the number of terms b_m that are $\leq 4i-1$ is $(4i-1)i - i = 3i-1$. By (13), the numbers $4i$ and $4i+1$ are also in b, so that the number of terms b_m in $[1, 4i+1]$ is $3i+1$. These numbers are $b_0, b_1, \ldots, b_{3i}$, so that $b_{3i} = 4i+1$.

Case 2: $n = 3i+1$. By the induction hypothesis,

$$4i+2 \leq a_i \leq 4i+3,$$

and $a_{i+1} \geq 4i+6$, so that there are exactly $i+1$ terms a_m in $[1, 4i+3]$. Therefore, there are exactly $4i+5-(i+1) = 3i+4$ terms b_m that are $\leq 4i+5$, specifically, $b_0, b_1, \ldots b_{3i+3}$. By the induction hypothesis,

$$4i+6 \leq a_{i+1} \leq 4i+7, \tag{14}$$

so that there are at most $i+1$ terms a_m in $[1, b_{i+3}]$. (Otherwise, $a_{i+1} \in [1, b_{i+3}]$, so that $a_{i+1} \leq b_{3i+3}$, contrary to the already proved inequality $b_{3i+3} \leq 4i+5 < a_{i+1}$). Since $b_{3i+3} \leq 4i+5$, and since, by (14), $4i+4$ and $4i+5$ are in b, we have $b_{3i+2} = 4i+4$ and $b_{3i+3} = 4i+5$, so that

$$b_{3i+1} \leq 4i+3. \tag{15}$$

We turn now to a proof that $b_{3i+1} \geq 4i+2$. Every integer in $[1, 4i]$ is in a or b, and since (by the induction hypothesis) $a_{i-1} \leq 4i-1$, there are at least i terms a_m that are $\leq 4i-1$, hence at most $(4i-1)-i = 3i-1$ terms b_m in $[1, 4i-1]$. Since

$$a_{i-1} \leq 4i-1 < 4i+2 \leq a_i,$$

the numbers $4i$ and $4i+1$ are in b. There are, therefore, at most $3i+1$ terms b_m in $[1, 4i+1]$. We have

$$b_0 < b_1 < b_2 < \cdots b_{3i} \leq 4i+1$$

and $b_{3i+1} > 4i+1$. Thus, $b_{3i+1} \geq 4i+2$.

Case 3: $n = 3i+2$. By the induction hypothesis, $a_{i+1} \geq 4i+6$, so that, since

$$4i+2 \leq a_i \leq 4i+3,$$

the number of terms a_m that are $\leq 4i+3$ is $i+1$. Since every integer in $[1, 4i+3]$ is in a or b, the number of terms b_m that are $\leq 4i+3$ is $4i+3-(i-1) = 3i+2$. The induction hypothesis implies

$$a_i \leq 4i+3 < 4i+6 < a_{i+1},$$

so that $4i+4$ is in b. We have thus counted a total of $3i+3$ terms b_m in $[1, 4i+4]$; they are

$$b_0, b_1, \ldots, b_{3i+2},$$

so that $b_{3i+2} = 4i+4$.

We now finish the induction for establishing (12):

$$\begin{aligned} 4(k+1) + \frac{10}{3} &= \frac{4(k+1)+5}{3} + \frac{4(2k+2)+5}{3} \\ &\geq b_{k+1} + b_{2k+2} \\ &= a_{k+1} \\ &\geq \frac{4(k+1)+2}{3} + \frac{4(2k+2)+2}{3} \\ &= 4(k+1) + \frac{4}{3}. \end{aligned}$$

Since a_{k+1} is an integer, the inequality

$$4(k+1) + \frac{10}{3} \geq a_{k+1} \geq 4(k+1) + \frac{4}{3}$$

implies

$$4(k+1)+2 \leq a_{k+1} \leq 4(k+1)+3.$$

□

The irregular subsequence of the sequence

$$b = (1, 3, 4, 5, 6, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 22, 24, 25, 27, \ldots$$

in Theorem 17 is

$$s(i) := (b_{3i+1}) = (3, 6, 11, 15, 19, 22, 27, 30, 35, 39, 42, 47, 51, 54, 67, \ldots), \qquad (16)$$

for which $4i + 2 \leq s(i) \leq 4i + 3$; indeed,

$$s_i = \begin{cases} 4i + 2, & \text{if } i = 2, 6, 8, 11, 14, 15, 18, 20, 24, 26, 29, 33, 35, 38, 42, \ldots \\ 4i + 3, & \text{if } i = 1, 3, 4, 5, 7, 9, 10, 12, 13, 16, 17, 19, 21, 22, 23, 25, \ldots. \end{cases}$$

