Journal of Integer Sequences, Vol. 24 (2021),
Article 21.3.5

Minimum Coprime Labelings of Generalized Petersen and Prism Graphs

John Asplund
Metron, Inc.
Reston, VA 20190
USA
asplund@metsci.com

N. Bradley Fox
Department of Mathematics and Statistics
Austin Peay State University
Clarksville, TN 37044
USA
foxb@apsu.edu

Abstract

A coprime labeling of a graph of order n is an assignment of distinct positive integer labels in which adjacent vertices have relatively prime labels. Restricting labels to only the set 1 to n results in a prime labeling. In this paper, we consider families of graphs in which a prime labeling cannot exist with the goal being to minimize the largest value of the labeling set, resulting in a minimum coprime labeling. In particular, prism graphs, generalized Petersen graphs with $k = 2$, and stacked prism graphs are investigated for minimum coprime labelings.

1 Introduction

Let G be a simple graph of order n with vertex set V. We denote an edge between two adjacent vertices v and w as vw. A *coprime labeling* of G is a labeling of V using distinct labels from the set $\{1, \ldots, m\}$ for some integer $m \geq n$ in which adjacent vertices are labeled by relatively prime integers. If the integers $1, \ldots, n$ are used as the labeling set, the labeling is called a *prime labeling*, and G is a *prime graph* or is simply referred to as *prime*. For graphs in which no prime labeling exists, our goal is to minimize the value m, the largest label in the coprime labeling. This smallest possible value m for a coprime labeling of G, denoted by $\mathfrak{pr}(G)$, is the *minimum coprime number* of G, and a coprime labeling with $\mathfrak{pr}(G)$ as the largest label is a *minimum coprime labeling* of G. A prime graph would have a minimum coprime number of $\mathfrak{pr}(G) = n$.

Prime labelings of graphs were developed by Roger Entringer and first introduced by Tout, Dabboucy, and Howalla [14]. Numerous classes of graphs over the past forty years have been shown to be prime, as well as many classes for which a prime labeling has been shown to not exist. A summary of these results can be seen in Gallian's dynamic survey of graph labelings [8]. Most of our upcoming results center around the concept of minimum coprime labelings, which were first studied by Berliner et al. [3] with their investigation of complete bipartite graphs of the form $K_{n,n}$. Asplund and Fox [1] continued this line of research by determining the minimum coprime number for classes of graphs such as complete graphs, wheels, the union of two odd cycles, the union of a complete graph with a path or a star, powers of paths and cycles, and the join of paths and cycles. Recently, Lee [11] made further progress on the minimum coprime number of the join of paths and complete bipartite graphs, in addition to investigating minimum coprime numbers of random subgraphs.

The focus of this paper is to determine the minimum coprime number of prism graphs, which are equivalent to the Cartesian product of a cycle of length n and a path with 2 vertices, denoted as $C_n \square P_2$. Additionally, a prism graph is equivalent to the generalized Petersen graphs when $k = 1$. In the next section, we include preliminary material regarding the classes of graphs we will investigate and previous research on prime labelings of these graphs. In Section 3, we construct minimum coprime labelings of prism graphs $\mathrm{GP}(n, 1)$ for several specific cases of odd n as well as present a conjecture for all sizes of odd prisms. Section 4 includes results on the minimum coprime number of the generalized Petersen graph $\mathrm{GP}(n, 2)$, a graph which is not prime for any value n. Section 5 consists of results on minimum coprime number of stacked prism graphs, and finally we investigate a variation of a generalized Petersen graph in Section 6.

2 Preliminary material

An important feature of a graph G that aides in determining whether a prime labeling may exist or if a minimum coprime labeling should instead be investigated is its *independence number*, denoted by $\alpha(G)$. Since even number labels must be assigned to independent vertices, the following criteria, first introduced by Fu and Huang [7], eliminates the possibility of a prime labeling on many classes of graphs.

Lemma 1. [7] *If G is prime, then the independence number of G must satisfy* $\alpha(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.

The generalized Petersen graph, denoted $\mathrm{GP}(n,k)$ where $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, consists of $2n$ vertices $v_1, \ldots, v_n, u_1, \ldots, u_n$. It has $3n$ edges described by $v_i v_{i+1}$, $u_i u_{i+k}$, and $v_i u_i$ where indices are calculated modulo n. In the particular case of $k = 1$, the two sets of vertices form n-gons that are connected to form a prism graph, which will be our first graph that we investigate.

When n is odd, $\mathrm{GP}(n,1)$ consists of two odd cycles connected by a perfect matching. Only $(n-1)/2$ vertices on each cycle can be independent, hence $\alpha(\mathrm{GP}(n,1)) = n - 1$ for odd n. Then by Lemma 1, $\mathrm{GP}(n,1)$ is not prime in this case, a property which extends to any value of k when n is odd. In fact, $\mathrm{GP}(n,k)$ was proven to not be prime by Prajapati and Gajjar [12] for any odd value of n as well as when n and k are both even. Independence numbers for generalized Petersen graphs for certain cases have been determined [2, 4, 6] that help provide bounds for the minimum coprime numbers of $\mathrm{GP}(n,k)$ in the non-prime cases.

The remaining case of $\mathrm{GP}(n,k)$ with n even and k odd is conjectured to be prime for all such n and k. When $k = 1$, the prism graph $\mathrm{GP}(n,1)$ has been proven to be prime in many specific cases by Haque et al. [9] such as when $2n + a$ or $n + a$ are prime for several small values of a. Additional cases of $\mathrm{GP}(n,1)$ were proven to be prime [12]. Dean [5] proved the conjecture that all ladders are prime. Since ladders are simply prism graphs with two edges removed, one might expect his prime labeling to carry over to $\mathrm{GP}(n,1)$. However, when applying this labeling to $\mathrm{GP}(n,1)$, these two additional edges do not maintain the relatively prime condition for all n.

While some results have been found on $\mathrm{GP}(n,3)$, see [10], most work involving prime labelings of the generalized Petersen graph has been focused on the prism graph. Schluchter et al. [13] made a number theoretic conjecture to bolster the conjecture that $\mathrm{GP}(n,1)$ is prime for all even n. Conjecture 2.1 [13] stated that for any even integer n, there exists an $s \in [1, n-1]$ such that $n + s$ and $2n + s$ are prime. By verifying this conjecture for all even n up to 2.468×10^9, they demonstrated $\mathrm{GP}(n,1)$ is prime with even n up to that value.

We conclude this section with the following observations, which will be used without citation in many of the theorems throughout this paper.

Observation 2. For positive integers a, b, and k, the following hold:

- $\gcd(a,b) = \gcd(ka,b)$.
- $\gcd(a,b) = \gcd(a-b,b)$.
- $\gcd(a,b) = \gcd(a+b,b)$.
- If $a + b$ is prime, then $\gcd(a,b) = 1$.
- If $a - b$ is a prime p and both a and b are not multiples of p, then $\gcd(a,b) = 1$.

3 Prism graphs

In this section, we provide several specific results for the minimum coprime number of $\mathrm{GP}(n,1)$ and conjecture that $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$ for all odd n. Many of the theorems follow a similar proof strategy, but a general construction extended from our techniques seems unlikely without the resolution of longstanding number theory conjectures. See Figure 1 for an example of our first result showing a minimum coprime labeling for the prism $\mathrm{GP}(11,1)$.

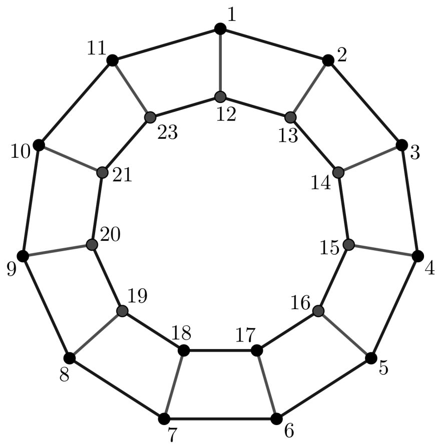

Figure 1: Minimum coprime labeling for $\mathrm{GP}(11,1)$

Theorem 3. *If n is prime, then $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.*

Proof. We construct a labeling ℓ in the following manner. We label $v_1,\ldots,v_n$ as $1,\ldots,n$ and $u_1,\ldots,u_n$ with $n+1,\ldots,2n-1,2n+1$ respectively. All adjacent pairs in $\{v_1,\ldots,v_n\}$ and in $\{u_1,\ldots,u_n\}$ have consecutive labels except for v_1v_n, $u_{n-1}u_n$, and u_1u_n. The first pair includes 1 as one of the labels, and the second pair is labeled by consecutive odd labels. Lastly, $\ell(u_1)$ and $\ell(u_n)$ are relatively prime since $\gcd(n+1,2n+1) = \gcd(2n+2,2n+1) = 1$.

It remains to show that the labels $\ell(u_i)$ and $\ell(v_i)$ are relatively prime for each i. For $i \leq n-1$, the difference between the labels on v_i and u_i is n. Since n is assumed to be prime, these pairs are relatively prime by Observation 2. Finally when $i = n$, we have $\gcd(2n+1,n) = \gcd(n+1,n) = 1$. Therefore, this is a coprime labeling, hence $\mathfrak{pr}(\mathrm{GP}(n,1)) \leq 2n+1$. Since the independence number of $\mathrm{GP}(n,1)$ is $n-1$ when n is odd, a prime labeling is not possible, making $\mathfrak{pr}(\mathrm{GP}(n,1)) > 2n$. Thus when n is prime, we have $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$. □

Theorem 4. *If $n+2$ is prime, then $\mathfrak{pr}(GP(n,1)) = 2n+1$.*

Proof. We construct a labeling ℓ in the following manner. First we label $v_1,\ldots,v_n$ with the numbers $1,\ldots,n$, respectively, and then the vertices $u_1,\ldots,u_n$ with the labels $n+3,\ldots,2n+1,n+2$, respectively.

Edges between vertices in $\{v_1, \ldots, v_n\}$ have vertices with consecutive labels or contain the label 1, and so $\gcd(\ell(v_i), \ell(v_{i+1})) = \gcd(\ell(v_1), \ell(v_n)) = 1$. Edges between vertices in $\{u_1, \ldots, u_n\}$ have vertices with consecutive labels or with labels $n+2$ and $2n+1$ in which $n+2$ is prime. Hence the labels between pairs of adjacent vertices in $\{u_1, \ldots, u_n\}$ are relatively prime. For $i = 1, \ldots, n-1$, since the difference of the labels on vertices u_i and v_i is $n+2$, which is prime, $\gcd(\ell(u_i), \ell(v_i)) = 1$. Lastly, u_n and v_n are labeled by consecutive odd integers. Therefore, we have a coprime labeling that is minimal since $\mathrm{GP}(n,1)$ is not prime for odd n. Thus, $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$ assuming $n+2$ is prime. □

Theorem 5. *If* $2n+1$ *is prime, then* $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.

