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The abc Conjecture Implies That Only Finitely Many s-Cullen Numbers Are Repunits

Jon Grantham and Hester Graves
Institute for Defense Analyses
Center for Computing Sciences
Bowie, MD 20715
USA
grantham@super.org
hkgrave@super.org

Abstract

Assuming the abc conjecture holds with $\epsilon = 1/6$, we use elementary methods to show that only finitely many s-Cullen numbers are repunits, aside from two known infinite families. More precisely, only finitely many positive integers s, n, b, and q with $s, b \geq 2$ and $n, q \geq 3$ satisfy $C_{s,n} = ns^n + 1 = (b^q - 1)/(b - 1)$.

1 Introduction

Definition 1. A *Cullen number* is an number of the form $C_n = n2^n + 1$, where n is a positive integer. The Cullen numbers are A002064 in the OEIS. An *s-Cullen number* is a number of the form $C_{n,s} = ns^n + 1$, where s, n are positive integers with $s \geq 2$. See A050914, for example, for the 3-Cullen numbers. Cullen and Dubner ([4, 5]) introduced the two families, respectively.

The first significant result on Cullen numbers occurred in 1976, when Hooley [7] showed that almost all Cullen numbers are composite. Caldwell [3] conjectured that infinitely many are prime.

Luca and Stănică [8] showed that the intersection of the Cullen numbers with the Fibonacci sequence is finite. Marques [9] generalized this result to s-Cullen numbers (for fixed

s). Bilu, Marques, and Togbé [2] (among other results) generalized the two previous papers' results to the intersection of s-Cullen numbers with other recurrence sequences.

In this paper, we consider the intersection of the s-Cullen numbers with a two–parameter family, namely the repunits. We are able to prove, conditionally, that the intersection of the repunits and Cullen numbers is finite, except for two infinite families. In particular, for fixed s, the intersection is finite.

Definition 2. A *repunit* is a positive integer n that we can write as

$$\frac{b^q-1}{b-1}=\sum_{j=0}^{q-1} b^j = b^{q-1}+b^{q-2}+\cdots+b+1=(11\cdots1)_b$$

for some integer *base* $b>1$ and some integer exponent q. Beiler [1] introduced the name in the 1960s.

For example, base-10 repunits are A002275 in the OEIS, base-15 repunits are A135518, and base-2 repunits are A000225. We impose the condition $q\geq 3$ to avoid the trivial representation of any number x as 11 in base $x-1$.

Definition 3. The *radical* of a positive integer n is the product of all the primes that divide n, so if $n=\prod_{p_i|n} p_i^{a_i}$, then $\operatorname{rad}(n)=\prod_{p_i|n} p_i$. For example, the radical of $90=2\cdot 3^2\cdot 5$ is $30=2\cdot 3\cdot 5$.

Conjecture 4. The *abc conjecture* of Oesterlé [11] and Masser [10] states that if a,b, and c are relatively prime integers such that $a+b=c$, then for any $\epsilon>0$, only finitely many (a,b,c) fail to satisfy the inequality

$$c<\operatorname{rad}(abc)^{1+\epsilon}.$$

In the next section, we use this conjecture with $\epsilon=1/6$.

2 Main result

Theorem 5. *The abc conjecture with $\epsilon=1/6$ implies that only finitely many s-Cullen numbers $C_{s,n}$, with $n\geq 3$, are repunits of length three or greater.*

We divide our theorem into two cases, each of which we prove as a proposition. The first proposition shows that only finitely many s-Cullen numbers can be written as repunits of length three with $n\geq 3$, and the second shows that only finitely many s-Cullen numbers can be written as repunits of length greater than three with $n\geq 2$. As the union of two finite sets is finite itself, the two propositions prove the theorem.

Proposition 6. *The abc conjecture with $\epsilon=1/6$ implies that only finitely many s-Cullen numbers are repunits of length three with $n\geq 3$.*

Proof. Suppose that $C_{s,n} = ns^n+1$ is a repunit of length three, i.e., $C_{s,n} = ns^n+1 = b^2+b+1$. We then infer that $ns^n = b(b+1)$. Let us first consider the case

$$b+1 < \mathrm{rad}(b(b+1))^{\frac{7}{6}} = \mathrm{rad}(ns^n)^{\frac{7}{6}} < (ns)^{\frac{7}{6}},$$

so that

$$ns^n = b(b+1) < (b+1)^2 < (ns)^{\frac{7}{3}} = n^{\frac{7}{3}} s^{\frac{7}{3}}.$$

By taking logarithms, we see that

$$n - \frac{7}{3} < \frac{4}{3} \log_s(n),$$

or

$$3n - 7 < 4 \log_s(n). \tag{1}$$

This equation cannot hold when $s \geq 9, n \geq 3$. For each s, $2 \leq s \leq 8$, only finitely many values of n satisfy Equation 1.

