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# Polynomials Whose Coefficients Are Stern Numbers

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**Abstract**

The main object in this paper is the sequence of polynomials $P_n(z)$ that have Stern numbers as their coefficients; that is, the terms of Stern's diatomic sequence. We derive certain basic properties of these polynomials, investigate the distribution of their real and complex zeros, and prove some results concerning factorizations and resultants. We also consider the $(0,1)$-polynomials obtained from $P_n(z)$ by taking their coefficients modulo 2. In spite of its simple form, the polynomial sequence is shown to possess some interesting algebraic and analytic properties. Finally, we discuss combinatorial interpretations of the polynomials $P_n(z)$ and indicate ways of generalizing them.

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# 1 Introduction

The Stern sequence, also known as Stern's diatomic sequence [16], is one of the most remarkable integer sequences in number theory and combinatorics. It can be defined by $s(0) = 0$, $s(1) = 1$, and

$$s(2n) = s(n), \qquad s(2n+1) = s(n) + s(n+1) \qquad (n \geq 1). \tag{1}$$

The first 30 Stern numbers, starting with $n = 1$, are

$$1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, 3, 7, 4. \tag{2}$$

The definition (1) gave rise to two different concepts of Stern polynomials, introduced in [10] and [15], respectively. Both preserve the "binary structure" of the sequence (1) and have various combinatorial interpretations. Subsequently these sequences of polynomials were generalized by the present authors [7], then unified as special cases of a class of polynomials in two variables [6], and more generally extended to polynomials in an arbitrary number of variables and with an underlying base $b \geq 2$ [8]. All this has applications to counting and classifying restricted binary and $b$-ary partitions of integers.

In the current paper we will, in the first place, consider polynomials of a very different type, namely those having the Stern numbers $s(n)$, and later some of their generalizations, as coefficients. Thus, we define

$$P_n(z) := s(1)z^n + s(2)z^{n-1} + \cdots + s(n)z + s(n+1), \tag{3}$$

and as a specific example we have, with (2),

$$P_8(z) = z^8 + z^7 + 2z^6 + z^5 + 3z^4 + 2z^3 + 3z^2 + z + 4. \tag{4}$$

The polynomials $P_n(z)$, as well, have a combinatorial interpretation which is related to binary overpartitions, and this leads to generalizations in different directions. The main purpose of this paper, however, is to derive various properties of the polynomials $P_n(z)$ and of a closely related polynomial sequence.

In the process we will encounter combinatorial objects, including recurrence relations, Fibonacci numbers, generating functions, and various $b$-ary partitions. From classical analysis we will come across several well-known theorems on the distribution of zeros of polynomials, and the algebraic topics occurring in this paper include cyclotomic polynomials, resultants, and divisibility and irreducibility.

We begin in Section 2 by recalling certain properties of the Stern sequence that will be needed later, followed by some basic properties of the polynomials $P_n(z)$. In Sections 3 and 4 we study the complex and real zeros, respectively, of these polynomials. Section 5 is devoted to algebraic properties of the polynomials $P_n(z)$, including factors and resultants. In Section 6 we consider a (0,1)-polynomial sequence whose coefficients are those of $P_n(z)$ taken modulo 2; we completely determine all factors of these polynomials and investigate their zeros. Section 7 deals with binary and $b$-ary overpartitions and with a combinatorial interpretation of the polynomials $P_n(z)$, including some generalizations.

## 2 Some basic properties

In this section we derive several basic properties of the polynomials $P_n(z)$, most of which will be required in later sections. We begin by recalling some easy facts about the Stern sequence.

### 2.1 Properties of the Stern sequence

Although at first sight the Stern sequence appears to behave rather erratically, it does in fact have a great deal of structure. First, we have the special values

$$s(2^m) = 1, \quad s(2^m + 1) = m + 1, \quad s(2^m + 2) = m, \quad s(2^m + 3) = 2m - 1 \tag{5}$$

($m \geq 2$), where the first three also hold for $m = 1$, and the first two for $m = 0$. All four identities are easy to prove by induction. The sequence $s(n)$ is also symmetric between two powers of 2, that is,

$$s(2^m + j) = s(2^{m+1} - j), \qquad 0 \leq j \leq 2^m. \tag{6}$$

In particular, with (5), this leads to values for $s(2^m - j)$, $j = 1, 2, 3$. Another identity that will be needed later is

$$s(3 \cdot 2^m + 1) = s(3 \cdot 2^m - 1) = 2m + 1, \qquad m \geq 0; \tag{7}$$

this, too, is easy to prove by induction. More properties like these are known, but they will not be needed here.

