Journal of Integer Sequences, Vol. 24 (2021),
Article 21.3.6

Enumeration of S-omino Towers and Row-Convex k-omino Towers

Alexander M. Haupt
Institute of Mathematics
Hamburg University of Technology
Am Schwarzenberg-Campus 3
21073 Hamburg
Germany
alexander.haupt@tuhh.de

Abstract

We first enumerate a generalization of domino towers that was proposed by Brown, which we call S-omino towers. We establish equations that the generating function must satisfy, and then apply the Lagrange inversion formula to find a closed formula for the number of towers. We also show a connection to generalized Dyck paths and describe an explicit bijection. Finally, we consider the set of row-convex k-omino towers, introduced by Brown, and calculate an exact generating function.

1 Introduction

In this paper, we enumerate a generalization of so-called *domino towers*. Domino towers are two-dimensional structures made out of n dominoes, i.e., rectangular blocks of width 2 and height 1, with the following properties:

1. The dominoes on the bottom level are contiguous, i.e., the row is convex;
2. Every domino above the bottom row is (half) supported on at least one domino in the row below it;
3. No domino lies directly on top of another domino, such as in a brickwork pattern.

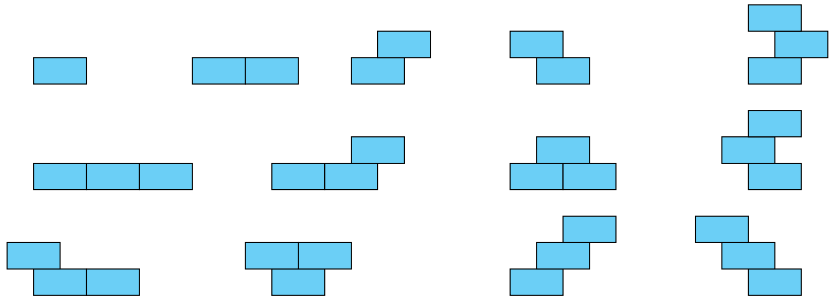


Figure 1: Small restricted domino towers for $n \in \{1, 2, 3\}$.

See Figure 1 for the domino towers with $n \in \{1, 2, 3\}$. The problem of counting domino towers was first mentioned by Viennot [9, Cor. 4]. Surprisingly, the number of domino towers made up of exactly n blocks is simply 3^{n-1}. Zeilberger showcased the result together with a proof using ordinary generating functions and a bijective proof [10]. The problem also appears as an example in the *Handbook of Enumerative Combinatorics* [1, p. 25].

In this paper, we drop the restriction that blocks cannot be placed directly on top of another and call the structures *unrestricted* towers. Brown proved that there are 4^{n-1} unrestricted towers [4, Cor. 2.3]. We give an alternative proof of this fact in Corollary 8 using symbolic methods and a construction called substitution. See Figures 2a and 2b for examples of restricted and unrestricted domino towers.

(a) A restricted domino tower with 8 blocks.

(b) An unrestricted domino tower with 10 blocks.

Figure 2: Examples of domino towers.

In 2016, Brown generalized the problem to unrestricted towers made up of rectangles of width k, which they called k-omino towers [4]. They also introduced a variable $b \geq 1$ for the number of blocks in the bottom row. The number of k-omino towers is $\binom{kn-1}{n-b}$.

Brown also suggests that enumerating towers using rectangles of mixed widths could be interesting for other applications [3, p. 17]. In this paper, we study this generalization by allowing rectangles with any width in a fixed finite list $S = (s_1, \dots, s_m)$ of positive integers. We call this set of towers S-*omino* towers. We additionally fix a list $(n_1, \dots, n_m)$, where n_i denotes the number of blocks of width s_i, and $b \geq 1$ the number of blocks in the bottom row. Furthermore, let $n := n_1 + \cdots + n_m \geq 1$ be the total number of blocks as before. We now state the first result of this paper. Note that for $S = (k)$ we, of course, recover the same formula as found by Brown.

Theorem 1. *The number of S-omino towers with n blocks of which n_i have width s_i, and b blocks in the bottom row, which has to be convex, equals*

$$\binom{n}{n_1, \dots, n_m}\binom{-1+\sum n_i s_i}{n-b}.$$

Summing over all $b \in [n]$ we get

$$\binom{n}{n_1, \dots, n_m}\binom{-1+\sum_{i=1}^m s_i n_i}{n-1} \cdot {}_2F_1\left(1, 1-n; 1+\sum_{i=1}^m (s_i-1)n_i; -1\right),$$

where ${}_2F_1$ is the Gaussian hypergeometric function.

Note that the heights of the blocks do not change the result, as we will explain in the next section. In particular, setting $S = (1, k)$ for $k \geq 2$ corresponds to stacking k-ominoes horizontally or vertically. In this paper, we assume that S does not contain duplicate entries for ease of notation. However, the methods would work and yield the same formula. Duplicate entries could be interpreted as having multiple distinguishable versions of dominoes with equal width.

At the end of the paper we turn our attention to convex k-omino towers, which are defined as follows:

Definition 2. A tower is called *column-convex* or *row-convex* if all its columns or respectively rows are convex. Further, a tower is called *convex* if it is both column- and row-convex.

In 2016, Brown calculated the generating function for convex towers and asked whether row-convex towers can be enumerated as well [3, p. 17].

Definition 3. Let $g(n)$ be the number of row-convex k-omino towers made up of n k-ominoes. We also define $f_\ell(n)$ to be the number of row-convex k-omino towers made up of n k-ominoes resting on a platform of width ℓk. In other words, the blocks on the bottom row need to rest on this platform, but the platform does not count towards the number of blocks.

By adapting a method that Privman and Švrakić used in 1988 to calculate so-called fully directed compact lattice animals [7], we calculate the ordinary generating functions $G(z) = \sum_{n=0}^\infty g(n)z^n$ and $F_\ell(z) = \sum_{n=0}^\infty f_\ell(n)z^n$.

Theorem 4. *We have*

$$G(z) = \sum_{\ell=1}^\infty z^\ell F_\ell(z),$$

where

$$
\begin{aligned}
F_\ell(z) &= \Big((1+kz)T_{1,\ell} + (kz^2-1)T_{2,\ell} + (k-1)z^3T_{3,\ell}\Big) \Big/ \Big((k-1)^2z^5T_{2,3} \\
&\quad + (1-(2k-1)(1+z)z+k^2z^3)T_{1,2} + (k-1)((2k-1)z-1)z^3T_{1,3}\Big),\\
T_{s,t}(z) &= A_sB_t - A_tB_s,\\
A_\ell(z) &= \sum_{j=0}^{\infty} \frac{z^{\ell j}h_j}{(z;z)_j^2},\\
B_\ell(z) &= \sum_{j=0}^{\infty} \frac{z^{\ell j}h_j}{(z;z)_j^2}\left(\ell + \sum_{m=1}^{j}\left(1+\frac{2}{1-z^m}-\frac{1}{1+(k-1)z^m}\right)\right), \text{ and}\\
h_j(z) &:= z^{j(j+1)}\left((1-k)z;z\right)_j .
\end{aligned}
$$

The structure of the paper is as follows: In Section 2, we will introduce a few ideas and notation, which we need in later sections. In Section 3, we prove Theorem 1 using ordinary generating functions and the Lagrange inversion formula. In Section 4, we will turn the proof into an explicit bijection and show the connection to generalized Dyck paths [8]. Finally, in Section 5, we prove Theorem 4.

2 Representation of towers as sequences

We can think of domino towers as being built by dropping single dominoes one by one straight down from an infinite height. However, there may be multiple ways of building the same tower. However, there is a unique order as described in Lemma 5.

Lemma 5. *For any domino tower there is a unique order* $b_1, \ldots, b_n$ *of its dominoes with the following properties:*

i) The tower can be built by dropping the dominoes straight down from an infinite height in the order $b_1, \ldots, b_n$*;*

ii) For all $i \in [n-1]$*, the left border of* b_{i+1} *is strictly to the left of the right border of* b_i*.*

Proof. Consider the set of blocks B, which have no other blocks above them, i.e., could be the last block according to condition i). Let b' be the left-most block in B and $b^\star \in B$ be the block, which is dropped last. Suppose, for a contradiction, that $b' \neq b^\star$. Then all blocks after b' must be strictly to the left of b'. In particular, $b^\star$ is to the left of b'. This contradicts the definition of b', and therefore b' is dropped last and we are done by induction on n. □

Hence, instead of enumerating towers, we can enumerate valid sequences of x-coordinates of the left borders of blocks $b_1, \ldots, b_n$. We fix $x_1 = 0$ to keep this sequence unique. For example, $\langle 0, 1, 0, 1, -2, -1, -3, -4 \rangle$ is the sequence for the tower in Figure 2a. We now define the set $\mathcal{W}_b$ and then prove that it contains exactly those sequences of x-coordinates corresponding to domino towers with b blocks in the bottom row.