Proof. We construct a labeling ℓ in the following manner. First we label $v_1, \ldots, v_n$ as $1, \ldots, n$ as in last theorem, but we label $u_1, \ldots, u_n$ in reverse order as $2n+1, 2n-1, 2n-2, \ldots, n+1$.

Edges connecting vertices in $\{v_1, \ldots, v_n\}$ have consecutive labels or contain the label 1, and so $\gcd(\ell(v_i), \ell(v_{i+1})) = \gcd(\ell(v_1), \ell(v_n)) = 1$. Edges between vertices in $\{u_1, \ldots, u_n\}$ have vertices with consecutive labels, with consecutive odd integer labels, or with the pair of labels $n+1$ and $2n+1$, and so the labels on these adjacent vertices are relatively prime. For each $i = 2, \ldots, n$, since $\ell(u_i) + \ell(v_i) = 2n+1$, we know that $\gcd(\ell(u_i), \ell(v_i)) = 1$ by Observation 2 because $2n+1$ is prime. Finally, the edge u_1v_1 includes the label 1 on u_1. Thus, this is a coprime labeling that shows $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$ when $2n+1$ is prime. □

Theorem 6. *If* $2n-1$ *is prime, then* $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.

Proof. Notice that when $n \equiv 2 \pmod 3$, then $2n-1$ is divisible by 3, so we may assume $n \not\equiv 2 \pmod 3$. If $n \equiv 0 \pmod 3$, we construct a labeling ℓ by first labeling $v_1, \ldots, v_n$ as $1, 2, \ldots, n-1, 2n-1$ and then label $u_1, \ldots, u_n$ as $2n-2, 2n-1, \ldots, n, 2n+1$. One can see the pairs u_iu_{i+1} and v_iv_{i+1} have relatively prime labels, where $\gcd(\ell(u_1), \ell(u_n)) = \gcd(2n+1, 2n-2) = 1$ since n is a multiple of 3 in this case. The pairs u_iv_i have relatively prime labels for $i = 1, \ldots, n-1$ since $\ell(u_i) + \ell(v_i) = 2n+1$, which is assumed to be prime. Whereas for $i = n$, $\gcd(\ell(u_n), \ell(v_n)) = 1$ since these are consecutive odd integers.

If $n \equiv 1 \pmod 3$, then use the same labeling as the previous case except $\ell(u_1) = 2n-2$ instead of $2n$. This is a coprime labeling for similar reasoning as our first case, except now the pair of labels $2n$ and $2n+1$ on u_1 and u_n are consecutive, and the labels $2n$ and $2n-3$ on u_1 and u_2 are relatively prime since $n \equiv 1 \pmod 3$. In each case, the labeling is a minimum coprime labeling, proving $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$ assuming $2n-1$ is prime. □

For further results regarding minimum coprime labelings of prism graphs in other specific cases, see Appendix A. Using Theorems 3–6, along with Theorems 17–23 in the Appendix, an explicit minimum coprime labeling is given for $\mathrm{GP}(n,1)$ for all odd n up to 1641. The following is a more general construction assuming a particular pair of prime numbers exists.

Theorem 7. *Let* $n \geq 3$ *be odd. If there exists an* $s \in [3, n-1]$ *such that* $n+s+1$ *and* $2n+s+2$ *are prime, then* $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.

Proof. We use the labeling defined in Table 1 where the top row represents the vertices $v_1, \ldots, v_n$ and the bottom row represents the vertices $u_1, \ldots, u_n$. The vertex pairs on edges of the form u_1v_1, u_iu_{i+1} and v_iv_{i+1} either contain the label 1, are consecutive integers, are consecutive odd integers, or are the relatively prime pair $n+1$ and $2n+1$. The adjacent pairs u_iv_i for $i = 2, \ldots, s$ have labels that add to $n + s + 1$, and the pairs u_iv_i for $i = s + 1, \ldots, n$ are labeled by integers whose sum is $2n + s + 2$. Since both of these sums are assumed to be prime, the labels on those pairs are relatively prime as well. Thus, since $\mathrm{GP}(n, 1)$ is not prime when n is odd, we have constructed a minimum coprime labeling proving that $\mathfrak{pr}(\mathrm{GP}(n, 1)) = 2n + 1$ if such a value s exists. □

1	2	$\cdots$	$s-1$	s	$s+1$	$s+2$	$\cdots$	$n-1$	n
$n+s+1$	$n+s-1$	$\cdots$	$n+2$	$n+1$	$2n+1$	$2n$	$\cdots$	$n+s+3$	$n+s+2$

Table 1: Labeling for Theorem 7

Recall that Conjecture 2.1 [13] states that for all even integers n, there is an $s < n$ such that $n + s$ and $2n + s$ are both prime. If this is true for all even integers, then the previous theorem would prove the subsequent conjecture for all odd n since applying Conjecture 2.1 to the even integer $n + 1$ would result in $n + s + 1$ and $2n + s + 2$ being prime. Results by Schluchter et al. [13] combine with Theorem 7 to confirm the following conjecture for odd $n < 2.468 \times 10^9$.

Conjecture 8. For all odd $n \geq 3$, $\mathfrak{pr}(\mathrm{GP}(n, 1)) = 2n + 1$.

4 Generalized Petersen graphs with $k = 2$

We next consider the generalized Petersen graph in the case of $k = 2$. The vertices of $\mathrm{GP}(n, 2)$ with $n \geq 5$ are still referred to as $v_1, \ldots, v_n$,$u_1, \ldots, u_n$ with the edges of the forms v_iv_{i+1}, u_iu_{i+2}, and v_iu_i in which indices calculated modulo n. An example of this type of graph with $n = 5$ is the well-known Petersen graph, which is shown with a minimum coprime labeling in Figure 2.

The independence number for generalized Petersen graphs when $k = 2$ is given by the formula $\lfloor \frac{4n}{5} \rfloor$ through the study of minimum vertex covers of $\mathrm{GP}(n, 2)$ by Behsaz et al [2]. This results in the generalized Petersen graph with $k = 2$ not being prime for any value of n. The denominator of this formula for the independence number provides a natural direction by which to create an independent set for this graph by dividing $\mathrm{GP}(n, 2)$ into blocks that include 5 of the v_i and 5 of the u_i vertices. We utilize this technique in the following proof but limit ourselves for now to the case when n is a multiple of 5.

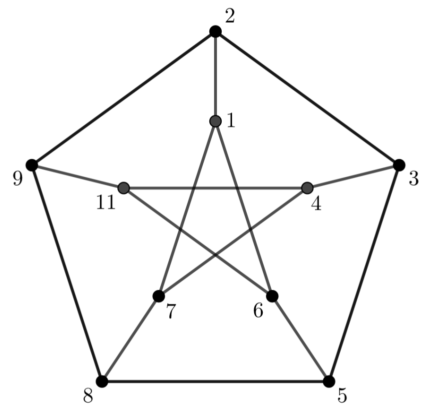

Figure 2: A minimum coprime labeling of the Petersen graph GP$(5,2)$

Lemma 9. *Let m be a positive integer. Then* $\mathfrak{pr}(\mathrm{GP}(5m,2)) = 12m-1$.

Proof. We aim to construct a coprime labeling ℓ and later will show that it is minimal. We begin by assigning $v_1,\ldots,v_5$ the labels $2,3,5,8,9$ and assigning $u_1,\ldots,u_5$ the labels $1,4,6,7,11$, respectively. One can verify these ten labels form a coprime labeling when $m=1$. For $m>1$ we then define the following labeling for the block of ten vertices $v_{5k+1},\ldots,v_{5k+5},u_{5k+1},\ldots,u_{5k+5}$ for each $1\le k<m$:

$$\begin{aligned}
&\ell(v_{5k+1}) = 12k+2, && \ell(v_{5k+4}) = 12k+8, && \ell(u_{5k+1}) = 12k+1, && \ell(u_{5k+4}) = 12k+7,\\
&\ell(v_{5k+2}) = 12k+3, && \ell(v_{5k+5}) = 12k+9, && \ell(u_{5k+2}) = 12k+4, && \ell(u_{5k+5}) = 12k+11.\\
&\ell(v_{5k+3}) = 12k+5, && && \ell(u_{5k+3}) = 12k+10, &&
\end{aligned} \tag{1}$$

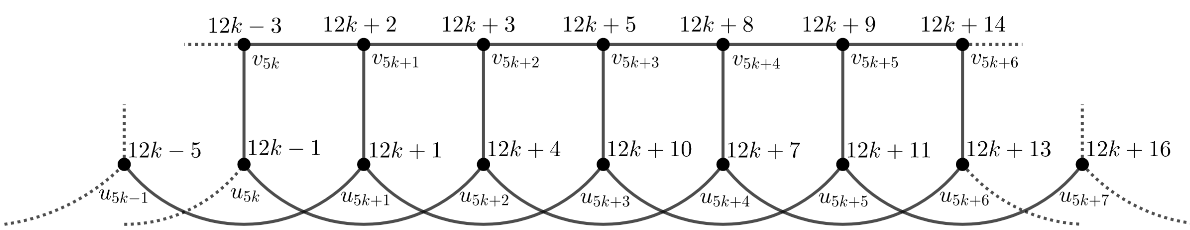

Figure 3: Visual representation of the labeling described in Equation (1)

As currently defined the labeling ℓ, which is displayed in Figure 3, is not enough to guarantee each pair of adjacent vertices has relatively prime labels, particularly for pairs of labels that have a difference of 5. We alter the labeling ℓ by addressing cases for specific k

values based on the divisibility of $12k-1$, $12k-3$, and $12k+5$. Before altering ℓ, first note that no adjacent vertices are both labeled by even integers. One can also observe that no labels that are multiples of 3 are assigned to adjacent vertices, including the adjacent pairs whose labels differ by 9. Additionally, the final vertices in the last block u_{n-1}, u_{5m}, and v_{5m} are adjacent to the vertices u_1, u_2, and v_1, respectively. Since $\ell(u_1) = 1$, it is relatively prime to the label of u_{n-1}. Likewise, $\ell(u_2) = 4$ and $\ell(v_1) = 2$, while $\ell(u_{5m}) = 12m - 1$ and $\ell(v_{5m}) = 12m - 3$ are both odd, making those adjacent pairs of labels also relatively prime.

As we define ℓ for the upcoming cases, the labels on vertices u_{5k+4}, u_{5k+5}, and v_{5k+5} are not changed except in Cases 4b, 4c, and 4d, and this occurs only as the subsequent block is labeled. These three vertices are the only ones within the block $v_{5k+1}, \ldots, v_{5k+5}, u_{5k+1}, \ldots, u_{5k+5}$ that are adjacent to vertices in the subsequent block, hence leaving these three vertices unchanged is essential to guaranteeing that adjacent labels on vertices in different blocks are relatively prime. Let $U_k = \{u_{5k+1}, \ldots, u_{5k+5}\}$ and $V_k = \{v_{5k+1}, \ldots, v_{5k+5}\}$.