Next, combining the two identities in (1), we have $s(2n + 1) = s(2n) + s(2n + 2)$; this implies that for $n \geq 1$ each odd-index Stern number $s(2n + 1)$ is strictly larger than its two neighbors. The Stern sequence is therefore a strict up-down sequence, beginning with the second term. This leads us to consider the ratio of neighboring terms. First, it is clear from (5) and by symmetry that ratios of neighboring terms can be arbitrarily large or arbitrarily small rational numbers. This is consistent with the well-known fact that the sequence $s(n+1)/s(n)$ is a one-to-one enumeration of all positive rationals; see, e.g., [1, 15].

We now show that the ratios $s(n + 1)/s(n)$ given by the first two identities in (5), and their reciprocals, are in fact extremal in each interval between two neighboring powers of 2.

**Lemma 1.** *For any integer $m \geq 1$ we have*

$$\frac{1}{m} \leq \frac{s(n+1)}{s(n)} \leq m, \quad \textit{where} \quad 2^{m-1} \leq n \leq 2^m - 1, \tag{8}$$

*and the lower and upper bound are attained exactly when $n = 2^m - 1$ and $n = 2^{m-1}$, respectively.*

*Proof.* We proceed by induction on $m$. The cases $m = 1$ and $m = 2$ are easily seen to be true. Suppose now that (8) holds up to some $m \geq 2$, and let $n$ be in the interval $2^m \leq n \leq 2^{m+1}-1$. First, if $n$ is even, we write $n = 2k$ and with (1) we get

$$\frac{s(n+1)}{s(n)} = \frac{s(k)+s(k+1)}{s(k)} = 1 + \frac{s(k+1)}{s(k)};$$

by the induction hypothesis we then get the upper bound $m+1$. Second, if $n$ is odd, we set $n = 2k+1$ and again with (1) we get

$$\frac{s(n+1)}{s(n)} = \frac{s(k+1)}{s(k)+s(k+1)} = \frac{1}{1+s(k)/s(k+1)} \geq \frac{1}{1+m},$$

having used the induction hypothesis. We also implicitly used the "up-down" property which implies that the quotients $s(n+1)/s(n)$ alternate between greater than and less than 1. This proves (8) by induction. The final statement is clear by (5) and (6), and uniqueness comes from the one-to-one enumeration mentioned above. □

We also require a remarkable property of the Stern sequence that was apparently first proved by D. H. Lehmer [16].

**Lemma 2** (Lehmer)**.** *The largest value of $s(n)$ between $n = 2^{m-1}$ and $n = 2^m$ is the Fibonacci number $F_{m+1}$, with the usual definition $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$. The maximum is attained at*

$$\alpha_m := \frac{1}{3}\left(2^{m+1} + (-1)^m\right) \quad \text{and} \quad \beta_m := \frac{1}{3}\left(5 \cdot 2^{m-1} - (-1)^m\right) \quad (m \geq 1). \tag{9}$$

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_m$ | 1 | 3 | 5 | 11 | 21 | 43 | 85 | 171 | 341 | 683 |
| $\beta_m$ | 2 | 3 | 7 | 13 | 27 | 53 | 107 | 213 | 427 | 853 |
| $F_{m+1}$ | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |

Table 1: $\alpha_m$, $\beta_m$, $F_{m+1}$ for $1 \leq m \leq 10$

## 2.2 Properties of $P_n(z)$

Now we turn to some basic properties of the polynomials $P_n(z)$. First, the definition (3) immediately gives

$$P_n(z) = z \cdot P_{n-1}(z) + s(n+1). \tag{10}$$

By separating even and odd powers of $z$ in (3), we obtain a different class of recurrence relations, as follows.