Definition 6. For $b \geq 1$ we define $\mathcal{W}_b$ to be the set of sequences $\langle x_1, \ldots, x_n \rangle$ with the following properties:

i) We have $x_1 = 0$;

ii) For all $i \in [n-1]$ we have $x_{i+1} < x_i + 2$;

iii) There exists a set $D \subset [n]$ containing b numbers $1 = d_1 < d_2 < \cdots < d_b$ such that

  a) for all $j \in [b-1]$ we have $x_{d_{j+1}} + 2 = x_{d_j}$,

  b) for all $i \notin D$ with $i < d_{j+1}$ we have $x_i \geq x_{d_j}$,

  c) for all $i \notin D$ with $d_b < i$ we have $x_i + 2 > \min_{j<i} x_j$.

Proposition 7. *The set $\mathcal{W}_b$ contains exactly the sequences corresponding to domino towers with b blocks in the bottom row. In other words, Lemma 5 always produces a sequence in $\mathcal{W}_b$ and any sequence in $\mathcal{W}_b$ corresponds to a valid domino tower.*

Proof. Given a tower with b blocks in the bottom row, let $b_1, \ldots, b_n$ be the order produced by Lemma 5. Let $\langle x_1, \ldots, x_n \rangle$ be the x-coordinates of the blocks $b_1, \ldots, b_n$ with $x_1 = 0$. We now show that $\langle x_1, \ldots, x_n \rangle \in \mathcal{W}_b$ by checking all properties:

i) We have $x_1 = 0$ by definition.

ii) From property ii) in Lemma 5 follows $x_{i+1} < x_i + 2$.

iii) Let D be the set of indices of the blocks in the bottom row of the tower. As the tower has b blocks in the bottom row, we have $|D| = b$. Order $D = \{d_1, \ldots, d_b\}$ as $d_1 < \cdots < d_b$.

  a) It follows immediately from the properties in Lemma 5 that the rightmost block in the bottom row is always dropped first. Hence we have $d_1 = 1$. Similarly, we have that the bottom row must be dropped from right to left. Hence block $b_{d_{j+1}}$ is dropped two units to the left of block b_{d_j}, because the bottom row is convex by definition. Hence we have $x_{d_{j+1}} + 2 = x_{d_j}$. See Figure 3 for an illustration.

  b) If a block b_i is not in the bottom row, dropped before $b_{d_{j+1}}$ and $x_i < x_{d_j}$, then we cannot anymore drop $b_{d_{j+1}}$ from an infinite height such that $b_{d_{j+1}}$ reaches the bottom row.

  c) If we have $x_i + 2 \leq \min_{j<i} x_j$, then block b_i falls down to the bottom row. This is not possible if $i > d_b$, as at this point the bottom row is complete.

For the other direction, fix a sequence $\langle x_1, \ldots, x_n \rangle \in \mathcal{W}_b$ and consider the structure built by dropping blocks at positions x_1 to x_n. We now show that the structure is a valid domino tower, such that the set D from property iii) contains the indices of blocks which land in the bottom row. Clearly, the first block lands in the bottom row. Properties iii. a) and iii. b) guarantee that the block with index d_{j+1} lands in the bottom row and keeps the row convex. In Figure 3 all blocks with indices strictly between d_j and d_{j+1} are dropped to the right of the vertical line. Finally, properties ii) and iii. c) guarantee that all other blocks land on top of a previous block. □

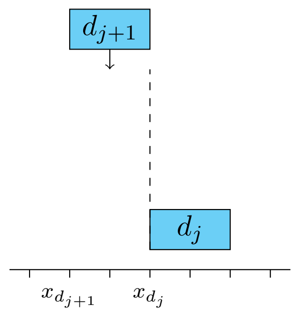


Figure 3: Illustration of property iii. a) in Definition 6.

Note that by thinking about sequences of x-coordinates instead of towers, it is now clear that the heights of blocks do not change the number of towers. Only the x-coordinates and order matter. Also note that a tower is restricted if and only if the corresponding sequence does not have repeated consecutive entries. Using this insight we can now explain the relation between restricted and unrestricted towers:

Corollary 8 ([4, Cor. 2.3])**.** *The number of unrestricted domino towers made out of n dominoes is equal to 4^{n-1}.*

Proof. Consider the sequences in $\mathcal{W} := \bigcup_b \mathcal{W}_b$ that have no repeated consecutive entries. We know from the introduction that there are 3^{n-1} such sequences of length n. The corresponding ordinary generating function is therefore $f(x) = x + 3x^2 + 9x^3 + \cdots = \frac{x}{1-3x}$. Now, if we take such a sequence and replace every entry x_i by a sequence $x_i, \ldots, x_i$ of arbitrary positive length, we get a sequence in $\mathcal{W}$, where repeated consecutive entries are allowed, i.e., a sequence corresponding to an unrestricted tower. This process is reversible: To recover the original sequence, we simply delete repeated consecutive entries. See Figure 4 for an illustration. This is an example of a *substitution* as defined by Flajolet [5, Def. I.14]. In terms of the generating functions this procedure therefore corresponds to replacing x with $x + x^2 + x^3 + \cdots = \frac{x}{1-x}$. We can now deduce the generating function for the unrestricted domino towers:

$$\frac{x}{1-3x} \rightsquigarrow \frac{\frac{x}{1-x}}{1 - 3 \cdot \frac{x}{1-x}} = \frac{x}{1-4x},$$

from which we can read off the number: 4^{n-1}. □

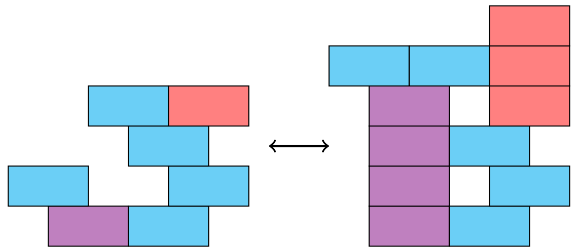


Figure 4: Illustration of the substitution of x with $\frac{x}{1-x}$. The corresponding sequences are $\langle 0,1,0,1,-2,-1,-3\rangle$ and $\langle 0,1,0,1,1,1,-2,-2,-2,-2,-1,-3\rangle$ respectively.

We can generalize sequences $\mathcal{W}_b$ for S-omino towers by also keeping track of the widths ℓ_i. For that we redefine $\mathcal{W}_b$ as follows:

Definition 9. We define $\mathcal{W}_b$ to be the set of sequences of pairs $\langle(x_1,\ell_1),\dots,(x_n,\ell_n)\rangle$ with $n \geq b$ with the following properties:

i) We have $x_1 = 0$;

ii) For all $i \in [n-1]$ we have $x_{i+1} < x_i + \ell_i$;

iii) There exists a set $D \subset [n]$ containing b numbers $1 = d_1 < d_2 < \cdots < d_b$ such that

  a) for $j \in [b-1]$ we have $x_{d_{j+1}} + \ell_{d_{j+1}} = x_{d_j}$,

  b) for all $i \notin D$ with $i < d_{j+1}$ we have $x_i \geq x_{d_j}$,

  c) for all $i \notin D$ with $d_b < i$ we have $x_i + \ell_i > \min_{j<i} x_j$.

Proposition 7 still holds analogously, as 2 was merely replaced by the length of the appropriate block. The purposes of the properties remain exactly the same.

As we want to keep track of how many blocks of each length we have used, we define the weight of a sequence $t = \langle(x_1,\ell_1),\dots,(x_n,\ell_n)\rangle \in \mathcal{W}$ as $w(t) := z^n y_{\ell_1} y_{\ell_2} \cdots y_{\ell_n}$. Hence, the exponent of y_ℓ is the number of pairs in t with second entry equal to ℓ and the exponent of z is the total number of pairs. We define the multivariate ordinary generating function of $\mathcal{W}_b$ with formal variables $z, y_1, y_2, y_3, \dots$ as $W = \sum_{t\in\mathcal{W}} w(t)$. Lemma 5 now immediately generalizes to:

Lemma 10. *Fix s_i, n_i and b as before. Then there is a bijection between such S-omino towers and elements in $\mathcal{W}_b$ of weight $z^n \prod_i y_{s_i}^{n_i}$. Furthermore, a tower is restricted if and only if there are no repeated consecutive elements in the corresponding sequence in $\mathcal{W}_b$.*

Proof. Similarly to Lemma 5, the last element of the sequence must correspond to the left-most block of the tower, among the blocks that do not have any other blocks vertically above it. The statement follows from induction on n. □

We will often need to offset sequences of pairs of the form (x, ℓ) horizontally, so we define

$$\langle (x_1, \ell_1), \ldots, (x_n, \ell_n) \rangle + \alpha := \langle (x_1 + \alpha, \ell_1), \ldots, (x_n + \alpha, \ell_n) \rangle.$$

Also, we define *concatenation* of two sequences as follows:

$$\langle a_1, \ldots, a_n \rangle \parallel \langle b_1, \ldots, b_m \rangle := \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle.$$

3 Proof using the Lagrange inversion formula

The main tool we use in this section is the following version of the Lagrange inversion formula [2, Section 2.6]. Here, $[x^n]G(x)$ denotes the coefficient of x^n in the formal power series $G(x)$.