Case 1: Suppose that $5 \nmid 12k-1, 5 \nmid 12k-3$, and $5 \nmid 12k+5$.

Label the vertices $U_k \cup V_k$ as in Equation 1. As previously observed, pairs of adjacent vertices in $U_k \cup V_k$ or adjacent pairs between the vertices in $U_k \cup V_k$ and $\{v_{5k}, u_{5k}, u_{5k-1}\}$ do not have labels that share a common factor of 2 or 3. The adjacent vertex pairs $u_{5k}u_{5k+2}$, $v_{5k}v_{5k+1}$, and $v_{5k+3}u_{5k+3}$ have labels that differ by 5. Our assumptions for this case ensure that these pairs are not both divisible by 5, resulting in the relatively prime condition being satisfied.

Case 2: Suppose that $5 \mid 12k+5$.

Use the labeling ℓ from Equation (1) with the following redefined label:

$$\ell(u_{5k+3}) = 12k + 6. \tag{2}$$

Since we assumed $5 \mid 12k+5$, it follows that $5 \nmid 12k-1$ and $5 \nmid 12k-3$, and thus after applying reasoning from Case 1, we need only check that $\ell(u_{5k+3})$ is relatively prime with the labels of all neighbors of u_{5k+3}. Since u_{5k+3} is adjacent to u_{5k+1}, u_{5k+5} and v_{5k+3}, we need that $\gcd(12k+1, 12k+6) = 1$, $\gcd(12k+6, 12k+11) = 1$, and $\gcd(12k+5, 12k+6) = 1$. The third equality is trivial, and the first two equalities follow immediately from the Case 2 assumption.

Case 3a: Suppose that $5 \mid 12k-1$ and $7 \nmid 12k-3$.

Use the labeling ℓ in Equation (1) with the following two redefined labels:

$$\ell(u_{5k+2}) = 12k + 2, \quad \ell(v_{5k+1}) = 12k + 4. \tag{3}$$

Notice that since 5 divides $12k-1$, we have $\gcd(12k+5, 12k+10) = 1$. As before, we need only to check that $\ell(u_{5k+2})$ and $\ell(v_{5k+1})$ are relatively prime with the labels of any adjacent vertices. Clearly, $\gcd(12k+2, 12k+3) = \gcd(12k+3, 12k+4) = 1$. Since both $12k+2$ and $12k+4$ are not divisible by 3, we know that $\gcd(12k-1, 12k+2) = \gcd(12k+1, 12k+4) = 1$.

Since 7 is assumed to not divide $12k-3$, $\gcd(12k-3, 12k+4) = 1$. Finally, our assumption of $5 \mid 12k-1$ implies $5 \nmid 12k+2$, hence $\gcd(12k+2, 12k+7) = 1$.

Case 3b: Suppose that $5 \mid 12k-1$ and $7 \mid 12k-3$.

Use the labeling ℓ in Equation (1) with the following three redefined labels:

$$\ell(u_{5k+2}) = 12k+6, \qquad \ell(v_{5k+2}) = 12k+5, \qquad \ell(v_{5k+3}) = 12k+3. \tag{4}$$

We need only check that these new labels are relatively prime with labels of any neighboring vertices. It is clear that

$$\gcd(12k+6, 12k+7) = \gcd(12k+5, 12k+6) = \gcd(12k+3, 12k+5) = 1.$$

Since $12k+2$ is not divisible by 3, $\gcd(12k+2, 12k+5) = 1$. By our assumption that $12k-1$ is divisible by 5, $\gcd(12k+3, 12k+8) = 1$. Since $12k-3$ is assumed to be divisible by 7, $\gcd(12k-1, 12k+6) = \gcd(12k+3, 12k+10) = 1$.

Case 4a: Suppose that $5 \mid 12k-3$ and $7 \nmid 12k-3$.

Use the labeling ℓ in Equation (1) with the following four redefined labels:

$$\ell(v_{5k+1}) = 12k+4, \qquad \ell(v_{5k+4}) = 12k+10, \qquad \ell(u_{5k+2}) = 12k+8, \qquad \ell(u_{5k+3}) = 12k+2. \tag{5}$$

Clearly we have

$$\gcd(12k+3, 12k+4) = \gcd(12k+1, 12k+2) = \gcd(12k+7, 12k+8) = \gcd(12k+9, 12k+10) = 1.$$

Additionally, since none of the four reassigned labels are divisible by 3, it is clear that

$$\begin{aligned}\gcd(12k+1, 12k+4) = \gcd(12k+2, 12k+5) &= \gcd(12k+7, 12k+10)\\ = \gcd(12k-1, 12k+8) &= \gcd(12k+2, 12k+11) = 1.\end{aligned}$$

Our assumptions in this case include that $7 \nmid 12k-3$ and also imply that $5 \nmid 12k+8$ or $12k+10$. Thus we have

$$\gcd(12k-3, 12k+4) = \gcd(12k+3, 12k+8) = \gcd(12k+5, 12k+10) = 1.$$

Case 4b: Suppose that $5 \mid 12k-3$, $7 \mid 12k-3$, and $11 \nmid 12k-3$.

Use the labeling ℓ in Equation (1) with the following four redefined labels:

$$\ell(v_{5k}) = 12k-1, \qquad \ell(u_{5k}) = 12k-3, \qquad \ell(u_{5k+2}) = 12k+8, \qquad \ell(v_{5k+4}) = 12k+4. \tag{6}$$

Since the two vertices indexed by $5k$ are in the previous block of ten vertices, it is important to consider whether that block falls within a case in which any labels were swapped from the initial labeling of that block. Since we assume $5 \mid 12k-3$, then $5 \mid 12k-13$, which is equal to $12(k-1)-1$. Whereas $7 \mid 12k-3$ implies $7 \nmid 12k-15$, which is equal to $12(k-1)-3$. Therefore, vertices in $U_{k-1} \cup V_{k-1}$ would be labeled according to Case 3a. Neither vertex whose label was swapped within Case 3a is adjacent to v_{5k} or u_{5k}, so the adjacent pairs of labels to consider from that block are $12k-3$ and $12k-2$, $12k-3$ and $12k-1$, and $12k-1$ and $12k-4$.

Overall, there are ten adjacent pairs of labels that need to be verified as relatively prime. It is clear that

$$\gcd(12k-3, 12k-2) = \gcd(12k-3, 12k-1) = \gcd(12k+7, 12k+8) = \gcd(12k+4, 12k+5) = 1.$$

Since the only reassigned label that is a multiple of 3 is $12k-3$, we have

$$\gcd(12k-4, 12k-1) = \gcd(12k-1, 12k+2) = \gcd(12k+4, 12k+7) = 1.$$

The assumption $5 \mid 12k-3$ implies 5 is not a factor of $12k+8$ or $12k+4$, hence

$$\gcd(12k+3, 12k+8) = \gcd(12k+4, 12k+9) = 1.$$

Finally, by our assumption that $11 \nmid 12k-3$, we know $\gcd(12k-3, 12k+8) = 1$.

Case 4c: Suppose that $5 \mid 12k-3$, $7 \mid 12k-3$, $11 \mid 12k-3$, and $13 \nmid 12k-3$.

Use the labeling ℓ in Equation (1) with the following five redefined labels:

$$\begin{array}{lll} \ell(v_{5k}) = 12k-1, & \ell(u_{5k}) = 12k-3, & \ell(u_{5k+2}) = 12k+10, \\ \ell(u_{5k+3}) = 12k+8, & \ell(v_{5k+4}) = 12k+4. & \end{array} \tag{7}$$

We can change the labels on v_{5k} and u_{5k} without causing any adjacent pairs of vertices to not be relatively prime by the same reasoning as given in Case 4a. There are ten additional pairs of labels that need to be shown to be relatively prime to complete this case. Clearly,

$$\begin{aligned} \gcd(12k-1, 12k+2) &= \gcd(12k+4, 12k+5) = \gcd(12k+7, 12k+10) \\ &= \gcd(12k+5, 12k+8) = \gcd(12k+8, 12k+11) \\ &= \gcd(12k+4, 12k+7) = 1. \end{aligned}$$

By our assumption that $5 \mid 12k-3$, we know 5 is not a factor of $12k+4$, resulting in $\gcd(12k+4, 12k+9) = 1$. Similarly, 7 is assumed to be a factor of $12k-3$, so $7 \nmid 12k+3$ and $7 \nmid 12k+8$; therefore, $\gcd(12k+3, 12k+10) = \gcd(12k+1, 12k+8) = 1$. Lastly, we assumed $13 \nmid 12k-3$, hence $\gcd(12k-3, 12k+10) = 1$, resulting in the relatively prime condition being satisfied.

Case 4d: Suppose that $5 \mid 12k-3$, $7 \mid 12k-3$, $11 \mid 12k-3$, and $13 \mid 12k-3$.

Use the labeling ℓ in Equation (1) with the following two redefined labels: u_{5k-2} as $12k+2$ and v_{5k+1} as $12k-2$. As in the last two cases, the previous block of vertices that contains u_{5k-2} falls within Case 3a, which involves swapping two labels on vertices that are not adjacent to u_{5k-2}. Its neighbors then are labeled by $12k-11$, $12k-7$, and $12k-1$, while the label $12k-2$ on v_{5k+1} is adjacent to $12k-3$, $12k+1$, and $12k+3$. Thus, by our assumptions in this case, our reassigned labels are relatively prime with the labels of adjacent vertices.

Note that it is only possible for 5 to divide at most one value of $12k-1$, $12k-3$, and $12k+5$ since none of the differences between pairs of these values are a multiple of 5. The four possibilities this leaves are that 5 divides none of these three values (Case 1), 5 only divides $12k+5$ (Case 2), 5 only divides $12k-1$ (Cases 3a and 3b depending on the divisibility of this value by 7), and 5 only divides $12k-3$ (Case 4a-4d depending on if the next three prime numbers divide this value). Hence our eight cases are sufficient to cover all possible situations.

Therefore, by our assumptions and case analysis, it is clear that all labels are relatively prime with their adjacent labels. Thus in each case the updated ℓ is a coprime labeling, making $\mathfrak{pr}(\mathrm{GP}(5m,2)) \leq 12m-1$. Since $\alpha(\mathrm{GP}(5m,2)) = \left\lfloor \frac{4(5m)}{5} \right\rfloor = 4m$, we need $6m$ odd numbers to label the vertices in $\mathrm{GP}(5m,2)$. Thus, $\mathfrak{pr}(\mathrm{GP}(5m,2)) \geq 12m-1$. Therefore, $\mathfrak{pr}(\mathrm{GP}(5m,2)) = 12m-1$.

