Journal of Integer Sequences, Vol. 24 (2021),
Article 21.9.6

On Composite Odd Numbers k for Which $2^n k$ is a Noncototient for All Positive Integers n

Marcos J. González and Alberto Mendoza
Departamento de Matemáticas Puras y Aplicadas
Universidad Simón Bolívar
Sartenejas, Apartado 89000
Caracas, 1080-A
Venezuela
mago@usb.ve
jakob@usb.ve

Florian Luca
School of Mathematics
University of the Witwatersrand
1 Jan Smuts Avenue Braamfontein 2050
Johannesburg
South Africa
and
Research Group in Algebraic Structures and Applications
King Abdulaziz University
Jeddah
Saudi Arabia
florian.luca@wits.ac.za

V. Janitzio Mejía Huguet
Departamento de Ciencias Básicas
Universidad Autónoma Metropolitana-Azcapotzalco
Av. San Pablo #180
Col. Reynosa Tamaulipas
C. P. 02200, Azcapotzalco DF
México
vjanitzio@gmail.com

Abstract

We provide some results relating Riesel numbers with families of cototients and noncototients.

1 Introduction

A positive integer k is called a *Riesel number* whenever k is odd and $k2^n - 1$ is composite for every nonnegative integer n. These numbers are named after Hans Riesel who in 1956 discovered this property for the number $k = 509203$ (see [6]). This is conjectured to be the smallest Riesel number. A conjecture due to Erdős asserts that if k is a Riesel number then the smallest prime factor p_n of $k2^n - 1$ is bounded as n tends to infinity (see [4]). We refer to such a k as an *Erdős-Riesel number*. As far as we know, each known example of a Riesel number is also an Erdős-Riesel number.

A positive integer n is a *noncototient* if $n \neq m - \varphi(m)$, for every integer $m \geq 2$, where $\varphi(m)$ denotes *Euler's totient function*. Notice that k is a Riesel number if and only if $k2^n \neq p + 1 = 2p - \varphi(2p)$, for every odd prime p. Sierpiński and Erdős asked whether or not there exist infinitely many noncototients. This question was answered affirmatively by Browkin and Schinzel [1], by showing that the number $k = 509203$ has the property that $k2^n$ is a noncototient for every $n \geq 1$. In 2005, Flammenkamp and Luca [3] found six more numbers with this property, namely:

$$2554843, 9203917, 9545351, 10645867, 11942443, 65484763.$$

The method used in [3] is based on the fact that *if k is a prime Riesel number, which is not a Mersenne prime, and $2k$ is a noncototient, then $2^n k$ is a noncototient, for every $n \geq 1$* (see [3, Proposition 1]). For prime values of k, Grytczuk and Mędryk [5] proved that the criterion employed in [3] are optimal; more precisely, they proved that *for a prime integer k, $2^n k$ is a noncototient for every $n \geq 1$, if and only if $2k$ is a noncototient and k is a Riesel number which is not a Mersenne prime* (see [5, Theorem 1]). They also proved that if k is prime and $2k$ is a cototient then $2k = m - \varphi(m)$ for some even, squarefree integer m (see [5, Theorem 3]). We conclude that in this approach—adopted to find infinite families of noncototients—it is essential to work with prime numbers. The sequence of Riesel numbers is A101036 and starts as

$$509203, 762701, 777149, 790841, 992077, 1106681, 1247173, 1254341, \ldots$$

Our first aim in this paper is to present a method also valid for composite Riesel numbers. As an application, we have obtain the following:

Theorem 1. *For each k in the set*

$$\{762701, 790841, 992077, 1247173, 1730653, 1744117, 1830187, 1976473, \ 3419789, 3423373\},$$

the number $2^n k$ is a noncototient, for every $n \geq 1$.

It is worth adding that the 10 Riesel numbers displayed in the statement of Theorem 1 are all composite. One may ask what can one say about Riesel numbers k such that $2^n k$ is a cototient for some $n \geq 1$. We next result addresses such Riesel numbers.