Proposition 11 (The Lagrange inversion formula [2])**.** *Let $Y(x) = x\Phi(Y)$, where $\Phi(Y)$ is a power series such that $\Phi(0) \neq 0$. Then for any power series $g(Y)$ and $n \geq 1$ we have*

$$[x^n]g(Y) = \frac{1}{n}[y^{n-1}]g'(y)(\Phi(y))^n.$$

We prove an immediate corollary:

Corollary 12. *Let $Y(x) = x\Phi(Y)$, where $\Phi(Y)$ is a power series such that $\Phi(0) \neq 0$. Then for any power series $h(Y)$ and $n \geq 1$ we have*

$$[x^n]x\frac{dY}{dx}h(Y) = [y^{n-1}]h(y)(\Phi(y))^n.$$

Proof. Let $g(Y) = \int h(Y)dY$ and apply Proposition 11 on g.

$$\begin{aligned}
&[x^n]x\frac{dY}{dx}h(Y) \\
&= n \cdot [x^n] \int \frac{dY}{dx}h(Y)dx \\
&= n \cdot [x^n] \int h(Y)dY \\
&= [y^{n-1}]h(y)(\Phi(y))^n.
\end{aligned}$$

□

The idea of the proof is to relate $\mathcal{W}_b$ to other sets of sequences.

Definition 13. We define $\mathcal{U}_\star$ to be the set of sequences of pairs $\langle (x_1, \ell_1), \ldots, (x_n, \ell_n) \rangle$ with $n \geq 1$ with the following properties:

i) We have $x_1 = 0$;

ii) For all $i > 1$ we have $x_i \geq 1$ and $x_i < x_{i-1} + \ell_{i-1}$.

We also define $\mathcal{U}_1 := \mathrm{Seq}_{\geq 1}(\mathcal{U}_\star)$, the set of sequences that are a concatenation of at least one sequence in $\mathcal{U}_\star$. For convenience we similarly define $\mathcal{U} := \mathrm{Seq}_{\geq 0}(\mathcal{U}_\star) = \{\langle\rangle\} \cup \mathcal{U}_1$, which also contains the empty sequence.

Lemma 14. *Let $\langle(x_1, \ell_1), \dots, (x_n, \ell_n)\rangle \in \mathcal{U}_\star$. Then there is exactly one choice of indices $d_1, \dots, d_{\ell_1}$ such that*

- *we have $2 = d_1 \leq d_2 \leq \dots \leq d_{\ell_1} = n + 1$ and*
- *for all $j \in [\ell_1 - 1]$ the subsequence $\langle(x_{d_j}, \ell_{d_j}), \dots, (x_{d_{j+1}-1}, \ell_{d_{j+1}-1})\rangle - \alpha_j \in \mathcal{U}$, where $\alpha_j := \ell_1 - j$. In particular, if $d_j < d_{j+1}$, then $x_{d_j} = \alpha_j$.*

Proof. Define the indices $d_j^\star := \min\{i \geq 2 : x_i \leq \alpha_j\}$, where $d_j^\star = n + 1$ if such an i does not exist. We now prove by induction on j that $d_{j+1} = d_{j+1}^\star$ is the unique choice for indices that satisfy the conditions in the lemma. Clearly, for $j = 1$ we are done, because $d_1 = 2 = d_1^\star$. Similarly, we have $d_{\ell_1} = n + 1 = d_{\ell_1}^\star$ for $j = \ell_1 - 1$. Now for $j \in [\ell_1 - 2]$ we can assume the induction hypothesis for $j - 1$, i.e., that $d_j = d_j^\star$. For a contradiction, we consider the cases $d_{j+1} > d_{j+1}^\star$ and $d_{j+1} < d_{j+1}^\star$ separately:

1. If $d_{j+1} > d_{j+1}^\star \geq d_j^\star = d_j$, then by definition of $d_{j+1}^\star$ we have $x_{d_{j+1}^\star} \leq \alpha_{j+1}$. However, by definition of $\mathcal{U}_\star$ and the fact that $\langle(x_{d_j}, \ell_{d_j}), \dots, (x_{d_{j+1}-1}, \ell_{d_{j+1}-1})\rangle - \alpha_j \in \mathcal{U}$ we have $x_{d_{j+1}^\star} \geq \alpha_j = \alpha_{j+1} + 1$, a contradiction.
2. If $d_{j+1} < d_{j+1}^\star$, then by definition of $d_{j+1}^\star$ we have $x_{d_{j+1}} > \alpha_{j+1}$. However, for $k := \max\{p : d_p = d_{j+1}\}$ we have $d_k < d_{k+1}$ and therefore $x_{d_{j+1}} = x_{d_k} = \alpha_k \leq \alpha_{j+1}$, a contradiction.

Hence $d_{j+1} = d_{j+1}^\star$ and we are done by induction on j. □

Example 15. The sequence $\langle(0, 4), (3, 2), (3, 2), (1, 2), (2, 2)\rangle \in \mathcal{U}_\star$ corresponds to the tower in Figure 5. The indices d_i in this example are: $d_1 = 2$, $d_2 = 4$, $d_3 = 4$ and $d_4 = 6$. We have $\langle(3, 2), (3, 2)\rangle - 3 \in \mathcal{U}$ drawn in violet, $\langle\rangle - 2 \in \mathcal{U}$, and finally $\langle(1, 2), (2, 2)\rangle - 1 \in \mathcal{U}$ drawn in red.

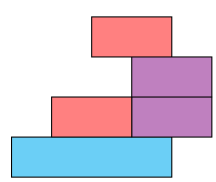


Figure 5: Illustration of Example 15.

We have just shown that every sequence in $\mathcal{U}_\star$ can be built by concatenating $\langle(0, \ell)\rangle$ with $\ell - 1$ sequences in $\mathcal{U}$, offset by $\ell - 1, \ell - 2, \dots, 1$ respectively and that this construction is unique. Similarly, every sequence in $\mathcal{U}_1$ can be constructed uniquely using ℓ sequences in $\mathcal{U}$. This motivates the following proposition.

Lemma 16. *Let $u \in \mathcal{U}$ be non-empty. Define ℓ and n such that the first element of u is $(0, \ell)$ and $n = |u|$. Then u satisfies the following two properties:*

i) There exists a unique pair (x, y) with $x \in \mathcal{U}_\star$ and $y \in \mathcal{U}$ such that $u = x \parallel y$.

ii) There is exactly one choice of indices $d_1, \ldots, d_{\ell+1}$ such that

- *we have $2 = d_1 \leq \cdots \leq d_{\ell+1} = n + 1$ and*
- *we have $u = \langle (0, \ell) \rangle \parallel (c_{\ell-1} + \ell - 1) \parallel (c_{\ell-2} + \ell - 2) \parallel \cdots \parallel (c_0 + 0)$, where $c_0, \ldots, c_{\ell-1}$ in $\mathcal{U}$ with $|c_{\ell-i}| = d_{i+1} - d_i$.*

Proof. For part i) suppose that $u = \langle (x_1, \ell_1), \ldots, (x_n, \ell_n) \rangle$. There are two cases:

1. Suppose that $\forall i > 1\ x_i \geq 1$. In this case $u \in \mathcal{U}_\star$. Note that x must be non-empty by definition and if y were non-empty its first element would be $(0, \ell')$ for some ℓ'. Hence $(u, \langle\rangle)$ is the unique pair (x, y) such that $u = x \parallel y$.

2. Suppose that there exists $i > 1$ with $x_i = 0$ and let $i^\star > 1$ be the minimal such i. In this case $u \notin \mathcal{U}_\star$. Hence y cannot be empty and needs to start with $(0, \ell')$ for some ℓ' by definition. By definition of $\mathcal{U}_\star$ we must have $|x| < i^\star$. Hence the pair $(\langle (x_1, \ell_1), \ldots, (x_{i^\star-1}, \ell_{i^\star-1}) \rangle, \langle (x_{i^\star}, \ell_{i^\star}), \ldots, (x_n, \ell_n) \rangle)$ is the unique pair (x, y) such that $u = x \parallel y$.

This completes the proof of part i). We now prove part ii). By part i) there exists a unique pair (x, y) with $x \in \mathcal{U}_\star, y \in \mathcal{U}$ and $u = x \parallel y$. By Lemma 14 there is exactly one choice of indices $d_1, \ldots, d_\ell$ such that $2 = d_1 \leq \cdots \leq d_\ell = |x| + 1$ and for all $j \in [\ell - 1]$ we have $c_{\ell-j} := \langle (x_{d_j}, \ell_{d_j}), \ldots, (x_{d_{j+1}-1}, \ell_{d_{j+1}-1}) \rangle - \ell + j \in \mathcal{U}$. Also define $c_0 := y$ and $d_{\ell+1} := n + 1$. Then we have

$$x = \langle (0, \ell) \rangle \parallel (c_{\ell-1} + \ell - 1) \parallel (c_{\ell-2} + \ell - 2) \parallel \cdots \parallel (c_1 + 1)$$

with $|c_{\ell-j}| = d_{j+1} - d_j$ and $|c_0| = |y| = n - |x| = d_{\ell+1} - d_\ell$. As $u = x \parallel y$, the result follows. $\square$

We also define two other sets of sequences $\mathcal{X}$ and $\mathcal{V}$.

