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Series Involving Euler's Eta (or Dirichlet Eta) Function

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Abstract

In this article, we evaluate in closed form a number of series involving values of the Dirichlet eta function, and also Fibonacci and Lucas numbers. We also introduce a special constant representing the values of several such series.

1 Introduction

Recall that the Riemann zeta function $\zeta(s), s \in \mathbb{C}$, is defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \Re(s) > 1.$$

[1]Disclaimer: Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

The closely related function $\eta(s), s \in \mathbb{C}$, defined by

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s}, \quad \Re(s) > 0,$$

is called alternating zeta, or Dirichlet's eta, or Euler's eta function. This function was used by Euler and it is possible that Euler preferred this function instead of $\zeta(s)$ for its better convergence [3]. The functions are linked by the following relation

$$\eta(s) = (1 - 2^{1-s})\zeta(s). \tag{1}$$

An integral representation for $\eta(s)$ is given by

$$\eta(s) = \frac{2^{s-1}}{\Gamma(s+1)} \int_0^{\infty} \frac{t^s}{\cosh^2(t)}\, dt, \qquad \Re(s) > -1, s \neq 1.$$

Several properties of the eta function were studied recently by Sondow [16, 17], Milgram [14] and Alzer and Kwong [2], among others.

This paper is concerned with infinite series involving the eta function. The classical Goldbach theorem says that

$$\sum_{n=2}^{\infty} (\zeta(n) - 1) = 1.$$

The analog theorem for $\eta(s)$ can be derived as follows:

$$\begin{aligned}
\sum_{n=2}^{\infty} (\eta(n) - 1) &= \sum_{n=2}^{\infty} \left(\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k^n} \right) \\
&= \sum_{k=2}^{\infty} (-1)^{k-1} \left(\sum_{n=2}^{\infty} \left(\frac{1}{k} \right)^n \right) \\
&= \sum_{k=2}^{\infty} (-1)^{k-1} \left(\frac{1}{k^2} \frac{1}{1 - \frac{1}{k}} \right) \\
&= \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k(k-1)} \\
&= \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k-1} - \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} \\
&= -\ln(2) - (\ln(2) - 1) \\
&= 1 - 2\ln(2).
\end{aligned}$$

In the next section we evaluate a number of series with eta values in terms of a special constant K. The third section presents a collection of series involving eta values together with Fibonacci and Lucas numbers. Their closed form evaluation is given in terms of such numbers in combination with trigonometric functions and the digamma function.

2 Eta series involving a special constant

Lemma 1. *We have*

$$|\eta(n+1)-1| \le \frac{C}{2^n}$$

for some constant C.

This estimate follows from (1) and a similar estimate for $\zeta(n+1)-1$ shown in [5].

Lemma 2. *(Abel) Let $\{a_n\}$ and $\{b_n\}, n \ge p$, be two sequences and let $A_n = a_p + a_{p+1} + \cdots + a_n$. Then for every $n > p$ we have*

$$\sum_{k=p}^{n} a_k b_k = b_n A_n + \sum_{k=p}^{n-1} A_k (b_k - b_{k+1}).$$

Let $\psi(x) = \Gamma'(x)/\Gamma(x)$ be the digamma function and let the constant K be defined by

$$K = \int_0^1 \frac{\psi(1+x) - \psi(1+x/2)}{x}\, dx. \tag{2}$$

The constant K exhibits strong similarity to another constant, M, that was defined and studied by the first author [5]:

$$M = \int_0^1 \frac{\psi(1+x)+\gamma}{x}\, dx, \tag{3}$$

with $\gamma = -\psi(1)$ being the Euler-Mascheroni constant. The digits of M are sequence A131688 in the OEIS [15]. The decimal form of the constant M can be found in Finch's book about mathematical constants [7, p. 62]. The article by Coffey [6] also contains important information about M. The constants K and M are strongly related to each other as

$$K = \int_{1/2}^1 \frac{\psi(1+x)+\gamma}{x}\, dx. \tag{4}$$

The numerical values are

$$M = 1.25774688694436963\cdots$$
$$K = 0.55212832208549207\cdots\ .$$

In this section it will be proved that several interesting eta series can be evaluated in terms of the new constant K. The results from the next theorem are similar to those of [5, Theorem 2]. We recall that harmonic numbers $(H_n)_{n\ge 0}$ are defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \qquad H_0 = 0,$$

and alternating or skew-harmonic numbers $(H_n^-)_{n\geq 0}$ are defined by

$$H_n^- = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}, \qquad H_0^- = 0.$$