□

Theorem 10. *The minimum coprime number for* $\mathrm{GP}(n,2)$ *for* $n \geq 5$ *is given by*

$$\mathfrak{pr}(\mathrm{GP}(n,2)) = \begin{cases} 12m-1, & \text{if } n = 5m; \\ 12m+3, & \text{if } n = 5m+1; \\ 12m+5, & \text{if } n = 5m+2; \\ 12m+7, & \text{if } n = 5m+3; \\ 12m+9, & \text{if } n = 5m+4. \end{cases}$$

Proof. When $n = 5m$, we constructed in Lemma 9 a minimum coprime labeling ℓ with $12m-1$ as the largest label. For the remaining four cases, we will build the labeling by using ℓ defined in Lemma 9 for the first $5m$ vertices in $v_1 \ldots v_n$ and the first $5m$ vertices in $u_1 \ldots u_n$. Note that the vertices v_{5m}, u_{5m}, and u_{5m-1} are not changed from the labeling defined in Equation (1). Hence, $\ell(v_{5m}) = 12m-3$, $\ell(u_{5m}) = 12m-1$, and $\ell(u_{5m-1}) = 12m-5$.

Case 1: Suppose that $n = 5m+1$. Label the remaining vertices as $\ell(u_{5m+1}) = 12m+3$ and $\ell(v_{5m+1}) = 12m+1$. By Lemma 9, we need only check that the following pairs of adjacent vertices have relatively prime labels: $u_{5m}u_1$, $v_{5m+1}v_1$, $u_{5m+1}u_2$, $u_{5m+1}v_{5m+1}$, $u_{5m-1}u_{5m+1}$, and $v_{5m}v_{5m+1}$. It is clear we have each of the following necessary relatively prime pairs:

$$\begin{aligned} \gcd(1, 12m-1) &= \gcd(2, 12m+1) = \gcd(4, 12m+3) = \gcd(12m+1, 12m+3) \\ &= \gcd(12m-5, 12m+3) = \gcd(12m-3, 12m+1) = 1. \end{aligned}$$

Since the independence number is $\alpha(\mathrm{GP}(5m+1,2)) = \left\lfloor \frac{4(5m+1)}{5} \right\rfloor = 4m$, we have used the maximum number of even labels less than $12m+3$. Since all odd integers were used from 1 to $12m+3$, we have that $\mathfrak{pr}(\mathrm{GP}(n,2)) = 12m+3$ when $n = 5m+1$.

Case 2: We now suppose that $n = 5m+2$. We label the vertices $v_1, \ldots, v_{5m}$ and $u_1, \ldots, u_{5m}$ as in Lemma 9. Label the remaining vertices as

$$\ell(u_{5m+1}) = 12m+4, \quad \ell(u_{5m+2}) = 12m+5, \quad \ell(v_{5m+1}) = 12m+1, \quad \ell(v_{5m+2}) = 12m+3.$$

As explained above, $\ell(u_{5m}) = 12m-1$, $\ell(v_{5m}) = 12m-3$, and $\ell(u_{5m-1}) = 12m-5$. Also note that our new labels on vertices adjacent v_1, u_1, and u_2 make relatively prime pairs since $\ell(u_{5m+2})$ and $\ell(v_{5m+2})$ are odd. For the remaining adjacent pairs, we have

$$\begin{aligned}\gcd(12m-5, 12m+4) &= \gcd(12m+1, 12m+4) = \gcd(12m-1, 12m+5)\\ &= \gcd(12m+3, 12m+5) = \gcd(12m-3, 12m+1)\\ &= \gcd(12m+1, 12m+3) = 1.\end{aligned}$$

The independence number in this case is $\alpha(\mathrm{GP}(5m+2, 2)) = \left\lfloor \frac{4(5m+2)}{5} \right\rfloor = 4m+1$, which shows we used the maximum number of even labels since one of the last four vertex labels is even. Thus, we have that $\mathfrak{pr}(\mathrm{GP}(n, 2)) = 12m+5$ when $n = 5m+2$.

Case 3: We next suppose that $n = 5m+3$. Again we label the vertices $v_1, \ldots, v_{5m}$ and $u_1, \ldots, u_{5m}$ as in Lemma 9. Label the remaining vertices as

$$\begin{aligned}&\ell(v_{5m+1}) = 12m+1, \quad \ell(v_{5m+2}) = 12m+2, \quad \ell(v_{5m+3}) = 12m+5,\\ &\ell(u_{5m+1}) = 12m+4, \quad \ell(u_{5m+2}) = 12m+3, \quad \ell(u_{5m+3}) = 12m+7.\end{aligned}$$

Since $\ell(v_{5m+3})$ and $\ell(u_{5m+3})$ are odd, they are relatively prime with $\ell(v_1)$ and $\ell(u_2)$, respectively. The remaining adjacent pairs satisfy the following:

$$\begin{aligned}\gcd(12m-5, 12m+4) &= \gcd(12m+1, 12m+4) = \gcd(12m+4, 12m+7)\\ &= \gcd(12m-1, 12m+3) = \gcd(12m+2, 12m+3)\\ &= \gcd(12m+5, 12m+7) = \gcd(12m-3, 12m+1)\\ &= \gcd(12m+1, 12m+2) = \gcd(12m+2, 12m+5) = 1.\end{aligned}$$

The independence number when $n = 5m+3$ is $\alpha(\mathrm{GP}(5m+3, 2)) = \left\lfloor \frac{4(5m+3)}{5} \right\rfloor = 4m+2$, implying our use of two even labels on the final six vertices is the maximum allowable. Therefore, we have that $\mathfrak{pr}(\mathrm{GP}(n, 2)) = 12m+7$ when $n = 5m+3$.

Case 4: Finally, we suppose that $n = 5m+4$. We need to consider three cases when labeling the final eight vertices. In each case, the labels on u_{5m+3}, u_{5m+4}, and v_{5m+4} trivially have no common factors with their respective adjacent vertices u_1, u_2, and v_1. First, assume $5 \nmid 12m+2$ and label the remaining vertices as

$$\begin{aligned}&\ell(v_{5m+1}) = 12m+1, \quad \ell(v_{5m+2}) = 12m+5, \ell(v_{5m+3}) = 12m+2, \quad \ell(v_{5m+4}) = 12m+7,\\ &\ell(u_{5m+1}) = 12m+4, \quad \ell(u_{5m+2}) = 12m+8, \ell(u_{5m+3}) = 12m+3, \quad \ell(u_{5m+4}) = 12m+9.\end{aligned} \tag{8}$$

As before, v_{5m}, u_{5m}, and u_{5m-1} are all constructed the same as in Equation (1). Thus,

$$\begin{aligned}\gcd(12m-5,12m+4) &= \gcd(12m+3,12m+4) = \gcd(12m+1,12m+4)\\ &= \gcd(12m-1,12m+8) = \gcd(12m+5,12m+8)\\ &= \gcd(12m+8,12m+9) = \gcd(12m+2,12m+3)\\ &= \gcd(12m+7,12m+9) = \gcd(12m-3,12m+1)\\ &= \gcd(12m+1,12m+5) = \gcd(12m+2,12m+5) = 1.\end{aligned}$$

Additionally, our final pair satisfies $\gcd(12m+2,12m+7)=1$ by our assumption of $5 \nmid 12m+2$.

Next we assume $5 \mid 12m+2$ and $7 \nmid 12m+2$. We label the eight vertices in the final block as in Equation (8) except reassign $\ell(u_{5m+1}) = 12m+2$ and $\ell(v_{5m+3}) = 12m+4$. The following show these two vertices have labels that are relatively prime with their neighbors, where the assumption of $7 \nmid 12m+2$ is necessary for the second line of gcd calculations:

$$\begin{aligned}v_{5m+3} &: \gcd(12m+4,12m+5) = \gcd(12m+3,12m+4) = \gcd(12m+4,12m+7) = 1,\\ u_{5m+1} &: \gcd(12m-5,12m+2) = \gcd(12m+2,12m+3) = \gcd(12m+1,12m+2) = 1.\end{aligned}$$

Finally, we assume $5 \mid 12m+2$ and $7 \mid 12m+2$. We label the vertices as follows:

$$\begin{array}{llll}\ell(v_{5m+1}) = 12m+5 & \ell(v_{5m+2}) = 12m+3 & \ell(v_{5m+3}) = 12m+4 & \ell(v_{5m+4}) = 12m+9,\\ \ell(u_{5m+1}) = 12m & \ell(u_{5m+2}) = 12m+2 & \ell(u_{5m+3}) = 12m+7 & \ell(u_{5m+4}) = 12m+1.\end{array}$$

Since $12m+2$ is divisible by 5 and 7, we know that $12m$ is not divisible by 5 or 7 and likewise $12m+4$ is not divisible by 5, resulting in

$$\gcd(12m-5,12m) = \gcd(12m,12m+5) = \gcd(12m,12m+7) = \gcd(12m+4,12m+9) = 1.$$

The remaining adjacent pairs have relatively prime labels based on the following:

$$\begin{aligned}\gcd(12m-1,12m+2) &= \gcd(12m+1,12m+2) = \gcd(12m+2,12m+3)\\ &= \gcd(12m+4,12m+7) = \gcd(12m+1,12m+9)\\ &= \gcd(12m-3,12m+5) = \gcd(12m+3,12m+5)\\ &= \gcd(12m+3,12m+4) = 1.\end{aligned}$$

In all three cases for the final eight vertices, our relatively prime condition is true while using three even labels in this block. Since the independence number when $n = 5m+4$ is $\alpha(\mathrm{GP}(5m+4,2)) = \left\lfloor \frac{4(5m+4)}{5} \right\rfloor = 4m+3$, we have that $\mathfrak{pr}(\mathrm{GP}(n,2)) = 12m+9$ when $n = 5m+4$, concluding our fifth and final case of n modulo 5. □

We conclude this section with conjectures for the minimum coprime number of $\mathrm{GP}(n,k)$ for larger cases of k. The independence number for the case of $k=3$ when n is odd is $\alpha(\mathrm{GP}(n,3)) = n-2$, and when $n = 3k$ we have $\alpha(\mathrm{GP}(3k,k)) = \left\lceil \frac{5k-2}{2} \right\rceil$, see [2, 6]. These values lead to the following conjectures, which we have verified for small values of n, although a minimum coprime labeling in each general case still alludes us.

Conjecture 11. For $n \geq 7$, the minimum coprime number for $\mathrm{GP}(n, 3)$ if n is odd is

$$\mathfrak{pr}(\mathrm{GP}(n, 3)) = 2n + 3.$$

Conjecture 12. For $k \geq 2$, the minimum coprime number for $\mathrm{GP}(3k, k)$ is given by

$$\mathfrak{pr}(\mathrm{GP}(3k, k)) = \begin{cases} 7k, & \text{if } k \text{ is odd;} \\ 7k + 1, & \text{if } k \text{ is even.} \end{cases}$$

5 Stacked prisms

We next turn our focus to the class of graphs known as the *stacked prism*, also known as the generalized prism graph. A stacked prism is defined as $Y_{m,n} = C_m \Box P_n$ for $m \geq 3$ and $n \geq 1$. See Figure 4 for an example of $Y_{3,6}$ with a minimum coprime labeling.