Definition 17. We define $\mathcal{X}_\ell$ to be the set of sequences

$$\mathcal{X}_\ell := \{u - \alpha : 0 \leq \alpha < \ell \text{ and } u \in \mathcal{U}_\star \text{ and } u \text{ starts with } (0, \ell)\}.$$

Let $\mathcal{X} := \bigcup_{\ell \in \mathbb{N}} \mathcal{X}_\ell$ be the union over all possible lengths ℓ. Further, we define $\mathcal{V}$ to be the minimal set with the following properties:

i) We have $\langle\rangle \in \mathcal{V}$;

ii) The set $\mathcal{V}$ is closed under the following procedure:

a) Pick any $\ell \in \mathbb{N}$;

b) Pick any elements $v \in \mathcal{V}$ and $x \in \mathcal{X}_\ell$ starting with, say, $(-\alpha, \ell)$;

c) Then $x \parallel (v - \alpha) \in \mathcal{V}$.

Remark 18. To aid readability of the following arguments, we now describe the towers corresponding to the sets of sequences $\mathcal{W}_b, \mathcal{U}_\star, \mathcal{U}_1, \mathcal{U}, \mathcal{X}_\ell, \mathcal{X}$ and $\mathcal{V}$ informally.

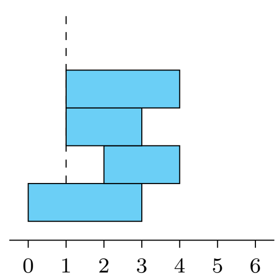
The sequences in $\mathcal{W}_b$ correspond to the towers we want to enumerate: Towers with b blocks in the bottom row as introduced in Section 1.

The sequences in $\mathcal{U}_\star$ correspond to towers with a single block in the bottom row with left edge at $x = 0$, and no other block crossing the vertical line $x = 1$. See Figure 6a for an example.

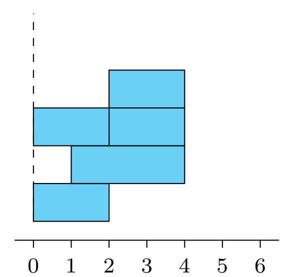
The sequences in $\mathcal{U}_1$ correspond to towers with a single block in the bottom row with left edge at $x = 0$, and no other block crossing the vertical line $x = 0$. See Figure 6b for an example. The set of sequences $\mathcal{U}$ equals $\mathcal{U}_1$ but including an empty tower with 0 blocks.

You can think of sequences in $\mathcal{X}_\ell$ as towers in $\mathcal{U}_\star$ with a base of length ℓ lying on top of a unit-length platform. The platform is fixed between 0 and 1 and its position relative to the base block corresponds to α in Definition 17. See Figure 6c for an example. The set $\mathcal{X}$ is simply the union over all possible lengths of the base.

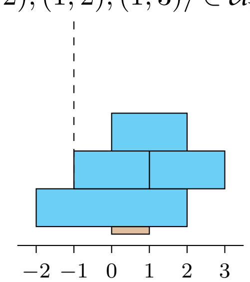
Similarly to $\mathcal{X}$, you can think of sequences in $\mathcal{V}$ as towers in $\mathcal{W}_1$ with their base lying on top of a unit-length platform. See Figure 6d for an example.



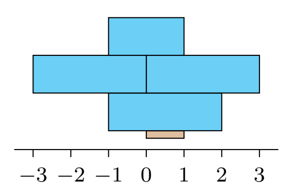
(a) The tower corresponding to $\langle (0,3), (2,2), (1,2), (1,3) \rangle \in \mathcal{U}_\star$.



(b) The tower corresponding to $\langle (0,2), (1,3), (2,2), (2,2), (0,2) \rangle \in \mathcal{U}_1$.



(c) The tower corresponding to $\langle (-2,4), (1,2), (-1,2), (0,2) \rangle \in \mathcal{X}_4$.



(d) The tower corresponding to $\langle (-1,3), (0,3), (-3,3), (-1,2) \rangle \in \mathcal{V}$.

Figure 6: Towers corresponding to sequences in $\mathcal{U}_\star, \mathcal{U}_1, \mathcal{X}_4$ and $\mathcal{V}$.

Lemma 19. *The set $\mathcal{W}_b$ is related to $\mathcal{X}$ and $\mathcal{V}$ as follows:*

i) *For all $b \geq 2$ we have a weight-preserving bijection $\mathcal{W}_b \leftrightarrow \mathcal{U}_1 \times \mathcal{W}_{b-1}$;*

ii) *We have a weight-preserving bijection $\mathcal{W}_1 \leftrightarrow \mathcal{U}_\star \times \mathcal{V}$.*

Proof. Consider any element $\langle (x_1, \ell_1), \ldots, (x_n, \ell_n) \rangle \in \mathcal{W}_b$. For $b \geq 2$ we know from Definition 9 that there exists an index $1 < d_2 = \min\{i \geq 2 : x_i < 0\}$ such that we have

$$\langle (x_1, \ell_1), \ldots, (x_{d_2-1}, \ell_{d_2-1}) \rangle \in \mathcal{U}_1 \text{ and } \langle (x_{d_2}, \ell_{d_2}), \ldots, (x_n, \ell_n) \rangle + \ell_{d_2} \in \mathcal{W}_{b-1}.$$

For $b = 1$ we let $d = \min\{i \geq 2 : x_i \leq 0\}$ and $d = n + 1$ if such an i does not exist. Then

$$\langle (x_1, \ell_1), \ldots, (x_{d-1}, \ell_{d-1}) \rangle \in \mathcal{U}_\star \text{ and } \langle (x_d, \ell_d), \ldots, (x_n, \ell_n) \rangle \in \mathcal{V}.$$

In both cases the function has an inverse: Concatenate both parts back together. □

Example 20. The sequence $\langle (0, 2), (1, 2), (2, 2), (-2, 2), (-1, 2), (-3, 2), (-4, 2) \rangle \in \mathcal{W}_2$ corresponds to the tower in Figure 7. We have $\langle (0, 2), (1, 2), (2, 2) \rangle \in \mathcal{U}_1$ drawn in blue, $\langle (-2, 2), (-1, 2) \rangle + 2 \in \mathcal{U}_\star$ drawn in violet, and finally $\langle (-3, 2), (-4, 2) \rangle + 2 \in \mathcal{V}$ drawn in red.

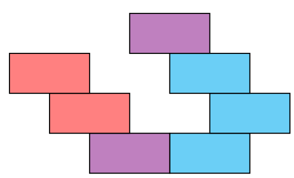


Figure 7: Illustration of Example 20.

Using the same weights as for W, let $U, U_1, U_\star, X_\ell, X$ and V be the multivariate ordinary generating functions of $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_\star, \mathcal{X}_\ell, \mathcal{X}$ and $\mathcal{V}$ respectively.

Theorem 21. *The ordinary generating functions satisfy the following equations:*

a) $U = 1 + U_1$ *and* $U_1 = U_\star \cdot U = \sum_{\ell \in \mathbb{N}} z y_\ell U^\ell$

b) $X_\ell = \ell z y_\ell U^{\ell-1}$, $X = \sum_{\ell \in \mathbb{N}} X_\ell$, *and* $V = 1 + X \cdot V$

c) $W_b = U_\star \cdot V \cdot U_1^{b-1}$

d) $z \frac{dU_1}{dz} = U_1 + X \cdot z \frac{dU_1}{dz}$

e) $z \frac{dU_1}{dz} = U_1 \cdot V$

f) $(1 + U_1) \cdot W_b = U_1^{b-1} z \frac{dU_1}{dz}$

Proof. Parts a), b) and c) follow from Lemma 16, Definition 17 and Lemma 19 respectively. For d) note that $z\frac{dU_1}{dz}$ is the ordinary generating function of $\Theta\mathcal{U}_1$, i.e., the set $\mathcal{U}_1$, where one element is marked. We can define $\Theta\mathcal{U}_1 := \{(u,k) : u \in \mathcal{U}_1, k \in [|u|]\}$. We now describe a bijection f between $\Theta\mathcal{U}_1$ and $\mathcal{U}_1 + X \times \Theta\mathcal{U}_1$. Let $(u,k) \in \Theta\mathcal{U}_1$. Now note that there are two cases:

1. For $k = 1$ we simply define $f((u,k)) := u \in \mathcal{U}_1$;

2. For $k \geq 2$ we know that by Lemma 16 there exists $\ell \in \mathbb{N}$ and sequences $c_0, \ldots, c_{\ell-1} \in \mathcal{U}$ and $2 = d_1 \leq \cdots \leq d_{\ell+1} = n+1$ such that

$$u = \langle (0,\ell) \rangle \parallel (c_{\ell-1} + \ell - 1) \parallel (c_{\ell-2} + \ell - 2) \parallel \cdots \parallel (c_0 + 0),$$

where $|c_{\ell-i}| = d_{i+1} - d_i$. As $k \geq 2$, there exists exactly one $p \in [\ell]$ such that $d_p \leq k < d_{p+1}$. Hence $(c_{\ell-p}, k - d_p + 1) \in \Theta\mathcal{U}_1$. Let

$$\begin{aligned} x = \langle (-\ell + p, \ell) \rangle \parallel (c_{\ell-1} + p - 1) \parallel \cdots \parallel (c_{\ell-p+1} + 1) \\ \parallel (c_{\ell-p-1}) \parallel \cdots \parallel (c_0 - \ell + p + 1). \end{aligned}$$

Then $x \in \mathcal{X}_\ell$. We define $f((u,k)) = (x, (c_{\ell-p}, k - d_p + 1)) \in \mathcal{X} \times \Theta\mathcal{U}_1$. This function is invertible, because x stores the variable p, which enables us to undo the shifts and reinsert $c_{\ell-p}$ into u at the correct position.