We also define the modified exponential integral $\mathrm{Ein}(x)$, an entire function, by

$$\mathrm{Ein}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n!n}.$$

Theorem 3. *With K as defined above, the following statements hold:*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\eta(n+1)}{n} = K, \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \ln\left(1+\frac{1}{n}\right) = K, \tag{6}$$

$$\sum_{n=1}^{\infty} H_n \left(\eta(n+1)-1\right) = K, \tag{7}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(n - 2\ln 2 - \eta(2) - \eta(3) - \cdots - \eta(n)\right) = K, \tag{8}$$

$$\sum_{n=1}^{\infty} H_n^- \left(\eta(n+1) - \eta(n+2)\right) = K - \ln 2, \tag{9}$$

$$\int_0^{\infty} \frac{\mathrm{Ein}(x)\, dx}{e^x + 1} = K, \tag{10}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} \left(\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \eta(k+1) \right) = K. \tag{11}$$

Proof. The well-known Taylor expansion for $\psi(1+x)+\gamma$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \zeta(n+1) x^n = \psi(1+x) + \gamma$$

implies

$$\sum_{n=1}^{\infty} (-1)^{n-1} \eta(n+1) x^n = \psi(1+x) - \psi(1+x/2).$$

Dividing by x both sides and integrating between 0 and 1 proves (5). Then (5) implies (6):

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\eta(n+1)}{n} &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{n+1}}\right)\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1}{k}\right)^{n}\right)\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\ln\left(1+\frac{1}{k}\right).\end{aligned}$$

Next we prove (6) $\rightarrow$ (7):

$$\begin{aligned}\sum_{n=1}^{\infty}H_n\left(\eta(n+1)-1\right) &= \sum_{n=1}^{\infty}H_n\left(\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k^{n+1}}\right)\\ &= \sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k}\left(\sum_{n=1}^{\infty}H_n\left(\frac{1}{k}\right)^{n}\right),\end{aligned}$$

and using the generating functions for the harmonic numbers

$$\sum_{m=1}^{\infty}H_m x^m = \frac{-\ln(1-x)}{1-x} \quad (|x|<1),$$

we get

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{-1}{1-\frac{1}{k}}\ln\left(1-\frac{1}{k}\right)\right) &= \sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k-1}\ln\left(\frac{k}{k-1}\right)\\ &= \sum_{m=1}^{\infty}\frac{(-1)^{m}}{m}\ln\left(\frac{m+1}{m}\right)\\ &= K.\end{aligned}$$

Now, starting from the first sum in the last equation we write

$$\begin{aligned}
K &= \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{-1}{1-\frac{1}{k}} \ln\left(1-\frac{1}{k}\right)\right) \\
&= \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k-1} \left(-\ln\left(1-\frac{1}{k}\right)\right) \\
&= \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k-1} \left(\sum_{n=1}^{\infty} \frac{1}{n\,k^n}\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k^n(k-1)}\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(n - 2\ln 2 - \eta(2) - \eta(3) - \cdots - \eta(n)\right)
\end{aligned}$$

by using the evaluation (proved by induction)

$$\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k^n(k-1)} = n - 2\ln 2 - \eta(2) - \eta(3) - \cdots - \eta(n).$$

Thus (8) is also proved. Next we prove (10). From the representation

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x+1}\,dx$$

we find with $s = n+1$ that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\eta(n+1)}{n} = \int_0^{\infty} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n!n}\right) \frac{dx}{e^x+1} = \int_0^{\infty} \frac{\mathrm{Ein}(x)\,dx}{e^x+1}.$$

We shall prove now the implication (5) $\to$ (9) by using Abel's lemma (Lemma 2) for transformation of series. We take $p = 1$, $a_k = \frac{(-1)^{k-1}}{k}$, and $b_k = \eta(k+1) - 1$. Then $A_k = H_k^-$ and we find

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(\eta(k+1)-1) = H_n^-(\eta(n+1)-1) + \sum_{k=1}^{n-1} H_k^-(\eta(k+1)-\eta(k+2)).$$