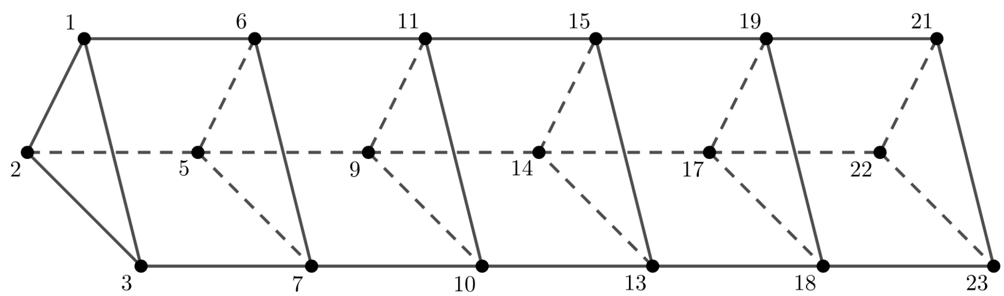

Figure 4: Example of a minimum coprime labeling of $Y_{3,6}$

We first focus on the stacked triangular prism, $Y_{3,n}$, which has $3n$ vertices. Its independence number is n since an independent set can contain at most one vertex from each triangle, and it is trivial to find such a set of n vertices. We demonstrate in the following result a way to apply a minimum coprime labeling based on how this independence number limits our use of even labels.

Theorem 13. *The minimum coprime number for the stacked triangular prism is given by*

$$\mathfrak{pr}(Y_{3,n}) = 4n - 1.$$

Proof. We refer to the vertices of $Y_{3,n}$ as $v_{i,j}$ where $i = 1, \ldots, n$ and $j = 1, 2, 3$. Then the edges of the graph are of the form $v_{i,j}v_{i+1,j}$ and $v_{i,j}v_{i,k}$ for $j \neq k$. We form a coprime labeling ℓ recursively by labeling the vertices on the $(i+1)^{\text{st}}$ triangle with $\ell(v_{i+1,r}), \ell(v_{i+1,s}), \ell(v_{i+1,t})$ based on the labels chosen for the i^{th} triangle, $\ell(v_{i,r}), \ell(v_{i,s}), \ell(v_{i,t})$. First assign the labels $\ell(v_{1,1}) = 1$, $\ell(v_{1,2}) = 2$, and $\ell(v_{1,3}) = 3$. Each subsequent $(i+1)^{\text{st}}$ triangle for $1 \leq i \leq n - 1$

will use the labels $4i+1$, $4i+2$, and $4i+3$ in some order (or $4i$, $4i+1$, and $4i+3$ in one case) depending on which labels from the ith triangle are multiples of 3 and/or 5. In each case we assume $\ell(v_{i,r}) = 4i-3$, $\ell(v_{i,s}) = 4i-2$, and $\ell(v_{i,t}) = 4i-1$.

Case 1: Suppose $5 \nmid 4i-3$ and $3 \nmid 4i-2$. In this case, we assign $\ell(v_{i+1,r}) = 4i+2$, $\ell(v_{i+1,s}) = 4i+1$, and $\ell(v_{i+1,t}) = 4i+3$. The three edges within the $(i+1)^{\text{st}}$ triangle have relatively prime labels since they are either consecutive integers or consecutive odd integers. Since we assumed $5 \nmid 4i-3$ and $3 \nmid 4i-2$, the edges between the i^{th} and $(i+1)^{\text{st}}$ triangle satisfy the following:

$$\gcd(4i-3, 4i+2) = \gcd(4i-2, 4i+1) = \gcd(4i-1, 4i+3) = 1.$$

Case 2: Assume $5 \mid 4i-3$ and $3 \nmid 4i-1$, which implies $5 \nmid 4i-2$. We then assign $\ell(v_{i+1,r}) = 4i+1$, $\ell(v_{i+1,s}) = 4i+3$, and $\ell(v_{i+1,t}) = 4i+2$. Again the vertices within the newly labeled triangle have pairs of relatively prime labels. Since $5 \nmid 4i-2$ and $3 \nmid 4i-1$, the edges between the two triangles satisfy the following:

$$\gcd(4i-3, 4i+1) = \gcd(4i-2, 4i+3) = \gcd(4i-1, 4i+2) = 1.$$

Case 3: Next suppose $5 \nmid 4i-3$ and $5 \nmid 4i-2$, in which we assign $\ell(v_{i+1,r}) = 4i+2$, $\ell(v_{i+1,s}) = 4i+3$, and $\ell(v_{i+1,t}) = 4i+1$. As in previous cases, we only need to verify the edges between the i^{th} and $(i+1)^{\text{st}}$ triangles have relatively prime labels on their endpoints, which is satisfied since our assumptions of $5 \nmid 4i-3$ and $5 \nmid 4i-2$ result in

$$\gcd(4i-3, 4i+2) = \gcd(4i-2, 4i+3) = \gcd(4i-1, 4i+1) = 1.$$

Case 4: There are two remaining other possible assumptions that can be made about factors of 3 or 5 that would allow all cases to be covered: either $5 \mid 4i-3$ and $3 \mid 4i-1$, or $5 \nmid 4i-3$ and both 3 and 5 divide $4i-2$. We handle these by combining them into one final case. This is because if $5 \mid 4i-3$ and $3 \mid 4i-1$, then this implies $5 \nmid 4i+1$ and $3, 5 \mid 4i+2$. Therefore, having the first of these possible cases implies the second occurs on the next triangle, so we assign labels to the $(i+1)^{\text{st}}$ and $(i+2)^{\text{nd}}$ triangles at once while assuming $5 \mid 4i-3$ and $3 \mid 4i-1$. We set $\ell(v_{i+1,r}) = 4i$, $\ell(v_{i+1,s}) = 4i+1$, and $\ell(v_{i+1,t}) = 4i+3$, as well as $\ell(v_{i+2,r}) = 4i+5$, $\ell(v_{i+2,s}) = 4i+6$, and $\ell(v_{i+2,t}) = 4i+7$. Note that the labels on one of the edges of the $(i+1)^{\text{st}}$ do not trivially satisfy the relatively prime condition. However, $\gcd(4i, 4i+3) = 1$ since the assumption of $3 \mid 4i-1$ implies $3 \nmid 4i$, so the condition is satisfied nonetheless. Also note that while the $(i+1)^{\text{st}}$ triangle does not use the three consecutive labels that have been used in other cases, the $(i+2)^{\text{nd}}$ triangle does use the three consecutive values that allow us to continue with our recursion to find the $(i+3)^{\text{rd}}$ triangle next.

We now verify the six edges from the i^{th} triangle to the $(i+1)^{\text{st}}$ and the $(i+1)^{\text{st}}$ triangle to the $(i+2)^{\text{nd}}$ have relatively prime endpoints. It is clear that $\gcd(4i-1, 4i+3) =$

$\gcd(4i+3, 4i+7) = 1$. Since we assumed $3 \mid 4i-1$, we know $3 \nmid 4i-2$ or $4i$, hence $\gcd(4i-2, 4i+1) = \gcd(4i-3, 4i) = 1$. Our assumption of $5 \mid 4i-3$ implies $5 \nmid 4i$ or $4i+1$, thus resulting in $\gcd(4i, 4i+5) = \gcd(4i+1, 4i+6) = 1$.

In all four cases, we have verified that each pair of adjacent vertices is labeled by relatively prime integers, thus resulting in a coprime labeling once our recursion approach reaches $i = n-1$. The final triangle will use $4(n-1)+3 = 4n-1$ as its largest label. Thus, since we used the maximum amount of n even labels based on the independence number and the smallest possible odd labels, we have proven $\mathfrak{pr}(Y_{3,n}) = 4n-1$. $\square$

We next investigate a minimum coprime labeling of the stacked pentagonal prism, $Y_{5,n}$, which has $5n$ vertices. Similarly to $Y_{3,n}$, its independence number is determined based on at most 2 independent vertices being on each pentagon. Thus, the independence number is $\alpha(Y_{5,n}) = 2n$, which leads to the following result on the minimum coprime number of the stacked pentagonal prism. See Figure 5 for an example of $Y_{5,6}$ with a minimum coprime labeling.

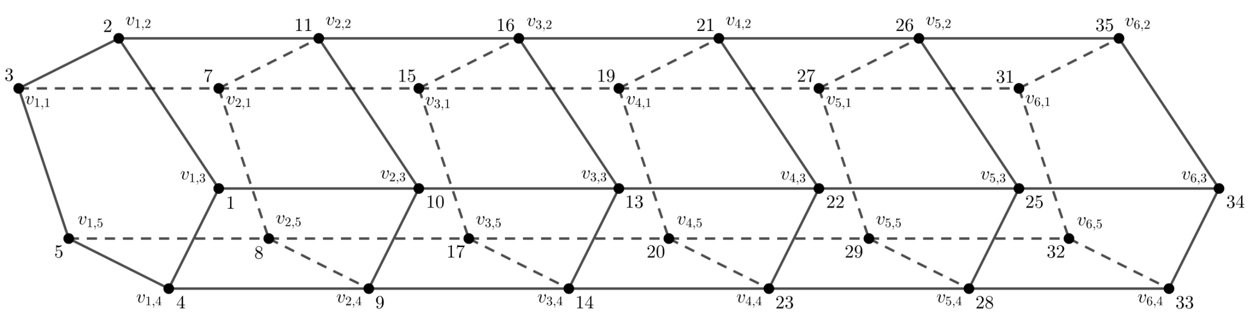

Figure 5: Example of a minimum coprime labeling of $Y_{5,6}$

Theorem 14. *The minimum coprime number for the stacked pentagonal prism graph is*

$$\mathfrak{pr}(Y_{5,n}) = 6n - 1.$$

Proof. We will call $v_{i,j}$ the vertices of $Y_{5,n}$ where $i = 1, \ldots, n$ and $j = 1, 2, 3, 4, 5$. We initially assign a labeling ℓ for $i = 1, \ldots, 70$ as follows if i is odd:

$$\ell(v_{i,1}) = 6i-3,\ \ell(v_{i,2}) = 6i-4,\ \ell(v_{i,3}) = 6i-5,\ \ell(v_{i,4}) = 6i-2,\ \ell(v_{i,5}) = 6i-1,$$

and using the following if i is even:

$$\ell(v_{i,1}) = 6i-5,\ \ell(v_{i,2}) = 6i-1,\ \ell(v_{i,3}) = 6i-2,\ \ell(v_{i,4}) = 6i-3,\ \ell(v_{i,5}) = 6i-4.$$

Since the labeling differs on odd- and even-indexed pentagons, there are twenty types of adjacent pairs to consider as having relatively labels. If i is odd and k is even, the edges

$v_{i,1}v_{i,2}$, $v_{i,2}v_{i,3}$, $v_{i,4}v_{i,5}$, $v_{k,2}v_{k,3}$, $v_{k,3}v_{k,4}$, $v_{k,4}v_{k,5}$, and $v_{k,5}v_{k,1}$ are labeled by consecutive integers. The edges $v_{i,5}v_{i,1}$, $v_{i,1}v_{i+1,1}$, $v_{k,1}v_{k,2}$, and $v_{k,1}v_{k+1,1}$ are labeled by odd integers that have a difference of 2, 4, or 8. Edges of the form $v_{i,3}v_{i,4}$, $v_{i,2}v_{i+1,2}$, $v_{i,3}v_{i+1,3}$, $v_{i,5}v_{i+1,5}$, $v_{k,2}v_{k+1,2}$, $v_{k,3}v_{k+1,3}$, or $v_{k,5}v_{k+1,5}$ have labels that differ by 3 or 9, in which these labels are not multiples of 3. At this point, we have shown that the vertices on eighteen edges are relatively prime.