For e), note that from b) and d) follows $\Theta\mathcal{U}_1 = \mathcal{U}_1 \times \mathrm{Seq}_{\geq 0}(\mathcal{X}) = \mathcal{U}_1 \times \mathcal{V}$. Finally f) follows from a), c) and e). □

Proof of Theorem 1. We consider the multivariate power series W_b in $z, y_1, y_2, y_3, \ldots$ as a power series in z with coefficients in the ring of multivariate formal power series in $y_1, y_2, y_3, \ldots$ and use Corollary 12:

$$\begin{aligned} [z^n]W_b &= [z^n] z \frac{dU_1}{dz} \frac{U_1^{b-1}}{1+U_1} \\ &= [u^{n-1}] \frac{u^{b-1}}{1+u} \left(\sum_{i \in [m]} y_{s_i} (1+u)^{s_i} \right)^n \\ &= [u^{n-b}] \frac{1}{1+u} \left(\sum_{i \in [m]} y_{s_i} (1+u)^{s_i} \right)^n . \end{aligned}$$

Now we also fix the number of occurrences of length s_i to be n_i:

$$\begin{aligned} [y_{s_1}^{n_1} \cdots y_{s_m}^{n_m}][z^n]W_b &= [y_{s_1}^{n_1} \cdots y_{s_m}^{n_m}][u^{n-b}] \frac{1}{1+u} \left(\sum_{i \in [m]} y_{s_i} (1+u)^{s_i} \right)^n \\ &= [u^{n-b}] \frac{1}{1+u} \binom{n}{n_1, \ldots, n_m} \prod_{i=1}^{m} (1+u)^{s_i n_i} \end{aligned}$$

$$= [u^{n-b}]\binom{n}{n_1,\ldots,n_m}(1+u)^{-1+\sum_{i=1}^m s_i n_i}$$
$$= \binom{n}{n_1,\ldots,n_m}\binom{-1+\sum_{i=1}^m s_i n_i}{n-b}.$$

Now, summing over all $b \in [n]$ we can express the total number of S-omino towers for given $(n_1,\ldots,n_m)$ in terms of the Gaussian hypergeometric function ${}_2F_1$:

$$[z^n y_{s_1}^{n_1}\cdots y_{s_m}^{n_m}]\sum_{b=1}^{n} W_b$$
$$= \sum_{b=1}^{n}\binom{n}{n_1,\ldots,n_m}\binom{-1+\sum_{i=1}^m s_i n_i}{n-b}$$
$$= \binom{n}{n_1,\ldots,n_m}\binom{-1+\sum_{i=1}^m s_i n_i}{n-1}\sum_{b=0}^{n-1}\frac{(n-1)!(-n+\sum_{i=1}^m s_i n_i)!}{(n-1-b)!(b-n+\sum_{i=1}^m s_i n_i)!}$$
$$= \binom{n}{n_1,\ldots,n_m}\binom{-1+\sum_{i=1}^m s_i n_i}{n-1}\sum_{b=0}^{n-1}\frac{(1)_b(1-n)_b}{(1-n+\sum_{i=1}^m s_i n_i)_b}\frac{(-1)^b}{b!}$$
$$= \binom{n}{n_1,\ldots,n_m}\binom{-1+\sum_{i=1}^m s_i n_i}{n-1}\cdot {}_2F_1\left(1, 1-n; 1+\sum_{i=1}^{m}(s_i-1)n_i; -1\right).$$

This completes the proof of Theorem 1. □

Remark 22. We can find closed formulas for the other sets analogously. For example, for $s := \sum n_i s_i$ we have

$$[y_{s_1}^{n_1}\cdots y_{s_m}^{n_m}][z^n]V = \binom{s}{n_1,\ldots,n_m,s-n}$$

and

$$[y_{s_1}^{n_1}\cdots y_{s_m}^{n_m}][z^n]U = \frac{1}{s+1}\binom{s+1}{n_1,\ldots,n_m,s+1-n}.$$

Therefore, the number of sequences in $\mathcal{U}$ for $m=1, n_1=n, s_1=2$ is given by the Catalan numbers – sequence A000108 in the *On-Line Encyclopedia of Integer Sequences* [11].

4 Bijective proof

In this section, we give an explicit bijection between $\mathcal{U}$ and generalized Dyck paths and then extend it to a bijection between $\mathcal{W}_b$ and $\mathcal{D}_{W_b}$.

Definition 23. We define the weighted set of generalized Dyck paths $\mathcal{D}_U$ as the set of all integer sequences $\langle u_1, u_2, \ldots, u_N \rangle$ with the following properties:

i) We have $\forall i \in [N]\ u_i \geq -1$;

ii) We have $\forall j \in [N]\ \sum_{i<j} u_i \geq 0$;

iii) We have $\sum_{i \in [N]} u_i = -1$.

We define the weight function to be

$$w(\langle u_1, u_2, \ldots, u_N \rangle) := \prod_{i \in [N]} \begin{cases} z y_{u_i+1}, & \text{if } u \geq 0; \\ 1, & \text{if } u = -1. \end{cases}$$

From the properties above, it follows that any sequence in $\mathcal{D}_U$ satisfies $u_N = -1$ and $\sum_{i \in [N-1]} u_i = 0$. We include the fixed -1 at the end of each sequence for later convenience. Also note that Rukavicka gave a different definition of generalized Dyck paths [8]. They also consider paths with *flaws*, which allows the path to fall below 0. When disregarding the fixed -1 at the end of our sequence, our definition corresponds to Dyck paths with 0 flaws as defined by Rukavicka. See Figure 8 for the correspondence between the two definitions: Every -1 corresponds to a vertical step of length 1 and every $u_i \geq 1$ corresponds to a horizontal step of length u_i. Note that our definition allows us to have $u_i = 0$ which would correspond to a horizontal step of length 0.

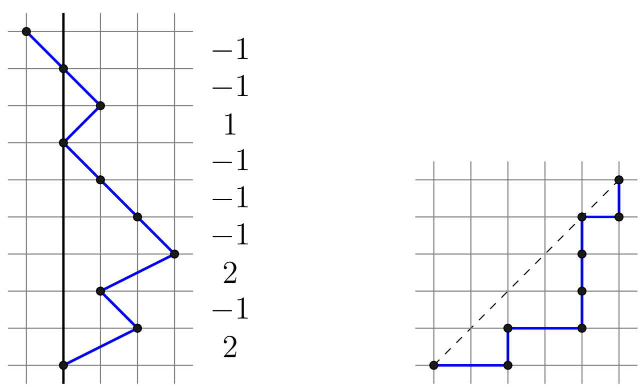


Figure 8: Correspondence between the sequence $\langle 2, -1, 2, -1, -1, -1, 1, -1, -1 \rangle \in \mathcal{D}_U$ on the left and a Dyck path with 0 flaws as defined by Rukavicka on the right.

Lemma 24. *We have a weight-preserving bijection f_U between $\mathcal{U}$ and $\mathcal{D}_U$.*

Proof. We define $f_U(\langle\rangle) := \langle -1\rangle$ and given a sequence $u = \langle (x_1, \ell_1), \ldots, (x_n, \ell_n)\rangle \in \mathcal{U}_1$, we define

$$f_U(u) := \big(\|_{p=1}^{n-1} \langle \ell_p - 1, \underbrace{-1, -1, \ldots, -1}_{x_p+\ell_p-1-x_{p+1} \text{ times}} \rangle\big) \,\|\, \langle \ell_n - 1, \underbrace{-1, -1, \ldots, -1}_{x_n+\ell_n \text{ times}}\rangle.$$

Note that f_U is clearly weight-preserving, as for every pair $(x, \ell) \in u$ we have got exactly one copy of $\ell - 1$ in $f_U(u)$, both of which account for the same weight: zy_ℓ.