Setting $n \to \infty$ and using the estimate from Lemma 1 and also the fact that $|H_n^-| \le H_n$ and $H_n \sim \ln n$ at infinity, we come to the equation

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(\eta(k+1)-1) = \sum_{k=1}^{\infty} H_k^-(\eta(k+1)-\eta(k+2)).$$

According to (5) we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(\eta(k+1)-1) = K - \ln 2$$

and (9) follows. Finally, we prove (11). The proof is based on Euler's series transformation [4]. Given a power series $f(x) = a_0 + a_1 x + \cdots$ we have for sufficiently small $|t|$ that

$$\frac{1}{1-t} f\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{n} \binom{n}{k} a_k\right).$$

We take $f(x) = \psi(1+x) - \psi(1+x/2)$ with the expansion $\psi(1+x) - \psi(1+x/2) = \sum_{n=1}^{\infty}(-1)^{n-1}\eta(n+1)x^n$ where $a_0 = f(0) = 0$. Using the substitution $x = \frac{t}{1-t}$, $dx = \frac{dt}{(1-t)^2}$ we compute

$$\begin{aligned}
K &= \int_0^1 \frac{\psi(1+x) - \psi(1+x/2)}{x}\, dx \\
&= \int_0^{1/2} \frac{1}{1-t} f\left(\frac{t}{1-t}\right) \frac{dt}{t} \\
&= \int_0^{1/2} \sum_{n=1}^{\infty} t^n \left(\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1}\eta(k+1)\right) \frac{dt}{t} \\
&= \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1}\eta(k+1)
\end{aligned}$$

after integrating term by term. □

3 Eta series with Fibonacci coefficients

The important Fibonacci numbers F_n (A000045) and the companion sequence of Lucas numbers L_n (A000032) are defined for $n \geq 0$ as $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$ with initial conditions $F_0 = 0, F_1 = 1$, $L_0 = 2$ and $L_1 = 1$, respectively. The Binet formulas are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n,$$

where α is the golden ratio, i.e., $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = -1/\alpha = \frac{1-\sqrt{5}}{2}$. For more details consult Koshy's book [13].

Very recently, some series evaluations involving the Riemann zeta function at positive integer arguments and Fibonacci (Lucas) numbers appeared in the mathematical literature

[1, 8, 9, 11, 12]. For the eta function, the following two identities were stated in a problem proposal [10]:

$$\sum_{n=1}^{\infty}\eta(2n)\frac{F_{2n}}{5^n}=\frac{\pi}{10\cos(\frac{\pi}{2\sqrt{5}})}\qquad\text{and}\qquad\sum_{n=1}^{\infty}\eta(2n)\frac{L_{2n}}{5^n}=\frac{\pi}{2\cos(\frac{\pi}{2\sqrt{5}})}-1. \tag{12}$$

In this section we prove some additional results of this nature. We will need the following lemma.

Lemma 4. *We have*

$$\sum_{n=1}^{\infty}\eta(2n)x^{2n}=\frac{1}{2}\Big(\frac{\pi x}{\sin(\pi x)}-1\Big),\quad |x|<1, \tag{13}$$

$$\sum_{n=1}^{\infty}\frac{\eta(2n)}{2n}x^{2n}=\frac{1}{2}\ln\Big(\frac{\pi x}{\sin(\pi x)}\Big)-\ln\Big(\frac{\frac{\pi x}{2}}{\sin(\frac{\pi x}{2})}\Big),\quad |x|<1, \tag{14}$$

and

$$\sum_{n=1}^{\infty}\eta(n+1)x^n=-\psi(1-x)+\psi(1-x/2),\quad |x|<1. \tag{15}$$

Proof. The first identity follows from the known series [18, p. 161]

$$\sum_{n=1}^{\infty}\zeta(2n)x^{2n}=\frac{1}{2}\Big(1-\pi x\cot(\pi x)\Big),\quad |x|<1,$$

whereas the second is deduced from [18, p. 160]

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2n}x^{2n}=\frac{1}{2}\ln(\Gamma(1-x)\Gamma(1+x)),\quad |x|<1.$$

Finally, the series [18, p. 160]

$$\sum_{n=2}^{\infty}\zeta(n)x^{n-1}=-\psi(1-x)-\gamma,\quad |x|<1,$$

implies the third eta series. □

We begin with a generalization of (12).