Following are two conjectures based on Program 7.4, with n ranging from 1 to 50000.

Conjecture 18. The difference sequence of the above sequence

$$(2, 6, 8, 11, 14, 15, 18, \ldots)$$

has only five distinct terms: $1, 2, 3, 4, 5$.

The least n for which all five differences occur in Conjecture 18 is 111.

Conjecture 19. The limiting proportion of s_i's of the form $4i + 2$ is $2/5$.

6 The equation $a_n = ub_n + vb_{n+k} + c$

The preceding sections show that some complementary equations with advanced subscripts are easily solved, whereas others involve a multitude of cases and irregularities. Here we consider equations of the form

$$a_n = ub_n + vb_{n+k} + c, \text{ where } b_0 = 1, \qquad (17)$$

where u, v, and c are integers satisfying $k \geq 1$ and $1 \leq u \leq v$, and we find the same sort of regularities and irregularities. Let

$$x = \lim_{n\to\infty} a_n/n \text{ and } y = \lim_{n\to\infty} b_n/n.$$

Then (17) together with $1/x + 1/y = 1$, yields

$$x = u + v + 1 \text{ and } y = (u + v + 1)/(u + v).$$

Table 5, generated by Program 7.5, shows in columns 4 and 5 the minimum and maximum values of $a_n - (u+v+1)n$ for $n = 0, 1, \ldots, 100000$. Also, Table 5 indicates that for $u = v = 1$,

$$\max\{a_n - 3n\} = a_0 = k + 2,$$

but that the sequence $(m(n)) = (\min\{a_n - 3n\})$ is not so predictable, as indicated by its first 22 terms:

$$3, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 10, 10, 11, 11, 12, 13, 13, 14, 14, 15, 16.$$

u	v	k	min	max	a_0
1	1	1	3	3	3
1	2	1	−1	10	5
1	3	1	−3	15	7
1	4	1	−5	18	9
1	5	1	−5	23	11
2	1	1	−1	9	4
3	1	1	−3	13	5
4	1	1	−6	15	6
5	1	1	−6	19	7
2	3	1	−6	17	8
3	2	1	−6	16	7
2	2	1	−3	14	6
3	3	1	−5	21	9
1	1	2	3	4	4
1	1	3	4	5	5
1	1	4	4	6	6
1	1	5	5	7	7
1	1	6	6	8	8
1	1	7	6	9	9

Table 5: $a_n = ub_n + vb_{n+k}, b_0 = 1$

As already noted, Table 5 and further explorations suggest that if $u > 1$ or $v > 1$, then both min and max are unbounded. Taking $(u, v, k) = (1, 2, 1)$ as an example, we have $(\min, \max) = (-1, 10)$ when $N = 100000$. The appearance of (N, m, M) in the following list means that as n ranges from 1 to N, the minimal and maximal differences $a_n - (u+v+1)n$ are m and M:

$$(100, 2, 7), (300, 1, 7), (600, 1, 7), (700, 1, 8),$$
$$(1900, 1, 8), (2000, 0, 8), (5000, 0, 8), (6000, 0, 9),$$
$$(20000, -1, 9), (10000, -1, 10), (200000, -2, 10)$$

Theorem 1, with $f(x, y) = ux + vy + c$, ensures a unique solution of (17) if $c \geq 2 - u - v$. This is the case, for example, for the equation $a_n = b_n + 2b_{n+1} + c$ when $c = -1$. It is easy to check, however, that this equation also has a unique solution for $c = -2$, but not for $c = -3$. This example raises a question to which we respond with another conjecture:

Conjecture 20. The equation (17), where $b_0 = 1$, has a unique solution if and only if

$$c \geq 2 - u - v + k - vk.$$

Results shown in Table 5 and experimentation suggest that if $(u, v) = (1, 1)$, then $\min\{a_n - (u + v + 1)n\}$ and $\max\{a_n - (u + v + 1)n\}$ are as indicated in Table 5 and that otherwise, both are unbounded.