**Lemma 3.** *For all $n \geq 1$ we have*

$$P_{2n}(z) = (1+z) \cdot P_{n-1}(z^2) + P_n(z^2), \tag{11}$$
$$P_{2n+1}(z) = z \cdot P_{n-1}(z^2) + (1+z) \cdot P_n(z^2). \tag{12}$$

*Proof.* By (3) we have

$$\begin{aligned} P_{2n}(z) &= \sum_{j=0}^{n} s(2j+1) z^{2n-2j} + \sum_{j=1}^{n} s(2j) z^{2n+1-2j} \\ &= \sum_{j=0}^{n} \big(s(j) + s(j+1)\big)(z^2)^{n-j} + z \sum_{j=1}^{n} s(j)(z^2)^{n-j} \\ &= (1+j) \sum_{j=0}^{n-1} s(j+1)(z^2)^{n-j-1} + \sum_{j=0}^{n} s(j+1)(z^2)^{n-j}, \end{aligned}$$

where we have used (1) and the fact that $s(0) = 0$. It is now obvious that this last line is the right-hand side of (11). The identity (12) can be obtained analogously. □

The next lemma deals with some special values of $P_n(z)$.

**Lemma 4.** *For any integer $m \geq 1$ we have*

$$P_{2^m-1}(1) = \sum_{j=1}^{2^m} s(j) = \frac{1}{2}\left(3^m + 1\right), \tag{13}$$

*and, for any $n \geq 1$,*

$$P_{2n}(-1) = P_n(1), \qquad P_{2n+1}(-1) = -P_{n-1}(1). \tag{14}$$

*Proof.* It is a well-known property of the Stern sequence that

$$\sum_{j=2^\nu+1}^{2^{\nu+1}} s(j) = 3^\nu, \qquad \nu \geq 0; \tag{15}$$

see, e.g., [16]. Summing (15) over all $\nu = 0, 1, \dots, m-1$ and adding $s(1) = 1$ to both sides, we obtain (13). Finally, if we set $z = -1$ in (11) and (12), we immediately get the two identities in (14). □

To conclude this section, we derive a generating function for the polynomials $P_n(z)$.

**Proposition 5.** *With $q$ a complex variable, we have*

$$\sum_{n=0}^{\infty} P_n(z) q^n = \frac{1}{1-zq} \prod_{k=0}^{\infty} \left(1 + q^{2^k} + q^{2^{k+1}}\right). \tag{16}$$

*Proof.* The Stern sequence has the well-known generating function

$$\prod_{k=0}^{\infty}\left(1+q^{2^k}+q^{2^{k+1}}\right)=\frac{1}{q}\sum_{n=1}^{\infty}s(n)q^n=\sum_{n=0}^{\infty}s(n+1)q^n; \tag{17}$$

see, e.g., [2, 19]. By the definition (3), $P_n(z)$ is a convolution of the sequences $s(1), s(2), \dots, s(n+1)$ and $z^0, z^1, \dots, z^n$. Hence

$$\sum_{n=0}^{\infty}P_n(z)q^n=\left(\sum_{k=0}^{\infty}z^kq^k\right)\left(\sum_{k=0}^{\infty}s(k+1)q^k\right),$$

and (16) follows from (17) and the sum of a geometric series. □

# 3 Complex zeros of $P_n(z)$

Plots of the zeros of $P_n(z)$ indicate that they lie relatively close to the unit circle (with some notable exceptions inside the unit circle) and have uniform angular distribution; see Figure 1. It is the purpose of this section to confirm these observations.

## 3.1 Some classical theorems

All our results will be based on some well-known classical theorems; we quote them here for the sake of completeness.

The first result is due to Eneström and Kakeya; see, for instance, [17, pp. 136–137].

**Theorem 6** (Eneström, Kakeya). *Let $f(z)=a_0+a_1z+\cdots+a_nz^n$ be a polynomial with positive coefficients. Then all the zeros of $f(z)$ lie in the annulus $\rho_1\le|z|\le\rho_2$, where*

$$\rho_1:=\min_{0\le k\le n-1}\left\{\frac{a_k}{a_{k+1}}\right\},\qquad \rho_2:=\max_{0\le k\le n-1}\left\{\frac{a_k}{a_{k+1}}\right\}. \tag{18}$$

Another well-known result of a similar nature is due to Cauchy; see [17, p. 123].