Theorem 5. *For each $k\geq 0$ we have the evaluations*

$$\sum_{n=1}^{\infty}\eta(2n)\frac{F_{2n+k}}{5^n}=\frac{\pi}{10}\frac{L_{k+1}}{\cos(\frac{\pi}{2\sqrt{5}})}-\frac{1}{2}F_k \tag{16}$$

and

$$\sum_{n=1}^{\infty}\eta(2n)\frac{L_{2n+k}}{5^n}=\frac{\pi}{2}\frac{F_{k+1}}{\cos(\frac{\pi}{2\sqrt{5}})}-\frac{1}{2}L_k. \tag{17}$$

Proof. Use (13) with $x = \alpha/\sqrt{5}$ and $x = \beta/\sqrt{5}$, respectively, to get

$$\sum_{n=1}^{\infty} \eta(2n)\frac{\alpha^{2n+k}}{5^n} = \frac{\pi}{2\sqrt{5}}\frac{\alpha^{k+1}}{\sin(\pi\alpha/\sqrt{5})} - \frac{\alpha^k}{2},$$

and

$$\sum_{n=1}^{\infty} \eta(2n)\frac{\beta^{2n+k}}{5^n} = \frac{\pi}{2\sqrt{5}}\frac{\beta^{k+1}}{\sin(\pi\beta/\sqrt{5})} - \frac{\beta^k}{2}.$$

Now,

$$\sin\left(\frac{\pi\alpha}{\sqrt{5}}\right) = \sin\left(\frac{\pi}{2\sqrt{5}} + \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2\sqrt{5}}\right)$$

and

$$\sin\left(\frac{\pi\beta}{\sqrt{5}}\right) = \sin\left(\frac{\pi}{2\sqrt{5}} - \frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2\sqrt{5}}\right).$$

The proof is finished after combining these results according to the Binet formulas. □

Theorem 6. *For each $k \geq 0$ we have*

$$\sum_{n=1}^{\infty} \eta(2n)\frac{F_{2n+k}}{20^n} = \frac{\pi\sqrt{2}}{20}L_{k+1}\frac{\cos(\frac{\pi}{4\sqrt{5}})}{\cos(\frac{\pi}{2\sqrt{5}})} - \frac{\pi\sqrt{2}}{4\sqrt{5}}F_{k+1}\frac{\sin(\frac{\pi}{4\sqrt{5}})}{\cos(\frac{\pi}{2\sqrt{5}})} - \frac{1}{2}F_k \tag{18}$$

and

$$\sum_{n=1}^{\infty} \eta(2n)\frac{L_{2n+k}}{20^n} = \frac{\pi\sqrt{2}}{4}F_{k+1}\frac{\cos(\frac{\pi}{4\sqrt{5}})}{\cos(\frac{\pi}{2\sqrt{5}})} - \frac{\pi\sqrt{2}}{4\sqrt{5}}L_{k+1}\frac{\sin(\frac{\pi}{4\sqrt{5}})}{\cos(\frac{\pi}{2\sqrt{5}})} - \frac{1}{2}L_k. \tag{19}$$

Proof. Use (13) with $x = \alpha/(2\sqrt{5})$ and $x = \beta/(2\sqrt{5})$, respectively, combined with

$$\sin\left(\frac{\pi\alpha}{2\sqrt{5}}\right) = \sin\left(\frac{\pi}{4\sqrt{5}} + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(\sin\left(\frac{\pi}{4\sqrt{5}}\right) + \cos\left(\frac{\pi}{4\sqrt{5}}\right)\right)$$

and

$$\sin\left(\frac{\pi\beta}{2\sqrt{5}}\right) = \sin\left(\frac{\pi}{4\sqrt{5}} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(\sin\left(\frac{\pi}{4\sqrt{5}}\right) - \cos\left(\frac{\pi}{4\sqrt{5}}\right)\right).$$

Simplify according to the Binet forms and keep in mind the trigonometric identity $\cos^2(x) - \sin^2(x) = \cos(2x)$. □

Example 7.