We now show $f_U(u) \in \mathcal{D}_U$ by checking all three properties of sequences in $\mathcal{D}_U$. For that let $\langle u_1, \ldots, u_N\rangle = f_U(u)$.

i) By definition we have $\forall i\ u_i \geq -1$.

ii) Let $k_1 < k_2 < \cdots < k_n \in [N]$ be all indices with $u_{k_m} \geq 0$. By construction of $f_U(u)$ it is enough to show property ii) for each $j = k_m$. We have

$$\begin{aligned}\sum_{i<k_m} u_i &= \sum_{p<m} ((\ell_p - 1) - 1 \cdot (x_p + \ell_p - 1 - x_{p+1})) \\ &= \sum_{p<m} (-1 \cdot (x_p - x_{p+1})) \\ &= x_m - x_1 = x_m \\ &\geq 0.\end{aligned}$$

iii) Finally, we have

$$\begin{aligned}\sum_{i\in[N]} u_i &= \sum_{p<n} \big((\ell_p - 1) - 1 \cdot (x_p + \ell_p - 1 - x_{p+1})\big) + (\ell_n - 1) - 1 \cdot (x_n + \ell_n) \\ &= \sum_{p<n} (-1 \cdot (x_p - x_{p+1})) - x_n - 1 \\ &= -1.\end{aligned}$$

Therefore $f_U(u) \in \mathcal{D}_U$. The inverse of above function can be described as follows: Fix any sequence $v = \langle u_1, \ldots, u_N\rangle \in \mathcal{D}_U$. Let $k_1 < k_2 < \cdots < k_n \in [N]$ be all indices with $u_{k_m} \geq 0$. Note that this determines the value of n. Define $f^{-1}(v) := \langle (x_1, \ell_1), \ldots, (x_n, \ell_n)\rangle$ with $x_m := \sum_{i<k_m} u_i$ and $\ell_m := u_{k_m} + 1$ for all $m \in [n]$. We now check both properties of sequences in $\mathcal{U}$:

i) We have $x_1 = \sum_{i<1} u_i = 0$.

ii) For all $m > 1$ we have $x_m = \sum_{i<k_m} u_i \geq 0$, by definition of $\mathcal{D}_U$. Also we have

$$\begin{aligned}x_m - x_{m-1} &= \sum_{k_{m-1}\leq i<k_m} u_i \\ &= u_{k_{m-1}} - 1 \cdot (k_m - k_{m-1} - 1) \\ &< u_{k_{m-1}} + 1 = \ell_{m-1}.\end{aligned}$$

Therefore $f^{-1}(v) \in \mathcal{U}$. $\square$

As we have proven Theorem 21 bijectively, we already know how we can relate sequences in $\mathcal{V}$ and $\mathcal{W}$ with sequences in $\mathcal{U}$. We now reuse the ideas from the previous section and replace $\mathcal{U}$ with $\mathcal{D}_U$. Also, we use Raney's lemma [6] instead of the Lagrange inversion formula.

Lemma 25 (Version of Raney's lemma)**.** *For any sequence of integers $\langle a_1, \dots, a_m \rangle$ with $a_i \geq -1$ and $\sum a_i = -1$, there exists exactly one $r \in [m]$ with the property that all proper partial sums, or in other words, the totals of all proper prefixes, of $\langle a_{r+1}, \dots, a_m, a_1, \dots, a_r \rangle$ are non-negative. Note that we must have $a_r = -1$, of course, as $a_{r+1} + \dots + a_m + a_1 + \dots + a_{r-1} \geq 0$, but $\sum a_i = -1$.*

We now generalize the idea from Dyck paths to the following sets $\mathcal{V}$ and $\mathcal{W}$:

Definition 26. Define

$$\mathcal{D}_V := \left\{ \langle u_1, u_2, \dots, u_N \rangle : \forall i \ u_i \geq -1 \text{ and } \sum_{i \in [N]} u_i = -1 \text{ and } u_N = -1 \right\} \text{ and}$$

$$\mathcal{D}_{W_b} := \left\{ \langle u_1, u_2, \dots, u_N \rangle : \forall i \in [b] \ u_i \geq 0 \text{ and } \sum_{i \in [N]} u_i = -b \text{ and } u_N = -1 \right\},$$

where the weight of a sequence is $w(\langle u_1, u_2, \dots, u_N \rangle) := \prod_{i \in [N]} \begin{cases} z y_{u_i+1}, & \text{if } u \geq 0; \\ 1, & \text{if } u = -1. \end{cases}$

We can easily enumerate elements in V and W_b with given weight.

Lemma 27. *Fix a weight $w = z^n \prod_{i \in [m]} y_{s_i}^{n_i}$ with $n = \sum_{i \in [m]} n_i$ and let $s = \sum_{i \in [m]} n_i s_i$. Then the number of elements in $\mathcal{D}_V$ with weight w equals*

$$\binom{s}{n_1, n_2, \dots, n_m, s-n}.$$

Also, the number of elements in $\mathcal{D}_{W_b}$ with weight w equals

$$\binom{n}{n_1, \dots, n_m} \binom{s-1}{n-b}.$$

Proof. For the first part, note that every element in $\mathcal{D}_V$ with weight w has $s - n + 1$ copies of -1 and total length $N = s + 1$. The number of sequences follows from the fact that only u_N is fixed to be -1.

For the second part, note that every element in $\mathcal{D}_{W_b}$ with weight w has $s - n + b$ copies of -1 and total length $N = s + b$. We have $u_N = -1$ and $u_i \neq -1$ for $i \in [b]$. The number of elements in $\mathcal{D}_{W_b}$ follows by first considering the order and then the position of the non-negative integers. $\square$

Corollary 28. *Define*

$$\mathcal{D}_Y := \{(v,k) : v = \langle u_1, u_2, \dots, u_N\rangle \in \mathcal{D}_U \text{ and } k \in [N] \text{ and } u_k = -1\},$$

which can be thought of as the set of sequences in $\mathcal{D}_U$, where one copy of -1 is marked. Then there is a bijection f_{V_2} between $\mathcal{D}_Y$ and $\mathcal{D}_V$.

Proof. Let f_{V_2} be the following bijection: Take an element $(v,k) \in \mathcal{D}_Y$ and let $f_{V_2}((v,k)) := \langle u_{k+1}, \dots, u_N, u_1, \dots, u_k\rangle \in \mathcal{D}_V$. In other words, we have moved the marked element of an element in $\mathcal{D}_Y$ to the end by doing a cyclic rotation of the sequence. This can be undone using Raney's lemma. □

Lemma 29. *We have a weight-preserving bijection f_{V_1} between $\mathcal{V}$ and $\mathcal{D}_Y$. Hence, $f_V := f_{V_2} \circ f_{V_1}$ is a weight-preserving bijection between $\mathcal{V}$ and $\mathcal{D}_V$.*

Proof. We construct an explicit bijection $f_{V_1} : \mathcal{V} \to \mathcal{D}_Y$ as follows. First, let $f_{V_1}(\langle\rangle) := (\langle -1\rangle, 1)$ and now consider any non-empty $v \in \mathcal{V}$. Then by definition $\exists \ell \in \mathbb{N}, \alpha \in \{0, \dots, \ell - 1\}, x \in \mathcal{X}_\ell, v' \in \mathcal{V}$, where x starts with $(-\alpha, \ell)$, such that $v = x \parallel (v' - \alpha)$. By definition of $\mathcal{X}_\ell$ and Lemma 14 we can find $c_1, \dots, c_{\ell-1} \in \mathcal{U}$ such that $x = \big(\langle(0,\ell)\rangle \parallel c_{\ell-1} + \ell - 1 \parallel c_{\ell-2} + \ell - 2 \parallel \cdots \parallel c_1 + 1\big) - \alpha$. Hence we have

$$v = \big(\langle(0,\ell)\rangle \parallel c_{\ell-1} + \ell - 1 \parallel c_{\ell-2} + \ell - 2 \parallel \cdots \parallel c_1 + 1 \parallel v'\big) - \alpha.$$

Let $(u,k) := f_{V_1}(v')$. We define $k' := k + 1 + |f_U(c_{\ell-1})| + \cdots + |f_U(c_{\alpha+1})|$ and

$$f_{V_1}(v) := \big(\langle \ell - 1\rangle \parallel f_U(c_{\ell-1}) \parallel \cdots \parallel f_U(c_{\alpha+1}) \parallel u \parallel f_U(c_\alpha) \parallel \cdots \parallel f_U(c_1), k'\big).$$

Note that k' is chosen such that the marked element in u remains marked. □

Lemma 30. *We have a weight-preserving bijection f_W between $\mathcal{W}_b$ and $\mathcal{D}_{W_b}$.*

Proof. From Lemma 19 we know that we can split an element $w \in \mathcal{W}_b$ into two parts. For $b = 1$ we have $w = u \parallel v$, where $u \in \mathcal{U}_\star$ and $v \in \mathcal{V}$. We define $f_W(w) := f_U(u) \parallel f_V(v)$. For $b \geq 2$ we have $w = u \parallel w' - \ell$, where $u \in \mathcal{U}_1, w' \in \mathcal{W}_{b-1}$ and u starts with $(0,\ell)$. Here we could choose to define $f_W(w) := f_U(u) \parallel f_W(w')$, but it would change the definition of $\mathcal{D}_W$ and make its enumeration more complicated. So we instead proceed as follows: Let $f_W(w') = \langle a_1, \dots, a_n\rangle$. Then define $f_W(w) := \langle a_1, \dots, a_{b-1}\rangle \parallel f_U(u) \parallel \langle a_b, \dots, a_n\rangle$. We can undo the function in both cases, as we know that the sequence $f_U(u)$ sums to -1. The bijection f_W is weight-preserving, because f_U and f_V are. □

We would like to conclude the section by providing examples of the bijection and show how the sequences are related to the towers described in Remark 18. Figure 9 shows the idea of the bijection for $\mathcal{D}_U$. The labels on the blocks correspond to the order of Lemma 5. The different colors correspond to the decomposition from Lemma 16.