**Theorem 7** (Cauchy). *All the zeros of $f(z)=a_0+a_1z+\cdots+a_nz^n$, $a_n\ne0$, lie in the disk*

$$|z|<1+\max_{0\le k\le n-1}\left\{\left|\frac{a_k}{a_n}\right|\right\}. \tag{19}$$

The next auxiliary result is due to Jentzsch [14]. It can be stated as follows.

**Theorem 8** (Jentzsch). *Let $f(z)=\sum_{k=0}^{\infty}a_kz^k$ be a power series with radius of convergence 1. Then every point on the unit circle is a limit point of zeros of the partial sums of $f(z)$.*

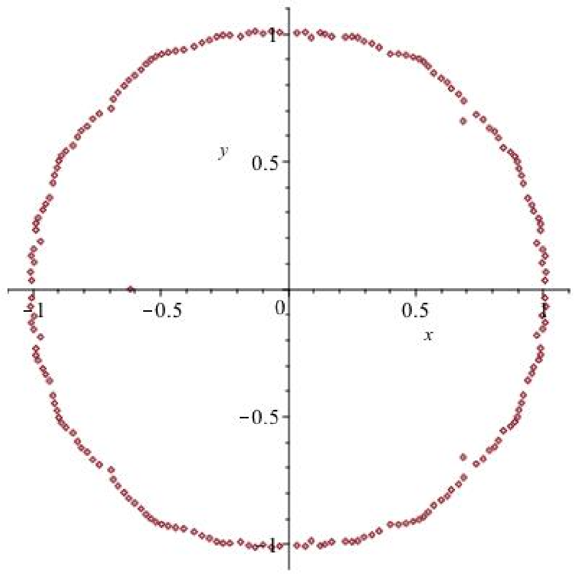


Figure 1: The zeros of $P_{201}(z)$

This remarkable theorem can be supplemented by the following well-known result by Erdős and Turán on the equidistribution of the arguments of the zeros of a polynomial; see [11], or [23] for a recent exposition and a new proof.

**Theorem 9** (Erdős and Turán). *Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with $a_0 a_n \neq 0$, and let $\alpha, \beta$ be real numbers such that $0 \leq \alpha < \beta \leq 2\pi$. If $N(\alpha, \beta)$ denotes the number of zeros $z_j$ with $\alpha \leq \arg z_j \leq \beta$, then*

$$\left| \frac{N(\alpha,\beta)}{n} - \frac{\beta-\alpha}{2\pi} \right| < 16\sqrt{\frac{\log P}{n}}, \quad \textit{where} \quad P = \frac{1}{\sqrt{|a_0 a_n|}} \sum_{k=0}^{n} |a_k|. \tag{20}$$

## 3.2 Applications to $P_n(z)$

We now apply these theorems to the polynomials $P_n(z)$. Our first result is an immediate consequence of Lemma 1 and Theorem 6.

**Proposition 10.** *Let $n \geq 1$ and $m \geq 1$ be such that $2^{m-1} \leq n \leq 2^m - 1$. Then the zeros of $P_n(z)$ all satisfy*

$$\frac{1}{m} \leq |z| \leq m. \tag{21}$$

We will see in the next section that in general the lower bound cannot be improved; however, see below for some special cases. We expect that the upper bound can be improved. In any case, we have the following result.

**Proposition 11.** *Every point on the unit circle is a limit point of the zeros of $P_n(z)$.*

*Proof.* For any $n \geq 0$ we consider the reciprocal polynomial

$$P_n^*(z) := z^n P_n(\tfrac{1}{z}) = 1 + z + 2z^2 + z^3 + 3z^4 + \cdots + s(n+1)z^n \tag{22}$$

and the power series

$$f(z) := \lim_{n\to\infty} P_n^*(z) = \sum_{k=0}^{\infty} s(k+1)z^k. \tag{23}$$

To determine the radius of convergence, we use the following bound due to Reznick [19, p. 472]: There is a constant $c > 0$ such that for all $n \geq 1$ we have

$$1 \leq s(n) \leq c \cdot n^\mu, \qquad \text{where} \quad \mu = \log_2\left(\tfrac{1+\sqrt{5}}{2}\right);$$

an asymptotic value for $c$ can also be found in [19]. Now we use the $n$th root test for the power series in (23) and find that

$$1 \leq \sqrt[n]{s(n)} \leq c^{1/n} \cdot n^{\mu/n} \to 1 \quad \text{as} \quad n \to \infty,$$

where the limit is clear if we consider logarithms. Hence the radius of convergence of the power series is 1, and Jentzsch's Theorem 8 shows that every point on the unit circle is a limit point of the zeros of $P_n^*(z)$. The same result is then true for the zeros of $P_n(z)$, which completes the proof. □