We turn now to equations in which $c \neq 0$. Indeed, we take c to be the least value for which the equation (17) has a solution; i.e., $c = 2 - u - v + k - vk$. Table 6, generated by Program 7.6, shows the minimum and maximum values of $a_n - (u + v + 1)n$ for $n = 0, 1, \ldots, 100000$.

u	v	k	c	min	max	a_0
1	2	1	−2	−2	9	3
1	2	2	−3	−1	10	4
1	2	3	−4	0	10	5
2	1	1	−1	−2	8	3
2	2	1	−3	−5	12	3
2	2	2	−4	0	9	4
2	1	3	−1	0	10	5
3	3	1	−6	−9	17	3
3	3	2	−8	−5	15	4
3	3	3	−10	−1	13	5

Table 6: $a_n = ub_n + vb_{n+k} + c, b_0 = 1$

As might be expected from the discussion about Table 5, it appears likely that minimum and maximum values represented in columns 5 and 6 of Table 6 are unbounded.

7 Mathematica programs

The examples and conjectures in the preceding sections are based on six `Mathematica programs` in which the parameters h, k, c, u, v, and z can be varied. Each program depends on the mex function mentioned near the end of Section 2.

7.1 Program for Table 2

The parameters z, c, h, k are set by the first two lines of code. Keep $c = 0$. Decrease z to 100 for a first run. To see row 4 of Table 2, for example, use $h = 3$ and $k = 6$. When experimenting, use larger z for larger h and k.

```
z = 1000; c = 0;
```

```
h = 2; k = 4;
mex[list_, start_] := (NestWhile[# + 1 &, start,
MemberQ[list, #] &]);
a = {}; b = {1}; AppendTo[a, c + mex[Flatten[{a, b}], 1]];
Do[Do[AppendTo[b, mex[Flatten[{a, b}], Last[b]]], {k}];
  AppendTo[a, c + Last[b] + b[[1 + (Length[b] - 1)/k h]]], {z}];
"Sequence (a(n)), from a(n)=a(h*n)+b(k*n):"
a ; Take[a, 100]
```

7.2 Program for Table 3

The parameters are set to generate row 7 of Table 3. When experimenting, use larger z for larger h, k, and c.

```
z = 1000;
h = 2; k = 4; c = 6;
mex[list_, start_] := (NestWhile[# + 1 &, start,
MemberQ[list, #] &]);
a = {}; b = {1}; AppendTo[a, c + mex[Flatten[{a, b}], 1]];
Do[Do[AppendTo[b, mex[Flatten[{a, b}], Last[b]]], {k}];
  AppendTo[a, c + Last[b] + b[[1 + (Length[b] - 1)/k h]]], {z}];
"Sequence (a^*(n)), from a(n)=a(h*n)+b(k*n)+c, where c = 1:"
a ; Take[a, 100]
```

Data supporting Conjecture 8 can be obtained by adding the following lines at the end of this program:

```
r = Select[Range[400], Mod[a[[# + 1]], 7] ==4 &]
(* r is the sequence just before Conjecture 8. *)
d = Differences[s]
Union[d]
```