$$\sum_{n=1}^{\infty} \eta(2n)\frac{F_{2n}}{20^n} = \frac{\pi\sqrt{2}}{20}\frac{\cos(\frac{\pi}{4\sqrt{5}}) - \sqrt{5}\sin(\frac{\pi}{4\sqrt{5}})}{\cos(\frac{\pi}{2\sqrt{5}})} \tag{20}$$

and

$$\sum_{n=1}^{\infty} \eta(2n)\frac{L_{2n}}{20^n} = \frac{\pi\sqrt{2}}{4}\frac{\cos(\frac{\pi}{4\sqrt{5}}) - \frac{1}{\sqrt{5}}\sin(\frac{\pi}{4\sqrt{5}})}{\cos(\frac{\pi}{2\sqrt{5}})} - 1. \tag{21}$$

It is worth mentioning that Theorem 5 can be also generalized into another direction.

Theorem 8. *Let k and j be integers with $k \geq 0$ and j odd. Then*

$$\sum_{n=1}^{\infty} \eta(2n) \frac{F_{2jn+k}}{5^n F_j^{2n}} = \frac{\pi}{10F_j} \frac{L_{j+k}}{\cos(\frac{\pi}{2}\frac{L_j}{\sqrt{5}F_j})} - \frac{1}{2}F_k \tag{22}$$

and

$$\sum_{n=1}^{\infty} \eta(2n) \frac{L_{2jn+k}}{5^n F_j^{2n}} = \frac{\pi}{2F_j} \frac{F_{j+k}}{\cos(\frac{\pi}{2}\frac{L_j}{\sqrt{5}F_j})} - \frac{1}{2}L_k. \tag{23}$$

Proof. From

$$\frac{\alpha^j}{\sqrt{5}F_j} = 1 + \frac{\beta^j}{\alpha^j - \beta^j},$$

it is obvious that $\alpha^j/\sqrt{5}F_j < 1$, if j is odd and $\alpha^j/\sqrt{5}F_j > 1$, if j is even. Also, using $L_j = F_{j+1} + F_{j-1}$,

$$\frac{\alpha^j}{\sqrt{5}F_j} = \frac{\alpha F_j + F_{j-1}}{\sqrt{5}F_j} = \frac{L_j}{2\sqrt{5}F_j} + \frac{1}{2}$$

and

$$\frac{\beta^j}{\sqrt{5}F_j} = \frac{\beta F_j + F_{j-1}}{\sqrt{5}F_j} = \frac{L_j}{2\sqrt{5}F_j} - \frac{1}{2},$$

so that

$$\sin\left(\frac{\alpha^j}{\sqrt{5}F_j}\right) = \cos\left(\frac{L_j}{2\sqrt{5}F_j}\right) \qquad \text{and} \qquad \sin\left(\frac{\beta^j}{\sqrt{5}F_j}\right) = -\cos\left(\frac{L_j}{2\sqrt{5}F_j}\right).$$

The expression then follows from (13). □

Example 9.

$$\sum_{n=1}^{\infty} \eta(2n) \frac{F_{6n}}{20^n} = \frac{\pi}{5\cos(\frac{\pi}{\sqrt{5}})} \tag{24}$$

and

$$\sum_{n=1}^{\infty} \eta(2n) \frac{L_{6n}}{20^n} = \frac{\pi}{2\cos(\frac{\pi}{\sqrt{5}})} - 1. \tag{25}$$

Theorem 10. *For each $k \geq 0$ we have*

$$\sum_{n=1}^{\infty} \frac{\eta(2n)}{2n} \frac{F_{2n+k}}{5^n} = \frac{F_k}{2} \ln\left(\frac{2\sqrt{5}}{\pi}\right) + \frac{L_k}{2\sqrt{5}} \ln\left(\frac{1}{\alpha}\frac{\cos(\frac{\pi}{4\sqrt{5}}) + \sin(\frac{\pi}{4\sqrt{5}})}{\cos(\frac{\pi}{4\sqrt{5}}) - \sin(\frac{\pi}{4\sqrt{5}})}\right) \tag{26}$$

and

$$\sum_{n=1}^{\infty} \frac{\eta(2n)}{2n} \frac{L_{2n+k}}{5^n} = \frac{L_k}{2} \ln\left(\frac{2\sqrt{5}}{\pi}\right) + \frac{\sqrt{5}F_k}{2} \ln\left(\frac{1}{\alpha}\frac{\cos(\frac{\pi}{4\sqrt{5}}) + \sin(\frac{\pi}{4\sqrt{5}})}{\cos(\frac{\pi}{4\sqrt{5}}) - \sin(\frac{\pi}{4\sqrt{5}})}\right). \tag{27}$$