As mentioned earlier, the lower bound in (21) is essentially best possible. On the other hand, for certain subscripts $n$, the zeros of $P_n(z)$ can be proven to be much larger.

**Proposition 12.** *For $m \geq 1$, let $\alpha_m$ and $\beta_m$ be the integers defined in (9). Then for all $n = \alpha_m - 1$ and $n = \beta_m - 1$, all the zeros of $P_n(z)$ satisfy $|z| > 1/2$.*

*Proof.* We consider the polynomial $P_n^*(z)$ as defined in (22). Then by Lemma 2, extended with (1), we see that the leading coefficient $s(\alpha_m)$ or $s(\beta_m)$ is greater than, or equal to, all other coefficients. Then by Cauchy's Theorem 7 all the zeros of $P_n^*(z)$ satisfy $|z| < 2$, which implies that for all the zeros of $P_n(z)$ we have $|z| > 1/2$, as claimed. □

The final result in this section is a direct consequence of the theorem of Erdős and Turán.

**Proposition 13.** *The zeros of $P_n(z)$ have uniform angular distribution. More precisely, if $\alpha, \beta \in \mathbb{R}$ are such that $0 \leq \alpha < \beta \leq 2\pi$, and $N(\alpha, \beta)$ denotes the number of zeros $z_j$ with $\alpha \leq \arg z_j \leq \beta$, then*

$$\left|\frac{N(\alpha,\beta)}{n} - \frac{\beta - \alpha}{2\pi}\right| < 16 \cdot \sqrt{\frac{\log 3}{\log 2}} \cdot \sqrt{\frac{1 + \log(n+1)}{n}}. \tag{24}$$

*Proof.* To determine $P$ in (20), we first note that by definition of $P_n(z)$ we have $a_n = 1$ and $a_0 \geq 1$; hence it remains to estimate the relevant sum of coefficients. Given the index $n$, let $m \geq 1$ be such that $2^{m-1} - 1 \leq n \leq 2^m - 2$. Then by (13), and upon subtracting $s(2^m) = 1$ from both sides, we get

$$P \leq \sum_{k=0}^{n} |a_k| = P_n(1) \leq P_{2^m-2}(1) = \frac{1}{2}\left(3^m - 1\right) < \frac{1}{2} \cdot 3^m,$$

so that

$$\log P < -\log 2 + m \log 3. \tag{25}$$

Now, since $2^{m-1} - 1 \leq n$, we have $2^m \leq 2(n+1)$, and thus

$$m \leq \frac{\log(2(n+1))}{\log 2} = 1 + \frac{\log(n+1)}{\log 2}.$$

Substituting this into (25), we get

$$\log P < -\log 2 + \log 3 + \frac{\log 3}{\log 2} \cdot \log(n+1) = \frac{\log 3}{\log 2} \cdot \big(1 + \log(n+1)\big).$$

This, with (20), gives (24). □

In closing we remark that the numerical value of the constant factor on the right of (24) is $16\sqrt{\log 3/\log 2} \simeq 20.14$. Also, (24) becomes meaningful only when $n$ is reasonably large.

# 4 Real zeros of $P_n(z)$

Since the polynomials $P_n(z)$ have only positive coefficients, the only possible real zeros will be negative. In fact, we have the following result.