7.3 Program for Table 4

The parameters are set to show frequencies $27, 61, 73, \ldots$, with $z = 1000$. The runtime is considerably longer for $z = 10000$, as in Table 4.

```
z = 1000; c = 0;
mex[list_, start_] := (NestWhile[# + 1 &, start,
MemberQ[list, #] &]);
h = 2; k = 4; a = {}; b = {1};
AppendTo[a, c + mex[Flatten[{a, b}], 1]];
Do[Do[AppendTo[b, mex[Flatten[{a, b}], Last[b]]], {k}];
```

```
  AppendTo[a, c + Last[b] + b[[1 + (Length[b] - 1)/k h]]], {z}];
"Sequence (a(n)), from a(n)=a(h*n)+b(k*n)+c, where c = 0:"
a; a1 = a; Take[a, 50]
c = 1; a = {}; b = {1};
AppendTo[a, c + mex[Flatten[{a, b}], 1]]; 90
Do[Do[AppendTo[b, mex[Flatten[{a, b}], Last[b]]], {k}];
  AppendTo[a, c + Last[b] + b[[1 + (Length[b] - 1)/k h]]], {z}];
"Sequence (a^*(n)), from a(n)=a(h*n)+b(k*n)+c"
a ; Take[a, 50]
"Distinct differences, a(n)^* - a(n):"
Union[a - a1]
p = Differences[Flatten[Position[a - a1, 0]]]; Take[ p, 100]
"Number of occurrences of each difference:"
t = Table[Count[p, k], {k, 1, 13}]
```

7.4 Program for Conjectures 18 and 19

The parameters are set to output the sequence on Conjecture 18. The code can be easily extended to support Conjecture 19.

```
mex[list_, start_] := (NestWhile[# + 1 &, start,
MemberQ[list, #] &]);
h = 1; k = 2; a = {}; b = {1}; z = 2000;
AppendTo[a, mex[Flatten[{a, b}], 1]];
Do[Do[AppendTo[b, mex[Flatten[{a, b}], Last[b]]], {k}];
  AppendTo[a, Last[b] + b[[1 + (Length[b] - 1)/k h]]], {z}];
Take[b, 50]
bb = b; Clear[b];(*next,re-index b with offset 0*)
b[n_] := bb[[n + 1]]; u = Table[b[3 i + 1],
{i, 0, -1 + Length[bb]/3}];
Take[u, 100]
t = Mod[u, 4]; p = Flatten[Position[t, 2]];
Take[p, 100]
d = Differences[p];
Union[d] (* set of distinct differences *)
```

7.5 Program for Table 5

The parameters u, v, k, c, and z are set to show $\{0, 9\}$ as minimal and maximal differences; to get $\{-1, 10\}$,m as in row 2 of Table 5, use $z = 100000$, and wait.

```
z = 10000;
mex[list_, start_] := (NestWhile[# + 1 &, start,
```

```
MemberQ[list, #] &]);
 {u, v, k, c} = {1, 2, 1, 0}; {a, b} = {{}, {1}};
Do[AppendTo[b, mex[Flatten[{a, b}], Last[b]]], {k}];
AppendTo[a, u b[[1]] + v b[[k + 1]] + c];
Do[AppendTo[b, mex[Flatten[{a, b}], Last[b]]];
  AppendTo[a, u b[[n]] + v b[[k + n]] + c], {n, 2, z}];
Take[a, 60];
Take[b, 60];
Intersection[a, b];
aa[n_] := a[[n + 1]];
t = Table[aa[n] - (u + v + 1) n, {n, 0, z - 2}];
{Min[t], Max[t]}
```

7.6 Program for Table 6

The parameters u, v, k, c, and z are set to show $\{-1, 8\}$ as minimal and maximal differences; to get $\{-2, 9\}$,m as in row 1 of Table 6, use $z = 100000$, and wait.

```
z = 10000;
mex[list_, start_] := (NestWhile[# + 1 &, start,
MemberQ[list, #] &]);
 {u, v, k, c} = {1, 2, 1, -2}; {a, b} = {{}, {1}};
Do[AppendTo[b, mex[Flatten[{a, b}], Last[b]]], {k}]
AppendTo[a, u b[[1]] + v b[[k + 1]] + c];
Do[AppendTo[b, mex[Flatten[{a, b}], Last[b]]];
  AppendTo[a, u b[[n]] + v b[[k + n]] + c], {n, 2, z}];
Take[a, 60]
Take[b, 60]
Intersection[a, b]
aa[n_] := a[[n + 1]];
t = Table[aa[n] - (u + v + 1) n, {n, 0, z - 2}];
{Min[t], Max[t]}
Flatten[Position[t, Min[t]]]
Flatten[Position[t, Max[t]]]
```

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