Proof. Inserting $x = \alpha/\sqrt{5}$ and $x = \beta/\sqrt{5}$, respectively, in (14) and working with the Binet forms we arrive after some steps of algebraic manipulations at

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\eta(2n)}{2n}\frac{F_{2n+k}}{5^n} &= -\frac{F_k}{2}\ln\Big(\frac{\pi}{2\sqrt{5}}\cos\Big(\frac{\pi}{2\sqrt{5}}\Big)\Big) - \frac{L_k}{2\sqrt{5}}\ln(\alpha) \\ &\quad+\frac{\alpha^k}{\sqrt{5}}\ln\Big(\cos\Big(\frac{\pi}{4\sqrt{5}}\Big)+\sin\Big(\frac{\pi}{4\sqrt{5}}\Big)\Big) - \frac{\beta^k}{\sqrt{5}}\ln\Big(\cos\Big(\frac{\pi}{4\sqrt{5}}\Big)-\sin\Big(\frac{\pi}{4\sqrt{5}}\Big)\Big).\end{aligned}$$

To simplify further use

$$\alpha^k = \frac{L_k+\sqrt{5}F_k}{2} \quad \text{and} \quad \beta^k = \frac{L_k-\sqrt{5}F_k}{2}.$$

The Lucas series evaluations are proved in the same manner and are omitted. □

Example 11.

$$\sum_{n=1}^{\infty}\frac{\eta(2n)}{2n}\frac{F_{2n}}{5^n} = \frac{1}{\sqrt{5}}\ln\Big(\frac{1}{\alpha}\frac{\cos(\frac{\pi}{4\sqrt{5}})+\sin(\frac{\pi}{4\sqrt{5}})}{\cos(\frac{\pi}{4\sqrt{5}})-\sin(\frac{\pi}{4\sqrt{5}})}\Big) \tag{28}$$

and

$$\sum_{n=1}^{\infty}\frac{\eta(2n)}{2n}\frac{L_{2n}}{5^n} = \ln\Big(\frac{2\sqrt{5}}{\pi}\Big). \tag{29}$$

Theorem 12. *For each $k \geq 0$ we have*

$$\sum_{n=1}^{\infty}\eta(n+1)\frac{F_{n+k}}{5^{n/2}} = F_k\Big(\frac{\alpha}{2}+1-\ln(2)\Big) - L_k\Big(\frac{\alpha}{2}-\frac{\ln(2)}{\sqrt{5}}\Big) - \frac{L_k}{\sqrt{5}}\Big(\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\alpha}{2\sqrt{5}}\Big)\Big) \tag{30}$$

and

$$\sum_{n=1}^{\infty}\eta(n+1)\frac{L_{n+k}}{5^{n/2}} = L_k\Big(\frac{\alpha}{2}+1-\ln(2)\Big) - 5F_k\Big(\frac{\alpha}{2}-\frac{\ln(2)}{\sqrt{5}}\Big) - \sqrt{5}F_k\Big(\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\alpha}{2\sqrt{5}}\Big)\Big). \tag{31}$$

Proof. Use (15) with $x = \alpha/\sqrt{5}$ and $x = \beta/\sqrt{5}$, respectively, to get

$$\begin{aligned}\sum_{n=1}^{\infty}\eta(n+1)\frac{F_{n+k}}{5^{n/2}} &= -\frac{L_k}{2\sqrt{5}}\Big(\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\beta}{\sqrt{5}}\Big)\Big) \\ &\quad-\frac{F_k}{2}\Big(\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)+\psi\Big(1-\frac{\beta}{\sqrt{5}}\Big)\Big) \\ &\quad+\frac{L_k}{2\sqrt{5}}\Big(\psi\Big(1-\frac{\alpha}{2\sqrt{5}}\Big)-\psi\Big(1-\frac{\beta}{2\sqrt{5}}\Big)\Big) \\ &\quad+\frac{F_k}{2}\Big(\psi\Big(1-\frac{\alpha}{2\sqrt{5}}\Big)+\psi\Big(1-\frac{\beta}{2\sqrt{5}}\Big)\Big).\end{aligned}$$