**Proposition 14.** *For all $n \geq 0$,*

*(a) $P_{2n}(x)$ has no real zeros; in fact, $P_{2n}(x) \geq \frac{3}{4}s(n) + s(n+1)$ for all $x \in \mathbb{R}$.*

*(b) $P_{2n+1}(x)$ is strictly increasing as a function of $x$, and therefore has exactly one real zero.*

*Proof.* From (10) we get $P_n(x^2) = x^2 P_{n-1}(x^2) + s(n+1)$, and substituting this into (11) gives

$$P_{2n}(x) = \big(1 + x + x^2\big) P_{n-1}(x^2) + s(n+1). \tag{26}$$

It is easy to see that $1 + x + x^2 \geq \frac{3}{4}$ and $P_{n-1}(x^2) \geq P_{n-1}(0) = s(n)$ for all $x$. This fact, along with (26), proves (a).

To prove part (b), we first take the derivative of (10) with respect to $z$, and then set $z = x^2$, to obtain

$$P'_n(x^2) = x^2 P'_{n-1}(x^2) + 2P_{n-1}(x^2). \tag{27}$$

Next, taking the derivative of (12), we get

$$P'_{2n+1}(x) = 2x^2 P'_{n-1}(x^2) + \big(2x + 2x^2\big) P'_n(x^2) + P_{n-1}(x^2) + P_n(x^2).$$

By substituting (27) into this last identity, we obtain, after some simplification,

$$P'_{2n+1}(x) = 2x^2\big(1 + x + x^2\big) P'_{n-1}(x^2) + \big(1 + 2x + 2x^2\big) P_{n-1}(x^2) + P_n(x^2).$$

We already noted that $1 + x + x^2 > 0$; it is also easy to verify that $1 + 2x + 2x^2 > 0$ for all $x \in \mathbb{R}$. Since all other terms are obviously positive, we have $P'_{2n+1}(x) > 0$ for all $x \in \mathbb{R}$, as desired. $\square$

| $n$ | $-r_n$ | $n$ | $-r_n$ | $n$ | $-r_n$ |
|---|---|---|---|---|---|
| 1 | 1 | 13 | .5607907569 | 25 | .5989653955 |
| 3 | .5698402910 | 15 | .2829027496 | 27 | .4179121334 |
| 5 | .6464478041 | 17 | .6442386664 | 29 | .5308746463 |
| 7 | .3767322046 | 19 | .4753681436 | 31 | .2253056724 |
| 9 | .6494575822 | 21 | .5862078204 | 33 | .6410098261 |
| 11 | .4468280932 | 23 | .3232890249 | 35 | .4907265638 |

Table 2: The real zeros $r_n$ of $P_n(x)$, $1 \le n \le 35$

Proposition 14(b) can be made more precise. For instance, Table 2 indicates that the real zeros of $P_n(x)$, for odd $n \ge 3$, lie strictly between $-1$ and 0. This, and more, is shown in the next result.

**Proposition 15.** *For $n \ge 1$, let $r_{2n+1}$ be the unique real zero of $P_{2n+1}(z)$; then $-1 < r_{2n+1} < 0$. Furthermore, if $m$ is such that $2^{m-1} + 1 \le 2n + 1 \le 2^m - 1$, then:*

*(a) If $n$ is odd, then*

$$-\frac{s(n+1)}{s(n)} < r_{2n+1} \le -\frac{1}{m}, \tag{28}$$

*and, if $n \equiv 3 \pmod 4$, then $-1/2 < r_{2n+1} \le -1/m$.*

*(b) If $n$ is even, then*

$$-\frac{s(2n+2)}{s(2n+1)} < r_{2n+1} \le -\frac{1}{m} \qquad (n \equiv 0 \pmod 4), \tag{29}$$

$$-\frac{s(n)}{s(n-1)} < r_{2n+1} \le -\frac{1}{m} \qquad (n \equiv 2 \pmod 4). \tag{30}$$

Before proving this result, we derive a consequence that explains the observation from Table 2 that the zeros $r_n$ are relatively small and decreasing in absolute value for $n = 7, 15, 31$. This becomes more pronounced when we compute $r_{63} \simeq -.1863219404$ and $r_{127} \simeq -.1583809009$. The following result partially explains these observations.

**Corollary 16.** *For any integer $m \geq 2$ we have*

$$-\frac{1}{m-1} < r_{2^m-1} \leq -\frac{1}{m}. \tag{31}$$

*Proof.* We use Proposition 15(a) with $n = 2^{m-1}-1$. Then by (5) and (6) we have $s(n+1) = 1$ and $s(n) = s(2^{m-2}+1) = m-1$, which gives the left inequality of (31). The right inequality remains the same as in (28). □

Corollary 16 shows that Proposition 10, in particular the left inequality, is best possible. It also shows that certain polynomials $P_n(z)$ can have zeros that are arbitrarily close to the origin. This does not contradict Proposition 11.