The edges $v_{i,4}v_{i+1,4}$ and $v_{k,4}v_{k+1,4}$ have labels that differ by 5 and 7, respectively, and hence may not be relatively prime for certain i and k. Rather than detail numerous cases of alterations that need to be made to the labeling to fix these, we instead list the reassigned labels of 46 vertices from the 350 labels in which $i = 1, \ldots, 70$, as seen in Table 2. The labels in bold are the 46 that were reassigned to avoid adjacent labels sharing multiples of 5 and 7.

3	2	1	4	5
7	11	10	9	8
15	**16**	13	**14**	17
19	**21**	22	**23**	20
27	26	25	28	29
31	35	34	33	32
39	38	37	40	41
43	**45**	46	**47**	44
51	**52**	49	**50**	53
55	59	58	57	56
63	62	61	64	65
67	71	70	69	68
75	74	73	76	77
79	83	82	81	80
87	86	85	88	89
91	95	94	93	92
101	98	97	100	**99**
105	107	106	**103**	104
113	110	109	112	**111**
115	119	118	117	116
123	122	121	124	125
127	131	130	129	128
135	134	133	136	137

139	143	142	141	140
147	146	145	148	149
151	**153**	154	**155**	152
159	**160**	157	**158**	161
163	167	166	165	164
171	170	169	172	173
175	179	178	177	176
183	**184**	181	**182**	185
187	**189**	190	**191**	188
195	194	193	196	197
199	203	202	201	200
207	206	205	208	209
211	**213**	214	**215**	212
219	**220**	217	**218**	221
223	227	226	225	224
231	230	229	232	233
235	239	238	237	236
243	242	241	244	245
247	251	250	249	248
255	254	253	256	257
259	263	262	261	260
267	**268**	265	**266**	269
275	**273**	274	**271**	272

279	278	277	280	281
287	**285**	286	**283**	284
291	**292**	289	**290**	293
295	299	298	297	296
303	302	301	304	305
307	311	310	309	308
315	314	313	316	317
319	323	322	321	320
327	326	325	328	329
331	335	334	333	332
339	338	337	340	341
343	**345**	346	**347**	344
351	**352**	349	**350**	353
355	**357**	358	**359**	356
363	362	361	364	365
367	371	370	369	368
375	374	373	376	377
379	383	382	381	380
387	386	385	388	389
391	395	394	393	392
399	398	397	400	401
403	**405**	406	**407**	404
411	**412**	409	**410**	413
415	419	418	417	416

Table 2: Labeling of the Stacked Pentagon for $i = 1$ to 70

Using a computer, we have verified that each reassigned label is relatively prime with any adjacent label, resulting in a coprime labeling of the graph up to $n = 70$. Another important fact that can be observed is that the largest difference between adjacent labels is 10.

To label the stacked pentagonal prism graph when $n > 70$, we assign for $i > 70$ and $j = 1, \ldots 5$ the label $\ell(v_{i,j}) = \ell(v_{i-70,j}) + 420$. Since the greatest difference between adjacent labels on the first 70 pentagons was 10, only common prime factors 2, 3, 5, and 7 need to be considered. The shift by 420, which only has these four prime numbers as factors, results in the relatively prime condition remaining satisfied for all vertex pairs with $i \geq 71$. The only exceptions that need to be verified are edges of the form $v_{70m,j}v_{70m+1,j}$ for some positive integer m. We have the following pairs of labels that are adjacent:

$$\begin{aligned}
&\{\ell(v_{70m,1}) = 420m - 5,\ \ell(v_{70m+1,1}) = 420m + 3\},\\
&\{\ell(v_{70m,2}) = 420m - 1,\ \ell(v_{70m+1,2}) = 420m + 2\},\\
&\{\ell(v_{70m,3}) = 420m - 2,\ \ell(v_{70m+1,3}) = 420m + 1\},\\
&\{\ell(v_{70m,4}) = 420m - 3,\ \ell(v_{70m+1,4}) = 420m + 4\},\\
&\{\ell(v_{70m,5}) = 420m - 4,\ \ell(v_{70m+1,5}) = 420m + 5\}.
\end{aligned}$$

The first pair is relatively prime since they are odd integers differing by a power of 2. The second, third, and fifth pairs are not multiples of 3 and are separated by either 3 or 9. Finally, the fourth pair are separated by 7, but since 7 divides 420, neither of these are multiples of 7. Thus, these adjacent pairs of vertices all have relatively prime labels, making this a coprime labeling for $n \geq 71$ as well.

Note that the largest label $\ell(v_{i,j})$ for $j = 1, \ldots, 5$ is $6i - 1$ for all i, hence the largest label for $Y_{5,n}$ is $6n - 1$. Since we used the maximum of 2 even labels on each pentagon along with the smallest possible odd labels, we have proven that $\mathfrak{pr}(Y_{5,n}) = 6n - 1$. □

For the graphs $Y_{3,n}$ and $Y_{5,n}$, the minimum coprime numbers were directly correlated to their independence number. This number is easy to obtain for stacked prisms involving larger odd cycles as well. Since $\alpha(C_{2k+1}) = k$, one can observe that $\alpha(Y_{2k+1,n}) = kn$. A minimum coprime labeling could use at most kn even labels, forcing $(2k + 1)n - kn = (k + 1)n$ odd labels to be used, leading to the following conjecture.

Conjecture 15. The minimum coprime number for stacked $(2k + 1)$-gon prism graph is

$$\mathfrak{pr}(Y_{2k+1,n}) = 2(k + 1)n - 1.$$

It should be noted to close this section that while we focused on the odd case, the stacked prisms with even-length cycles, $Y_{2k,n}$, satisfies $\alpha(Y_{2k,n}) = kn = \frac{|V(Y_{2k,n})|}{2}$. Therefore, one would hope a prime labeling exists in this case, but this remains an open problem.

6 Variation of generalized Petersen graphs

We extend the definition of the generalized Petersen graph for the case of even n and $k = \frac{n}{2}$. We let $\mathrm{GP}^*(2k, k)$ denote this graph and use the same notation for the vertices $v_1, \ldots, v_{2k}$ and $u_1, \ldots, u_{2k}$ with edges v_iv_{i+1}, v_iu_i, and u_iu_{i+k} where indices are calculated modulo n. Note that this graph differs from generalized Petersen graphs since $deg(u_i) = 2$ for all $i = 1, \ldots, n$ instead of the usual degree of 3 for $\mathrm{GP}(n, k)$. For examples of minimum coprime labeling of $\mathrm{GP}^*(2k, k)$ with $k = 9$ and $k = 10$, see Figure 6.

Theorem 16. *For the graph* $\mathrm{GP}^*(2k, k)$ *in which* $k \geq 2$*, we have the following:*

1. $\mathrm{GP}^*(2k, k)$ *is prime if* k *is odd,*
2. $\mathfrak{pr}(\mathrm{GP}^*(2k, k)) = 4k + 1$ *if* k *is even.*

Proof. We first assume k is odd. We begin by labeling the vertices of $\mathrm{GP}^*(2k, k)$ as follows:

$$\ell(v_i) = \begin{cases} 4i - 3, & \text{for } i = 1, 3, 5, \ldots, k; \\ 4i, & \text{for } i = 2, 4, 6, \ldots, k-1; \\ 4i - 4k, & \text{for } i = k+1, k+3, k+5, \ldots, 2k; \\ 4i - 3 - 4k, & \text{for } i = k+2, k+4, k+6, \ldots, 2k-1; \end{cases} \tag{9}$$

$$\ell(u_i) = \begin{cases} 4i - 2, & \text{for } i = 1, 3, 5, \ldots, k; \\ 4i - 1, & \text{for } i = 2, 4, 6, \ldots, k-1; \\ 4i - 1 - 4k, & \text{for } i = k+1, k+3, k+5, \ldots, 2k; \\ 4i - 2 - 4k, & \text{for } i = k+2, k+4, k+6, \ldots, 2k-1. \end{cases}$$

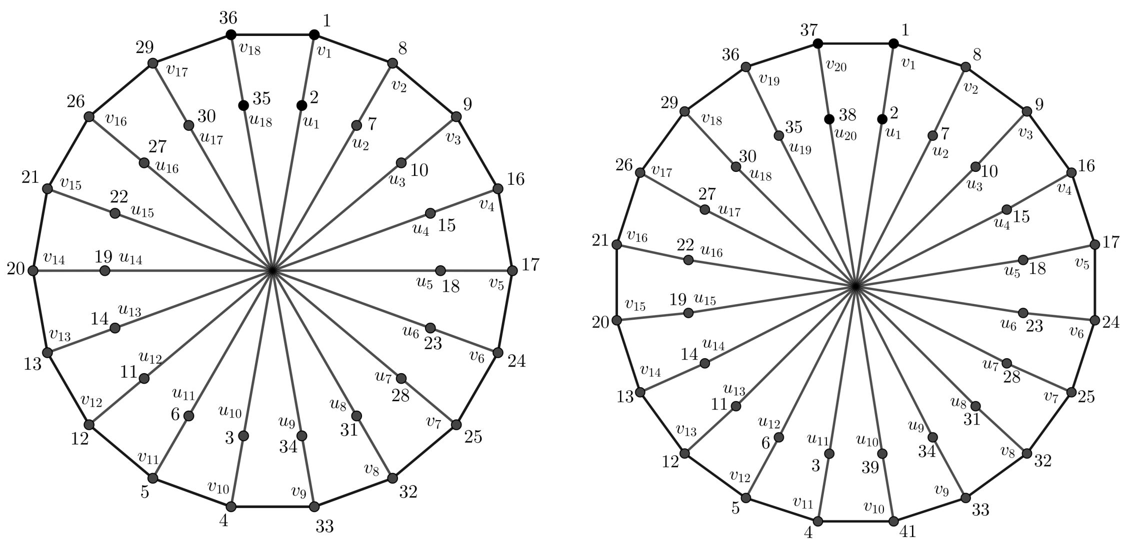

Figure 6: Examples of a prime labeling of $\mathrm{GP}^*(18, 9)$ on the left and a minimum coprime labeling of $\mathrm{GP}^*(20, 10)$ on the right.