The abc conjecture with $\epsilon = 1/6$ asserts that only finitely many values b satisfy $b+1 \geq \mathrm{rad}(b(b+1))^{\frac{7}{6}}$, so certainly only finitely many b satisfy $b+1 \geq \mathrm{rad}(b(b+1))^{\frac{7}{6}}$, where b^2+b+1 is an s-Cullen number.

We conclude that only finitely many $C_{s,n}$, with $n \geq 3$, are repunits of length three. □

Proposition 7. *The abc conjecture with $\epsilon = 1/6$ implies that only finitely many s-Cullen numbers with $n \geq 2$ are repunits of length greater than three.*

Proof. Suppose that $C_{s,n} = ns^n + 1$ is a repunit of length greater than three, i.e., that $C_{s,n}$ can written as $\frac{b^q-1}{b-1} = b^{q-1} + b^{q-2} + \cdots + b + 1$ for some $b \geq 2, q \geq 4$. Assuming this supposition,

$$ns^n = b(b^{q-2} + \cdots + b + 1) = b\left(\frac{b^{q-1}-1}{b-1}\right),$$

which we rewrite as

$$(b-1)ns^n = b(b^{q-1}-1).$$

In the case where $b^{q-1} < \mathrm{rad}(b(b^{q-1}-1))^{\frac{7}{6}}$, then

$$b^{q-1} < \mathrm{rad}((b-1)ns^n)^{\frac{7}{6}}$$

and thus

$$b^{q-1} - 1 < (ns)^{\frac{7}{6}}(b-1)^{\frac{7}{6}},$$

which we can rewrite as

$$\frac{b(b^{q-1}-1)}{b-1} < (ns)^{\frac{7}{6}} b(b-1)^{\frac{1}{6}}.$$

The previous inequality shows us that

$$ns^n = \frac{b(b^{q-1}-1)}{b-1} < (nsb)^{\frac{7}{6}},$$

or

$$s^{n-\frac{7}{6}} < n^{\frac{1}{6}} b^{\frac{7}{6}}. \tag{2}$$

We know that $ns^n > b^{q-1}$, so $(ns^n)^{\frac{7}{6(q-1)}} > b^{\frac{7}{6}}$. This inequality then gives us a further upper bound on Inequality 2, as

$$s^{n-\frac{7}{6}} < n^{\frac{1}{6}} b^{\frac{7}{6}} < n^{\frac{q+6}{6(q-1)}} s^{\frac{7n}{6(q-1)}}$$

so

$$s^{n-\frac{7}{6}-\frac{7n}{6(q-1)}} < n^{\frac{q+6}{6(q-1)}}.$$

If we take the log base s of both sides, we see that

$$n - \frac{7}{6} - \frac{7n}{6(q-1)} < \frac{q+6}{6(q-1)} \log_s n,$$

or

$$n - \frac{7(q-1)}{6q-13} < \frac{q+6}{6q-13} \log_s n.$$

We assumed that $q \geq 4$, giving us

$$n - \frac{21}{11} < \frac{10}{11} \log_s n,$$

or

$$11n - 21 < 10 \log_s n. \tag{3}$$

We can see that this equation cannot hold if $s \geq 3$ and $n \geq 3$, or if $s \geq 1025$ and $n = 2$. Thus, the only potential repunits are $C_{s,2}$, with $2 \leq s \leq 1024$.

Only finitely many b^{q-1} satisfy $b^{q-1} > \text{rad}(b(b^{q-1}-1))^{\frac{7}{6}}$, so only finitely many s-Cullen numbers can be written as $\frac{b^q-1}{b-1} = b^{q-1} + b^{q-2} + \cdots + b + 1$ for said values of b, and thus we can conclude that only finitely many s-Cullen numbers with $n \geq 2$ are repunits of length greater than three.

□

3 Known examples

Two infinite families of s-Cullen repunits exist, but each produces at most one example for each s.

We see that $C_{s,1}$ is a repunit of length at least 3 exactly when $s+1$ is.

When $C_{s,2}$ is a repunit of length 3, we know that $2s^2 + 1 = x^2 + x + 1$ for some s. Thus $s^2 = x(x + 1)/2$, and s^2 is a square triangular number. See A001110. These numbers were characterized by Euler [6], and help us find the infinite family $C_{6,2}$, $C_{35,2}$, $C_{204,2}, \ldots$ of s-Cullen numbers which are length-three repunits.

Other than the two families noted above, no s-Cullen numbers $C_{s,n}$ are also repunits with $s \leq 100$ and $n \leq 100$ or with $s \leq 10^6$ and $n \leq 10$. The PARI [12] code for this computation is at github.com/31and8191/Cullen. It is an open problem whether any s-Cullen repunits exist outside the two families characterized above.

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