Now,

$$\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\beta}{\sqrt{5}}\Big)=-\sqrt{5}\alpha$$

and

$$\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)+\psi\Big(1-\frac{\beta}{\sqrt{5}}\Big)=2\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)+\sqrt{5}\alpha.$$

The terms involving $2\sqrt{5}$ in the denominator can be manipulated using Legendre's duplication formula:

$$2\psi(2x)=2\ln(2)+\psi(x)+\psi(x+1/2).$$

Set $x=1-\alpha/2\sqrt{5}$. Then $1-\beta/2\sqrt{5}=x+1/2$ and

$$\psi(x+1/2)+\psi(x)=2\psi(2x)-2\ln(2),$$

$$\psi(x+1/2)-\psi(x)=2\psi(2x)-2\psi(x)-2\ln(2)$$

and

$$\psi(2x)=\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)+\sqrt{5}\alpha.$$

Finally, we gather terms and rearrange and the first expression follows. The Lucas counterpart is similar. □

Example 13.

$$\sum_{n=1}^{\infty}\eta(n+1)\frac{F_n}{5^{n/2}}=-\alpha+\frac{2\ln(2)}{\sqrt{5}}-\frac{2}{\sqrt{5}}\Big(\psi\Big(1-\frac{\alpha}{\sqrt{5}}\Big)-\psi\Big(1-\frac{\alpha}{2\sqrt{5}}\Big)\Big) \tag{32}$$

and

$$\sum_{n=1}^{\infty}\eta(n+1)\frac{L_n}{5^{n/2}}=\alpha+2-2\ln(2). \tag{33}$$

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References

[1] K. Adegoke and S. Ghosh, Fibonacci-Zeta infinite series associated with the polygamma functions, preprint, 2021. Available at https://arxiv.org/abs/2103.09799v1.

[2] H. Alzer and M. K. Kwong, On the concavity of Dirichlet's eta function and related functional inequalities, *J. Number Theory* **151** (2015), 172–196.

[3] R. Ayoub, Euler and the zeta function, *Amer. Math. Monthly* **81** (1974), 1067–1086.

[4] K. N. Boyadzhiev, Series transformation formulas of Euler type, Hadamard product of functions, and harmonic number identities, *Indian J. Pure and Appl. Math.* **42** (2011), 371–387.

[5] K. N. Boyadzhiev, A special constant and series with zeta values and harmonic numbers, *Gazeta Matematica, Seria A* **115** (2018), 1–16.

[6] M. W. Coffey, Series of zeta values, the Stieltjes constants, and a sum $S_\gamma(n)$, preprint, 2007. Available at https://arxiv.org/abs/0706.0345.

[7] S. R. Finch, *Mathematical Constants*, Cambridge University Press, 2003.

[8] R. Frontczak, Problem B-1267, *Fibonacci Quart.* **58** (2020), 179.

[9] R. Frontczak, Problem H-859, *Fibonacci Quart.* **58** (2020), 281.

[10] R. Frontczak, Problem H-872, *Fibonacci Quart.* **59** (2021), 90.

[11] R. Frontczak, Infinite series involving Fibonacci numbers and the Riemann zeta function, *Notes Number Theory Discrete Math.* **26** (2020), 159–166.

[12] R. Frontczak and T. Goy, General infinite series evaluations involving Fibonacci numbers and the Riemann zeta function, *Mat. Stud.* **55** (2021), 115–123.

[13] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, 2001.

[14] M. S. Milgram, Integral and series representations of Riemann's zeta function and Dirichlet's eta function and a medley of related results, *J. Math.* **2013**, Article ID 181724.

[15] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, 2021. Available at https://oeis.org.

[16] J. Sondow, Zeros of the alternating zeta function on the line $R(s) = 1$, *Amer. Math. Monthly* **110** (2003), 435–437.

[17] J. Sondow, Double integrals for Euler's constant and $\ln 4/\pi$ and an analog of Hadjicostas's formula, *Amer. Math. Monthly* **112** (2005), 61–65.

[18] H. M. Srivastava and J. Choi, Series associated with the Zeta and related functions, Springer Science+Business Media, 2001.

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