For the pairs $v_i v_{i+1}$ and $v_{i+k} v_{i+1+k}$ for $i \in \{1, \ldots, k-1\}$, it is clear that $\gcd(\ell(v_i), \ell(v_{i+1})) \in \{1, 7\}$ and $\gcd(\ell(v_{i+k}), \ell(v_{i+k+1})) \in \{1, 7\}$. Notice that $\gcd(\ell(v_k), \ell(v_{k+1})) = \gcd(4k-3, 4) = 1$ and $\gcd(\ell(v_{2k}), \ell(v_1)) = \gcd(4k-1, 1) = 1$. For the edges $u_i u_{i+k}$, it is easily verified that when $i \in \{1, 3, \ldots, k\}$,

$$\gcd(\ell(u_i), \ell(u_{i+k})) = \gcd(4i - 2, 4(i+k) - 1 - 4k) = \gcd(4i - 2, 4i - 1) = 1.$$

Similarly, $\gcd(\ell(u_i), \ell(u_{i+k})) = 1$ when $i \in \{2, 4, \ldots, k-1\}$ and $\gcd(\ell(u_i), \ell(v_i)) = 1$ for all $i = 1, \ldots, 2k$ since these pairs are consecutive integers. Thus, our only concern with the labeling is when $\ell(v_i)$ and $\ell(v_{i+1})$ are both divisible by 7. We handle these instances by breaking the proof into several cases based on the remainder of $\ell(v_i)/3$ and whether or not 5 divides $\ell(v_i) + 2$. For the sake of simplicity, let $\ell(v_i) = a$ where $i \in \{1, 3, \ldots, k-2, k+2, k+4, \ldots, 2k-1\}$. Notice that in our labeling, $\ell(v_i) = a$ is odd in these cases in which $\ell(v_{i+1}) - \ell(v_i) = 7$. In each of the cases below, we are assuming that $a \equiv 0 \pmod 7$.

Case 1: Suppose that $a \equiv 1 \pmod 3$.

In this case, swap the labels a and $a-2$ on the vertices v_i and u_{i-1}. Since a is not divisible by 3, $\gcd(a-3, a) = 1$. Since $a-2$ is also not divisible by 3, $\gcd(a-2, a+7) = \gcd(a-2, a+1) = 1$. Thus, the two labels involved in the swap are relatively prime with all adjacent labels.

Case 2: Suppose that $a \equiv 0 \pmod 3$ and $5 \nmid a$.

In this case, swap the labels $a+5$ and $a+7$ on the vertices v_{i+1} and u_{i+1+k}. Since a is not divisible by 5, $\gcd(a, a+5) = 1$. Since both $a+5$ and $a+7$ are not divisible by 3, $\gcd(a+5, a+8) = \gcd(a+4, a+7) = 1$. Thus, all newly adjacent pairs of labels after making this swap are relatively prime.

Case 3: Suppose that $a \equiv 0 \pmod{15}$.

The labels $a-1, a, a+1, a+5, a+6, a+7$ will be relabeled as $a+7, a+6, a+5, a-1, a, a+1$, respectively. Based on the placement of the swapped labels, we only need to check the gcd between six newly adjacent pairs of labels. Since a is a multiple of 3, 5, and 7, we have

$$\gcd(a+1, a+8) = \gcd(a-1, a+4) = \gcd(a+2, a+5) = \gcd(a-2, a+7) = \gcd(a+1, a+6) = 1.$$

Note that if $i = 2k-1$, instead of the label $a+8$, we have label 1 on v_{i+2}, which is still relatively prime with $a+1$. Our last remaining pair of adjacent labels involves $a+7$, which is now adjacent to a vertex labeled $a-8$. Since $a \equiv 0 \pmod{15}$, $\gcd(a-8, a+7) = 1$. Therefore, the reassigned labels are relatively prime with any newly adjacent label.

Case 4: Suppose that $a \equiv 2 \pmod 3$ and $5 \nmid a+2$.

In this case, swap the labels a and $a+2$. Since both a and $a+2$ are not divisible by 3 and $a+2$ is not divisible by 5, it follows that

$$\gcd(a, a+3) = \gcd(a+2, a+7) = \gcd(a-1, a+2) = 1.$$

Hence any new adjacencies after the swap consist of relatively prime labels.

Case 5: Suppose that $a \equiv 2 \pmod 3$ and $5 \mid a+2$.

The labels $a, a+1, a+2, a+6, a+7, a+8$ are reassigned as $a+6, a+7, a+8, a+2, a+1, a$, respectively. Based on the placement of the swapped labels, as in Case 3, we need only check the gcd of six pairs of labels. Since $a \equiv 2 \pmod 3$, $a+2 \equiv 0 \pmod 5$, and $a \equiv 0 \pmod 7$,

$$\gcd(a-1, a+6) = \gcd(a+3, a+8) = \gcd(a+2, a+5) = \gcd(a+1, a+6) = \gcd(a, a+9) = 1.$$

If $\ell(v_{i+2}) = a+8$ in the original labeling of the vertices, then since a is odd, $a \equiv 2 \pmod 3$, and $a+2 \equiv 0 \pmod 5$, it follows that $\gcd(a+15, a) = 1$. Since k is odd, it is not possible for $\ell(v_{i+2})$ to be 4. It is possible that $\ell(v_{i+2}) = 1$ in the case of $i = 2k-1$, but in this case, $\ell(v_{i+3}) = 8$ and the label a is adjacent to 8, which again proves our claim. Thus, all new pairs of adjacent labels are relatively prime.

Combining all five cases, we have proven that $\mathfrak{pr}(\mathrm{GP}^*(2k,k)) = 4k$ and hence $\mathrm{GP}^*(2k,k)$ is prime for all odd k.

Now suppose that k is an even integer. To show that $\mathrm{GP}^*(2k,k)$ is not prime when k is even, we consider the independence number $\alpha(\mathrm{GP}^*(2k,k))$. An independent set can contain at most k vertices from the k-cycle formed by the vertices $v_1, \ldots, v_{2k}$, either the set of vertices with odd indices or the set with even indices. Without loss of generality, consider the independent set $\{v_1, v_3, \ldots, v_{2k-1}\}$. Since we have edges u_iv_i for all i, only vertices of the form u_{2j} can be added to the independent set. However, the edges u_iu_{i+k} prevent all k of those edges from being independent. Thus, the size of our independent set is less than $2k$ if k of the vertices $v_1, \ldots, v_{2k}$ are included. If we instead begin with fewer than k of these vertices in an independent set, we cannot use more than k vertices from $u_1, \ldots, u_{2k}$ since only one of u_i or u_{i+k} can be included for each $i = 1, \ldots, k$. Hence we cannot create an independent set of size $2k$, showing that $\alpha(\mathrm{GP}^*(2k,k)) < 2k = \frac{1}{2}|V|$. Therefore, by Lemma 1, this graph is not prime when k is even.

We then create a minimum coprime labeling when k is even by starting with a labeling ℓ similar our odd case, defined by the following:

$$\ell(v_i) = \begin{cases} 4i-3, & \text{for } i = 1,3,5,\ldots,k-1; \\ 4i, & \text{for } i = 2,4,6,\ldots,k-2; \\ 4i-4k, & \text{for } i = k+1,k+3,k+5,\ldots,2k-1; \\ 4i-3-4k, & \text{for } i = k+2,k+4,k+6,\ldots,2k; \end{cases} \tag{10}$$

$$\ell(u_i) = \begin{cases} 4i-2, & \text{for } i = 1,3,5,\ldots,k-1; \\ 4i-1, & \text{for } i = 2,4,6,\ldots,k; \\ 4i-1-4k, & \text{for } i = k+1,k+3,k+5,\ldots,2k-1; \\ 4i-2-4k, & \text{for } i = k+2,k+4,k+6,\ldots,2k. \end{cases}$$

Note that the label for v_k was not listed in Equation (10) since continuing the pattern $\ell(v_i) = 4i$ for even i would assign it as $\ell(v_k) = 4k$, which is not relatively prime with its neighboring label $\ell(v_{k+1}) = 4$. Hence we instead assign $\ell(v_k)$ to be $4k+1$, which we observe is relatively prime with the adjacent labels since $\gcd(4, 4k+1) = \gcd(4k-1, 4k+1) = \gcd(4k-7, 4k+1) = 1$.

We note by similar reasoning to the odd case that our labeling when k is even is coprime with the exception of some adjacent vertices v_iv_{i+1} whose labels share a common factor of 7. We proceed by making the same alterations to ℓ as described in Cases 1–5, which by analogous arguments to those made in the odd k case results in a coprime labeling if the label $\ell(v_k)$ and its neighbors are not involved in the alterations to ℓ. The only situation in which any of the five cases involve swapping these labels would be if $a = \ell(v_{k-1})$. However, with $\ell(v_k) = 4k+1$, we have $\ell(v_k) - \ell(v_{k-1}) = 8$, so these labels cannot share a common factor of 7. Hence we have shown this is a coprime labeling for the case of even k with largest label $4k+1$. Therefore, since the graph is not prime in this case, we have proven $\mathfrak{pr}(\mathrm{GP}^*(2k,k)) = 4k+1$. □

A Additional generalized Petersen results

For Theorems 17–23, the proof for why the labeling is a coprime labeling is omitted as the pattern is clearly given in the tables and can be verified in each case. The techniques used are all outlined in Observation 2. Recall that we are working under the assumption that n is odd for the following theorems.

Theorem 17. *If $n+4$ is prime, then $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.*

Proof. We may assume $n \not\equiv 2 \pmod 3$, otherwise $n+4$ would be divisible by 3 and hence not prime. If $n \equiv 0 \pmod 3$, then we use the labeling defined in Table 3. If $n \equiv 1 \pmod 3$, then we alter the previous labeling and instead use the labeling defined in Table 4.