As $\mathcal{U}$ corresponds to towers with no overhang to the left, the sequences in $\mathcal{D}_U$ are Dyck paths. To deal with overhang we use Lemma 29 to mark an element -1 in the sequences.

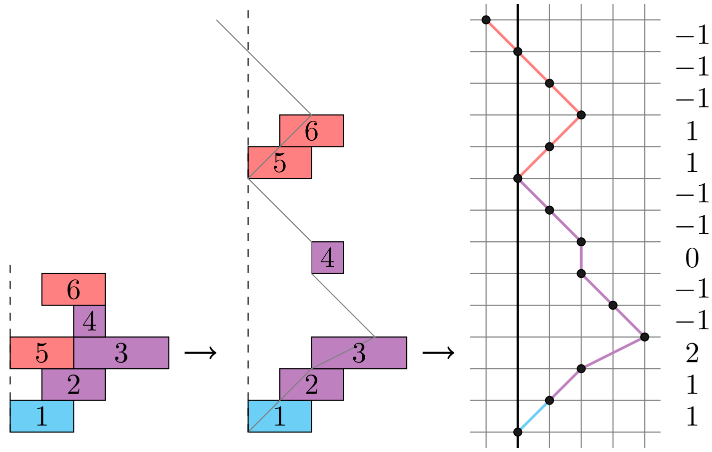
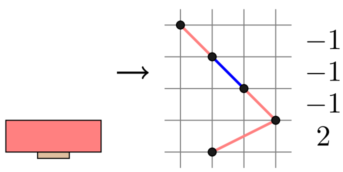
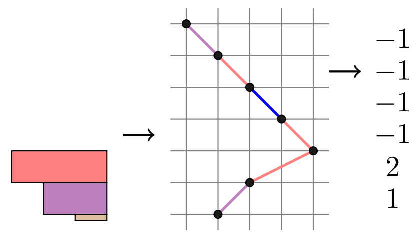


Figure 9: Illustration of the bijection f_U with the element $\langle 1, 1, 2, -1, -1, 0, -1, -1, 1, 1, -1, -1, -1\rangle \in \mathcal{D}_U$.



(a) $(\langle 2, -1, -1, -1\rangle, 3) \in \mathcal{D}_Y$.



(b) $(\langle 1, 2, -1, -1, -1, -1\rangle, 4) \in \mathcal{D}_Y$.

Figure 10: Illustration of the bijection f_{V_1}.

The position of the marked element then determines the overhang to the left. See Figures 10 and 11 for an illustration, where the marked element is colored dark blue. Figure 10 shows two examples of the bijection f_{V_1}, where in Figure 10b the result of Figure 10a is used.

Figure 11 shows a complete example of f_V. Note that the result from Figure 10b is used.

Finally, see Figure 12 for an example of the bijection between $\mathcal{W}_2$ and $\mathcal{D}_{W_2}$. The violet blocks are in $\mathcal{U}$, the blue blocks in $\mathcal{U}_\star$ and the red blocks in $\mathcal{V}$. Note that the result of f_{V_2} applied to Figure 10b is used.

5 Row-convex k-omino towers

In this section, we consider row-convex k-omino towers, as defined in Definition 2. By conditioning on the width of the bottom row, we see that f and g are related by the equation

$$g(n) = \sum_{\ell=1}^{n} f_\ell(n-\ell), \text{ or equivalently } G(z) = \sum_{\ell=1}^{\infty} z^\ell F_\ell(z).$$

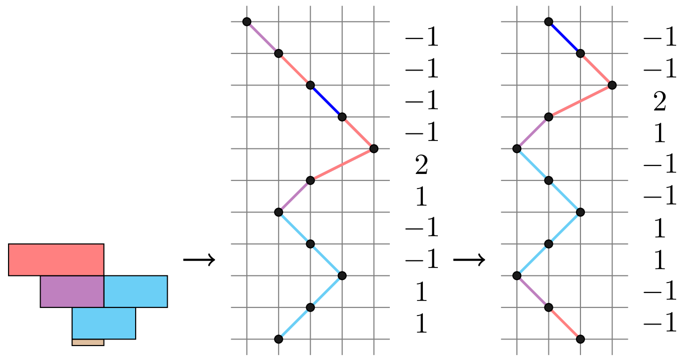


Figure 11: Illustration of the bijection $f_V := f_{V_2} \circ f_{V_1}$ with the element $\langle -1, -1, 1, 1, -1, -1, 1, 2, -1, -1 \rangle \in \mathcal{D}_V$.

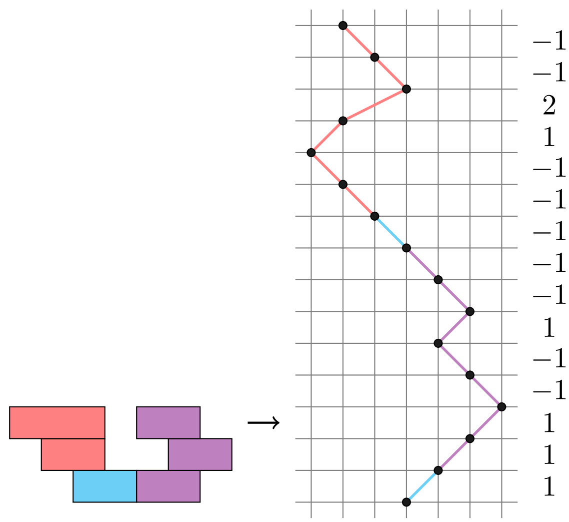


Figure 12: Illustration of the bijection f_W with the element $\langle 1, 1, 1, -1, -1, 1, -1, -1, -1, -1, -1, 1, 2, -1, -1 \rangle \in \mathcal{D}_{W_2}$.

To improve readability, from now on we write $F_\ell(z)$ as F_ℓ and similarly for $G(z)$ and $h_n(z), \alpha(z)$ and $\beta(z)$ which are yet to be introduced. If the platform has width $k\ell$ and the bottom row consists of i blocks, then there are $(\ell+2-i)k-1$ positions the blocks in the row above could take such that the row is convex and they do not fall off the sides. This can be seen by an argument similar to the one given by Brown [3, Prop. 2.5]. We immediately find the recurrence:

$$f_\ell(n) = \sum_{i=1}^{\ell+1} \Big((\ell+2-i)k - 1 \Big) f_i(n-i), \text{ for } n \geq 1, \text{ where we define} \tag{1}$$

$f_\ell(0) = 1$ and $f_\ell(n) = 0$, for $n < 0$.

We now simplify this recurrence relation of f and rewrite it in terms of the generating functions F_ℓ.

Lemma 31. *The generating functions F_ℓ satisfy the following recurrence relation and boundary conditions:*

$$F_{\ell+2} - 2F_{\ell+1} + F_\ell = z^{\ell+2} F_{\ell+2} + (k-1) z^{\ell+3} F_{\ell+3}, \tag{2}$$

$$\begin{aligned} F_1 &= 1 + (2k-1) z F_1 + (k-1) z^2 F_2, \\ F_2 &= 1 + (3k-1) z F_1 + (2k-1) z^2 F_2 + (k-1) z^3 F_3. \end{aligned} \tag{3}$$

Proof. First, we calculate

$$f_{\ell+1}(n) - f_\ell(n) = (k-1) f_{\ell+2}(n - \ell - 2) + k \sum_{i=1}^{\ell+1} f_i(n-i)$$

and then use this result twice as follows:

$$\begin{aligned} &f_{\ell+2}(n) - 2f_{\ell+1}(n) + f_\ell(n) \\ &= \big(f_{\ell+2}(n) - f_{\ell+1}(n)\big) - \big(f_{\ell+1}(n) - f_\ell(n)\big) \\ &= (k-1) f_{\ell+3}(n - \ell - 3) + f_{\ell+2}(n - \ell - 2). \end{aligned}$$

The corresponding recurrence in terms of F_ℓ is

$$\begin{aligned} &F_{\ell+2} - 2F_{\ell+1} + F_\ell \\ &= \sum_{n=0}^{\infty} \Big(f_{\ell+2}(n) - 2f_{\ell+1}(n) + f_\ell(n) \Big) z^n \\ &= \sum_{n=0}^{\infty} \Big(f_{\ell+2}(n - \ell - 2) + (k-1) f_{\ell+3}(n - \ell - 3) \Big) z^n \\ &= z^{\ell+2} F_{\ell+2} + (k-1) z^{\ell+3} F_{\ell+3}. \end{aligned}$$

The boundary conditions are obtained by setting $\ell = 1$ and $\ell = 2$ in (1). □

To solve the recurrence (2), we first guess that there is a solution of the form

$$\sum_{j=0}^{\infty} \frac{z^{\ell j} h_j(z)}{(z;z)_j^2}$$

and then determine an $h_j(z)$ such that the recurrence relation holds. Here, $(a;q)_n = \prod_{i=0}^{n-1}(1 - aq^i)$ denotes the q-Pochhammer symbol. That this method works is not surprising: In 1988, Privman and Švrakić successfully found an exact generating function for *fully directed compact lattice animals* using this approach [7]. The two problems are related,

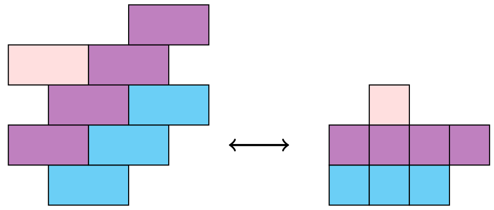


Figure 13: Illustration of the bijection between *restricted* row-convex domino towers and fully directed compact lattice animals

as there is a bijection between fully directed compact lattice animals and *restricted* row-convex domino towers. The number of dominoes in the bottom row maps to the number of compact sources of the directed animal. For an illustration of this bijection see Figure 13.