Theorem 2. *There are infinitely many Riesel numbers k such that $2^n k$ is a cototient for some $n \geq 1$.*

2 The proof of Theorem 1

Given non-zero integers a, p, with p prime, we let $\nu_p(a)$ denote the exponent of p in the prime factorization of a. Further, we use $P(n)$ for the largest prime factor of n. The following lemma is our workhorse when attempting to detect Riesel numbers k such that $2^n k$ is a noncototient for some positive integer n.

Lemma 3. *Let k be a Riesel number. Put $p := P(k)$ and assume*

$$2^{\nu_2(p+1)} k \neq p^{\nu_p(k)}(p+1). \tag{1}$$

The following conditions are equivalent:

(i) *for some $n \geq 1$, the number $2^n k$ is a cototient;*

(ii) *for every $n \geq 1$, the number $2^n k$ is a cototient.*

Proof. It is clear that (ii) implies (i), so it is enough to show that (i) implies (ii).

Assume that (i) holds and let $n_0 \geq 1$ be minimal such that $2^{n_0} k = m - \phi(m)$ for some positive integer m. Since $2^2 - 1 = 3$ is prime and k is a Riesel number, it follows that $k \geq 3$, so $m \geq 6$. In particular, $\phi(m)$ is even. The equation $2^{n_0} k = m - \phi(m)$ implies that m is even.

Consider the case $n_0 > 1$. If $4 \mid \phi(m)$, then $4 \mid m$ and $2^{n_0 - 1} k = (m/2) - \phi(m/2)$, contradicting the minimality of n_0. So, $m \geq 6$ is even and $2 \| \phi(m)$ showing that $m = 2q^e$ for some odd prime q and positive integer exponent e. But then

$$2^{n_0} k = m - \phi(m) = 2q^e - q^{e-1}(q-1) = q^{e-1}(q+1).$$

If $e > 1$, then $q = P(k) = p$ and $e - 1 = \nu_p(k)$. Since k is odd, we get $\nu_2(p+1) = n_0$. Thus,

$$2^{\nu_2(p+1)} k = p^{\nu_p(k)}(p+1),$$

which we assumed not to hold. If $e = 1$, then $2^{n_0} k - 1 = q$ is prime, contradicting the fact that k is Riesel. Thus, $n_0 = 1$. Since m is even, we have $2k = m - \phi(m)$ and m is even. Hence, for all $n \geq 1$, we have $2^n k = (2^{n-1} m) - \phi(2^{n-1} m)$ is a cototient, which is (ii). □

In particular, if we want k to be a Riesel number such that $2^n k$ is a noncototient for all n, it suffices to do two things:

(i) Check condition (1).

(ii) Check that $2k$ is a noncototient.

Given k, condition (i) is immediate to check. For (ii), we need to see whether there is a representation $2k = m - \phi(m)$ for some positive integer m. As we said, if this is so then m is even, so $\phi(m) \le m/2$; hence, $2k = m - \phi(m) \ge m/2$, showing that $m \le 4k$. Further, $m \equiv 2 \pmod 4$. Entry A101036 lists 28 Riesel numbers, with the largest one being $3580901 < 4 \times 10^6$. If one of them say k is such that $2k = m - \phi(m)$, then $m = 2(2\ell + 1)$ for some $\ell < 4 \times 10^6$. We generated with Mathematica the set of numbers of the form

$$\{m - \phi(m) : m \equiv 2(2\ell + 1),\ \ell < 4 \times 10^6\}.$$

There are 2117016 such numbers. We intersected this set with the set of numbers of the form $2k$, where k runs through the 28 Riesel numbers appearing in . Of these 28, only 16 numbers k survived. All of them passed condition (1) but 6 of them were primes. The remaining 10 are displayed in the statement of Theorem 1.

3 The proof of Theorem 2

Proof. Let us recall how one constructs Riesel numbers. Consider a system of triples $\{(a_i, b_i, p_i)\}_{i=1}^t$ of integers that satisfies the following two properties:

cov For every $n \in \mathbb{Z}$, there exists $i \in \{1, 2, \ldots, t\}$ such that $n \equiv a_i \pmod{b_i}$,

ord the numbers $p_1, \ldots, p_t$ are prime, pairwise distinct and $p_i \mid 2^{b_i} - 1$ for $i = 1, 2, \ldots, t$.