□

Theorem 18. *If $n-2$ is prime and $n > 5$, then $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.*

Proof. We may assume $n \not\equiv 2 \pmod 3$ to avoid $n-2$ being a multiple of 3. If $n \equiv 0 \pmod 3$, then we use the labeling defined in Table 5. If $n \equiv 1 \pmod 3$, we instead label the graph using the labeling in Table 6. Note that in both cases, the labels on most edges u_iv_i are relatively prime since the difference of the labels is the prime $n-2$, but a swap is necessary to avoid both labels being multiples of that prime number. □

Theorem 19. *If $n-4$ is prime and $n > 7$, then $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.*

Proof. We may assume $n \not\equiv 1 \pmod 3$, else $n-4$ is a multiple of 3. If $n \equiv 0 \pmod 3$, then we use the labeling defined in Table 7. If $n \equiv 2 \pmod 3$, then use the same labeling defined in Table 8. □

Theorem 20. *If $2n+3$ is prime, then $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.*

Proof. We may assume $n \not\equiv 0 \pmod 3$ to avoid $2n+3$ being divisible by 3. If $n \equiv 1 \pmod 3$, then we use the labeling defined in Table 9. If $n \equiv 2 \pmod 3$, then use the same labeling defined in Table 10 except $n+1$ is labeled as $n-1$ instead. □

Theorem 21. *If $2n-3$ is prime, then $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.*

Proof. We may assume $n \not\equiv 0 \pmod 3$, otherwise $2n-3$ is a multiple of 3. If $n \equiv 2 \pmod 3$, then we use the labeling defined in Table 11. If $n \equiv 1 \pmod 3$, then we use the labeling defined in Table 12. □

Theorem 22. *If $2n-5$ is prime, then $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.*

Proof. We may assume $n \not\equiv 1 \pmod 3$ to avoid 3 dividing into $2n-5$. If $n \equiv 0$ or $2 \pmod 3$ and $n \equiv 1, 2$, or $4 \pmod 5$, then we use the labeling defined in Table 13. When $n \equiv 0$ or $2 \pmod 3$ and $n \equiv 0 \pmod 5$, then $2n-5 \equiv 0 \pmod 5$, so we ignore these cases. When $n \equiv 0 \pmod 3$ and $n \equiv 3 \pmod 5$, then we use the labeling defined in Table 14. When $n \equiv 2 \pmod 3$ and $n \equiv 3 \pmod 5$, then we use the labeling defined in Table 15. □

Theorem 23. *If $n+6$ is prime, then $\mathfrak{pr}(\mathrm{GP}(n,1)) = 2n+1$.*

Proof. We may assume that $n \not\equiv 0 \pmod 3$, otherwise $n+6$ is divisible by 3. If $n \equiv 1$ or $2 \pmod 3$ and $n \equiv 0, 2$, or $3 \pmod 5$, then we use the labeling defined in Table 16. If $n \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 5$, then we use the labeling defined in Table 17. If $n \equiv 1 \pmod 3$ and $n \equiv 1 \pmod 5$, then we use the labeling defined in Table 18. If $n \equiv 1$ or $2 \pmod 3$ and $n \equiv 4 \pmod 5$, then $n+6$ is divisible by 5, so these cases are removed. □

1	2	$\cdots$	$n-3$	$n-2$	$n-1$	n
$n+5$	$n+6$	$\cdots$	$2n+1$	$n+1$	$n+2$	$n+4$

Table 3: Labeling for Theorem 17 when $n \equiv 0 \pmod 3$

1	2	$\cdots$	$n-3$	$n-2$	$n+2$	$n+1$
$n+5$	$n+6$	$\cdots$	$2n+1$	n	$n+3$	$n+4$

Table 4: Labeling for Theorem 17 when $n \equiv 1 \pmod 3$

1	2	3	4	$\cdots$	$n-3$	$n-2$	$n-1$	n
$2n-1$	$2n+1$	$n+1$	$n+2$	$\cdots$	$2n-5$	$2n-2$	$2n-3$	$2n-4$

Table 5: Labeling for Theorem 18 when $n \equiv 0 \pmod 3$

1	2	3	4	$\cdots$	$n-5$	$n-4$	$n-3$	$n-2$	$n-1$	n
$2n-1$	$2n+1$	$n+1$	$n+2$	$\cdots$	$2n-7$	$2n-4$	$2n-5$	$2n-6$	$2n-3$	$2n-2$

Table 6: Labeling for Theorem 18 when $n \equiv 1 \pmod 3$

1	2	3	4	5	6	$\cdots$	$n-7$	$n-6$	$n-5$	$n-4$	$n-3$	$n-2$	$n-1$	n
$2n-3$	$2n-1$	$2n-2$	$2n+1$	$n+1$	$n+2$	$\cdots$	$2n-11$	$2n-8$	$2n-9$	$2n-10$	$2n-7$	$2n-6$	$2n-5$	$2n-4$

Table 7: Labeling for Theorem 19 when $n \equiv 0 \pmod 3$

1	2	3	4	5	6	$\cdots$	$n-5$	$n-4$	$n-3$	$n-2$	$n-1$	n
$2n-3$	$2n-1$	$2n$	$2n+1$	$n+1$	$n+2$	$\cdots$	$2n-9$	$2n-6$	$2n-7$	$2n-8$	$2n-5$	$2n-4$

Table 8: Labeling for Theorem 19 when $n \equiv 2 \pmod 3$

1	2	3	4	$\cdots$	$n-2$	$n+1$	$n+2$
n	$2n+1$	$2n$	$2n-1$	$\cdots$	$n+5$	$n+4$	$n+3$

Table 9: Labeling for Theorem 20 when $n \equiv 1 \pmod 3$

1	2	3	4	$\cdots$	$n-2$	n	$n+3$
$n-1$	$2n+1$	$2n$	$2n-1$	$\cdots$	$n+5$	$n+4$	$n+2$

Table 10: Labeling for Theorem 20 when $n \equiv 2 \pmod 3$

1	2	3	4	$\cdots$	$n-2$	$2n-3$	$2n$
$2n-2$	$2n-5$	$2n-6$	$2n-7$	$\cdots$	$n-1$	$2n-1$	$2n+1$

Table 11: Labeling for Theorem 21 when $n \equiv 2 \pmod 3$

1	2	3	4	$\cdots$	$n-2$	$2n$	$2n+1$
$2n-4$	$2n-5$	$2n-6$	$2n-7$	$\cdots$	$n-1$	$2n-3$	$2n-1$

Table 12: Labeling for Theorem 21 when $n \equiv 1 \pmod 3$

1	2	3	$\cdots$	$n-3$	$2n-5$	$2n+1$	$2n$
$2n-6$	$2n-7$	$2n-8$	$\cdots$	$n-2$	$2n-3$	$2n-2$	$2n-1$

Table 13: Labeling for Theorem 22 when $n \equiv 0, 2 \pmod 3$ and $n \equiv 1, 2,$ or $4 \pmod 5$

1	2	3	$\cdots$	$n-3$	$2n-5$	$2n-1$	$2n$
$2n-4$	$2n-7$	$2n-8$	$\cdots$	$n-2$	$2n-3$	$2n-2$	$2n+1$

Table 14: Labeling for Theorem 22 when $n \equiv 0 \pmod 3$ and $n \equiv 3 \pmod 5$

1	2	3	$\cdots$	$n-3$	$2n-5$	$2n-1$	$2n$
$2n-6$	$2n-7$	$2n-8$	$\cdots$	$n-2$	$2n+1$	$2n-2$	$2n-3$

Table 15: Labeling for Theorem 22 when $n \equiv 2 \pmod 3$ and $n \equiv 3 \pmod 5$

1	2	3	$\cdots$	$n-5$	$n-4$	$n-3$	$n-2$	$n+2$	$n+3$
$n+7$	$n+8$	$n+9$	$\cdots$	$2n+1$	$n+1$	n	$n-1$	$n+4$	$n+6$

Table 16: Labeling for Theorem 23 when $n \equiv 1$ or $2 \pmod 3$ and $n \equiv 0, 2,$ or $3 \pmod 5$

1	2	3	$\cdots$	$n-5$	$n-4$	$n-3$	$n-2$	$n+2$	$n+5$
$n+7$	$n+8$	$n+9$	$\cdots$	$2n+1$	$n+1$	n	$n+3$	$n+4$	$n+6$

Table 17: Labeling for Theorem 23 when $n \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 5$

1	2	3	$\cdots$	$n-5$	$n-4$	$n-3$	$n-2$	$n+4$	$n+5$
$n+7$	$n+8$	$n+9$	$\cdots$	$2n+1$	$n+1$	n	$n+3$	$n+2$	$n+6$

Table 18: Labeling for Theorem 23 when $n \equiv 1 \pmod 3$ and $n \equiv 1 \pmod 5$

References

[1] J. Asplund and N. B. Fox, Minimum coprime labelings for operations on graph, *Integers* **19** (2019), Article A24.

[2] B. Behsaz, P. Hatami, and E. Mahmoodian, On minimum vertex covers of generalized Petersen graph, *Australas. J. Combin.* **40** (2008), 253–264.

[3] A. Berliner, N. Dean, J. Hook, A. Marr, A. Mbirika, and C. McBee, Coprime and prime labelings of graphs, *J. Integer Sequences* **19** (2016), Article 16.5.8.

[4] N. Besharati, J. B. Ebrahimi, and A. Azadi, Independence number of generalized Petersen graphs, *Ars Combin.* **124** (2016), 239–255.

[5] N. Dean, Proof of the prime ladder conjecture, *Integers* **17** (2017), Article A40.

[6] J. Fox, R. Gera, and P. Stănică, The independence number for the generalized Petersen graphs, *Ars Combin.* **103** (2012), 439–451.

[7] H.-L. Fu and K.-C. Huang, On prime labelling, *Discrete Math.* **127** (1994), 181–186.

[8] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* **DS6** (2020).

[9] K. Haque, X. Lin, Y. Yang, and P. Zhao, On the prime labeling of generalized Petersen graph $P(n, 1)$, *Util. Math.* **83** (2010), 95–106.

[10] K. Haque, X. Lin, Y. Yang, and P. Zhao, On the prime labeling of generalized Petersen graphs $P(n, 3)$, *Int. J. Contemp. Math. Sci.* **6** (2011), 1783–1800.

[11] C. Lee, Minimum coprime graph labelings, *J. Integer Sequences* **23** (2020), Article 20.11.4.

[12] U. Prajapati and S. Gajjar, Prime labeling of generalized Petersen graph, *Internat. J. of Math. and Soft Comput.* **5** (2015), 65–71.

[13] S. Schluchter, J. Schroeder, K. Cockus, R. Ellingson, H. Harris, E. Rarity, and T. Wilson, Prime labelings of generalized Petersen graphs, *Involve* **10** (2017), 109–124.

[14] A. Tout, A. N. Dabboucy, and K. Howalla, Prime labeling of graphs, *Nat. Acad. Sci. Lett.* **11** (1982), 365–368.

2010 *Mathematics Subject Classification*: Primary 05C78, Secondary 11A05.
Keywords: coprime labeling, prime labeling, generalized Petersen graph, prism graph.

Received July 22 2020; revised version received February 10 2021; February 14 2021. Published in *Journal of Integer Sequences*, February 15 2021.

Return to Journal of Integer Sequences home page.