After adapting their method we end up with two solutions A_ℓ and B_ℓ, which we now check:

Lemma 32. *Two solutions of* (2) *are:*

$$A_\ell := \sum_{j=0}^{\infty} \frac{z^{\ell j} h_j}{(z;z)_j^2} \text{ and}$$

$$B_\ell := \sum_{j=0}^{\infty} \frac{z^{\ell j} h_j}{(z;z)_j^2} \left(\ell + \sum_{m=1}^{j} \left(1 + \frac{2}{1-z^m} - \frac{1}{1+(k-1)z^m} \right) \right), \text{ where}$$

$$h_j := z^{j(j+1)} \left((1-k)z;z\right)_j .$$

Proof. First, we calculate the ratio

$$\frac{h_j}{h_{j-1}} = \frac{z^{j(j+1)}}{z^{(j-1)j}} \frac{\prod_{i=1}^{j}(1-(1-k)z^i)}{\prod_{i=1}^{j-1}(1-(1-k)z^i)} = z^{2j}(1+(k-1)z^j).$$

Then we show that the recurrence holds

$$\begin{aligned}
&A_{\ell+2} - 2A_{\ell+1} + A_\ell \\
&= \sum_{j=0}^{\infty} \frac{h_j}{(z;z)_j^2} \left(z^{(\ell+2)j} - 2z^{(\ell+1)j} + z^{\ell j} \right) \\
&= \sum_{j=1}^{\infty} \frac{h_{j-1}\left(z^{2j} + (k-1)z^{3j}\right)}{(z;z)_j^2} z^{\ell j} \left(1 - z^j\right)^2 \\
&= \sum_{j=1}^{\infty} \frac{h_{j-1}\left(z^{(\ell+2)j} + (k-1)z^{(\ell+3)j}\right)}{(z;z)_{j-1}^2} \\
&= \sum_{j=0}^{\infty} \frac{h_j\left(z^{(\ell+2)(j+1)} + (k-1)z^{(\ell+3)(j+1)}\right)}{(z;z)_j^2} \\
&= z^{\ell+2} A_{\ell+2} + (k-1) z^{\ell+3} A_{\ell+3}.
\end{aligned}$$

Similarly, we can prove that B_ℓ is a solution. The interested reader can find the detailed calculation for this in the appendix. □

We have yet to find a solution that satisfies the boundary conditions. A suitable linear combination of A_ℓ and B_ℓ, however, does the trick. In general, setting $F_\ell = \alpha A_\ell + \beta B_\ell$ and solving the simultaneous equations $c_1F_1 + c_2F_2 + c_3F_3 = 1$ and $d_1F_1 + d_2F_2 + d_3F_3 = 1$ for α and β yields after some algebra:

$$\begin{aligned}
\alpha &= \big(- (c_1 - d_1)B_1 - (c_2 - d_2)B_2 - (c_3 - d_3)B_3\big)/d,\\
\beta &= \big((c_1 - d_1)A_1 + (c_2 - d_2)A_2 + (c_3 - d_3)A_3\big)/d, \text{ where}\\
d &= (c_1A_1 + c_2A_2 + c_3A_3)(d_1B_1 + d_2B_2 + d_3B_3)\\
&\quad - (c_1B_1 + c_2B_2 + c_3B_3)(d_1A_1 + d_2A_2 + d_3A_3).
\end{aligned}$$

Now setting

$$\begin{aligned}
c_1 &= 1 - (2k-1)z, & c_2 &= -(k-1)z^2, & c_3 &= 0,\\
d_1 &= -(3k-1)z, & d_2 &= 1 - (2k-1)z^2, & d_3 &= -(k-1)z^3
\end{aligned}$$

as in (3) and plugging α and β into $F_\ell = \alpha A_\ell + \beta B_\ell$ yields the result of Theorem 4.

Remark 33. Note that for $k = 1$ the sequence $(g(n))_{n\geq1} = 1, 2, 4, 8, 15, \ldots$ is given by sequence A001523. It also counts the weakly unimodal compositions of n. Similarly, $f_\ell(n)$ counts the number of unimodal ℓ-tuples of positive integers summing to $n + \ell$. For example the sequence $(f_3(n))_{n\geq0}$ is given by sequence A000212. For $k = 2$, the sequence $(g(n))_{n\geq1} = 1, 4, 16, 61, 225, \ldots$ is given by sequence A338531 in the *On-line Encyclopedia of Integer Sequences* [11].

6 Comments and open questions

1. One might reconsider the *restricted* problem. Using a substitution similar to the one mentioned in Corollary 8, we can deduce the generating function for the restricted case, by replacing zy_i with $\frac{zy_i}{1+zy_i}$. However, is it possible to find a direct way of enumerating the generating function of restricted towers and a closed formula for its coefficients?

2. In this paper, we have counted S-omino towers and row-convex towers. Is it possible to combine the two ideas and count row-convex S-omino towers?

7 Acknowledgments

I would like to thank my supervisor Anusch Taraz and Julian Großmann for many helpful comments and suggestions regarding both content and presentation.

Additional thanks go to the anonymous referee for carefully reading the manuscript and their many valuable comments.

Appendix

For the sake of completeness, we show that B_ℓ is a solution to (2).

$$
\begin{aligned}
&B_{\ell+2} - 2B_{\ell+1} + B_\ell \\
&= \sum_{j=0}^{\infty} \frac{h_j}{(z;z)_j^2} \Bigg((\ell+2) z^{(\ell+2)j} - 2(\ell+1) z^{(\ell+1)j} + \ell z^{\ell j} \\
&\qquad + \left(z^{(\ell+2)j} - 2z^{(\ell+1)j} + z^{\ell j} \right) \sum_{m=1}^{j} \left(1 + \frac{2}{1-z^m} - \frac{1}{1+(k-1)z^m} \right) \Bigg) \\
&= \sum_{j=1}^{\infty} \frac{h_j}{(z;z)_j^2} z^{\ell j} \left(1 - z^j\right)^2 \left(2 + \ell - \frac{2}{1-z^j} + \sum_{m=1}^{j} \left(1 + \frac{2}{1-z^m} - \frac{1}{1+(k-1)z^m} \right) \right) \\
&= \sum_{j=1}^{\infty} \frac{h_{j-1} \left(z^{(\ell+2)j} + (k-1) z^{(\ell+3)j} \right)}{(z;z)_{j-1}^2} \Bigg(2 + \ell - \frac{2}{1-z^j} \\
&\qquad + \sum_{m=1}^{j} \left(1 + \frac{2}{1-z^m} - \frac{1}{1+(k-1)z^m} \right) \Bigg) \\
&= \sum_{j=1}^{\infty} \frac{h_{j-1} \left(z^{(\ell+2)j} + (k-1) z^{(\ell+3)j} \right)}{(z;z)_{j-1}^2} \Bigg(3 + \ell - \frac{1}{1+(k-1)z^j} \\
&\qquad + \sum_{m=1}^{j-1} \left(1 + \frac{2}{1-z^m} - \frac{1}{1+(k-1)z^m} \right) \Bigg) \\
&= \sum_{j=1}^{\infty} \frac{h_{j-1}}{(z;z)_{j-1}^2} \Bigg((\ell+2) z^{(\ell+2)j} + (k-1)(\ell+3) z^{(\ell+3)j} \\
&\qquad + \left(z^{(\ell+2)j} + (k-1) z^{(\ell+3)j} \right) \sum_{m=1}^{j-1} \left(1 + \frac{2}{1-z^m} - \frac{1}{1+(k-1)z^m} \right) \Bigg) \\
&= \sum_{j=0}^{\infty} \frac{h_j}{(z;z)_j^2} \Bigg((\ell+2) z^{(\ell+2)(j+1)} + (k-1)(\ell+3) z^{(\ell+3)(j+1)} \\
&\qquad + \left(z^{(\ell+2)(j+1)} + (k-1) z^{(\ell+3)(j+1)} \right) \sum_{m=1}^{j} \left(1 + \frac{2}{1-z^m} - \frac{1}{1+(k-1)z^m} \right) \Bigg) \\
&= z^{\ell+2} B_{\ell+2} + (k-1) z^{\ell+3} B_{\ell+3}.
\end{aligned}
$$

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2010 *Mathematics Subject Classification:* Primary 05A15; Secondary 05A19.
Keywords: domino, bijection, convex polyomino, Lagrange inversion.

(Concerned with sequences A000108, A001523, A000212, and A338531.)

Received November 1 2020; revised versions received November 9 2020; February 10 2021; February 15 2021. Published in *Journal of Integer Sequences*, February 16 2021.

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