Such covering progressions appeared in Erdős work [2] in the context of proving that there are infinitely many odd integers not of the form $2^n + p$, where p is some prime. As for Riesel numbers, any odd positive integer k solving the system of congruences

$$k \equiv 2^{-a_i} \pmod{p_i}, \quad i = 1, 2, \ldots, t, \tag{2}$$

fulfilling in addition the inequality $k \ge \max\{p_1, \ldots, p_t\}$, is a Riesel number. In addition to these congruences, we also want k to satisfy (1). Assuming that $\nu_p(k) = 1$, we impose the additional condition $2^{\nu_2(p+1)}k = p(p + 1)$. Note that if k satisfies the above equation, then $p = P(k)$. Consider the system of triples

$$\{(a_i, b_i, p_i)\}_{i=1}^7 = \{(1, 2, 3), (2, 4, 5), (4, 8, 17), (8, 16, 257), (16, 32, 65537),$$
$$(32, 64, 6700417), (0, 64, 641)\},$$

which is known to fulfill conditions **cov** and **ord** (see [4]). We note that $2^{a_i} \equiv \pm 1 \equiv 2^{-a_i} \pmod{p_i}$, for every $i = 1, \ldots, 7$. Choosing $\nu_2(p + 1) = 30$, the equation $2^{30}k = p(p + 1)$ implies

$$2^{30-a_i} \equiv p(p + 1) \pmod{p_i}$$

for $i = 1, 2, \dots, 7$. One checks that the quadratic equations $2^{30-a_i} \equiv x(x+1) \pmod{p_i}$ have an integer solution x for each $i = 1, \dots, 7$. The condition $\nu_2(p+1) = 30$ is equivalent to $p \equiv 2^{30} - 1 \pmod{2^{31}}$. Thus, by the Chinese Remainder Lemma there is an odd integer x coprime to $p_1 p_2 \cdots p_7$ such that the residue class x modulo $2^{31} p_1 \cdots p_7$ satisfies all the above congruences. By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many prime numbers p in the above progression. For example,

$$p = 1516736073584198554394222919 67$$

satisfies all the above congruences. With each such prime p, the number $k = p(p+1)/2^{30}$ is a Riesel number satisfying the required property, namely that $2^n k$ is a cototient for all $n \geq 1$. These numbers resemble Riesel numbers that are perfect powers, which have been studied in [4]. □

4 Acknowledgments

We thank the referee for a careful reading of the manuscript and suggestions that led to the current formulation of Theorem 2 and proofs of our results. During the preparation of this paper, V. J. M. H. was supported by Grant UAM-A.

References

[1] J. Browkin and A. Schinzel, On integers not of the form $n - \varphi(n)$, *Colloq. Math.* **68** (1995), 55–58.

[2] P. Erdős, On integers of the form $2^k + p$ and some related problems, *Summa Brasil. Math.* **2** (1950), 113–123.

[3] A. Flammenkamp and F. Luca, Infinite families of noncototients, *Colloq. Math.* **86** (2000), 37–41.

[4] M. Filaseta, C. Finch, and M. Kozek, On powers associated with Sierpiński numbers, Riesel numbers and Polignac's conjecture, *J. Number Theory* **128** (2008), 1916–1940.

[5] A. Grytczuk and B. Mędryk, On a result of Flammenkamp-Luca concerning noncototient sequence, *Tsukuba J. Math.* **29** (2005), 533–538.

[6] H. Riesel, Några stora primtal, *Elementa* **39** (1956), 258–260.

2010 *Mathematics Subject Classification*: Primary 11A51; Secondary 11A07, 11N25.
Keywords: Riesel number, cototient.

(Concerned with sequence A101036.)

Received March 9 2021; revised version received March 10 2021; October 15 2021. Published in *Journal of Integer Sequences*, October 17 2021.

Return to Journal of Integer Sequences home page.