*Proof of Proposition 15.* By the second identity in (14) we have $P_{2n+1}(-1) < 0$, while by definition we have $P_{2n+1}(0) = s(2n+2) > 0$. Hence the unique zero $r_{2n+1}$ must lie between $-1$ and 0, as claimed. In fact, by Proposition 10, $r_{2n+1}$ cannot be greater than $-1/n$, which proves the right-hand inequalities in (28)–(30).

For the remainder of the proof it will be convenient to deal with the slightly modified polynomial $Q_{2n+1}(x) := -P_{2n+1}(-x)$, which remains monic but changes the signs of the real zeros. We have, therefore,

$$Q_{2n+1}(x) = x^{2n+1} - x^{2n} + 2x^{2n-1} - \cdots + s(2n+1)x - s(2n+2). \tag{32}$$

To prove part (a), we rewrite (32) as

$$\begin{aligned} Q_{2n+1}(x) = x^{2n+1} + \sum_{j=1}^{n} \big(s(2j+1) - s(2j)x\big) \cdot x^{2n+1-2j} \\ - s(2n)x^2 + s(2n+1)x - s(2n+2). \end{aligned} \tag{33}$$

Since $s(2j+1) > s(2j)$ for all $j \geq 1$, all summands in the first line of (33) are positive for $0 < x < 1$. The last three summands taken together are nonnegative exactly when

$$x^2 - \frac{s(2n+1)}{s(2n)}x + \frac{s(2n+2)}{s(2n)} \leq 0.$$

With (1), this quadratic inequality can be rewritten as

$$x^2 - \left(1 + \frac{s(n+1)}{s(n)}\right)x + \frac{s(n+1)}{s(n)} \leq 0$$

and be factored as

$$(x-1)\cdot\left(x-\frac{s(n+1)}{s(n)}\right)\le 0. \tag{34}$$

Since $n$ is odd, $s(n+1)/s(n)<1$, and with $x=s(n+1)/s(n)$ the left of (34) obviously vanishes, and by (33) we have $Q_{2n+1}(s(n+1)/s(n))>0$, while $Q_{2n+1}(0)<0$. This proves the left inequality of (28).

For the final statement of part (a) we set $n=4k+3$ and obtain with (1),

$$\frac{s(n+1)}{s(n)}=\frac{s(4k+4)}{s(4k+3)}=\frac{s(2k+2)}{s(2k+1)+s(2k+2)}<\frac{1}{2},$$

where the inequality is easy to verify, using the fact that $s(2k+1)>s(2k+2)$.

To prove part (b), we modify (33) and obtain

$$\begin{aligned} Q_{2n+1}(x) &= x^{2n+1}+\sum_{j=1}^{n-1}\big(s(2j+1)-s(2j)x\big)\cdot x^{2n+1-2j} \\ &\quad+\big(-s(2n-2)x^2+s(2n-1)x-s(2n)\big)\,x^2+\big(s(2n+1)x-s(2n+2)\big). \end{aligned} \tag{35}$$

We proceed as in part (a) and note that all summands in the first line of (35) are positive. Next, in analogy to (34) we can see that

$$-s(2n-2)x^2+s(2n-1)x-s(2n)\ge 0$$

if and only if

$$(x-1)\cdot\left(x-\frac{s(n)}{s(n-1)}\right)\le 0, \tag{36}$$

which is the case when $s(n)/s(n-1)\le x\le 1$; here we note that $s(n)/s(n-1)<1$ since $n$ is even. For the final term in (35) we have $s(2n+1)x-s(2n+2)\ge 0$ if and only if $x\ge s(2n+2)/s(2n+1)$, which is also less than 1. So altogether, by (35) we have $Q_{2n+1}(x_0)>0$, while again $Q_{2n+1}(0)<0$, where

$$x_0:=\max\left\{\frac{s(n)}{s(n-1)},\frac{s(2n+2)}{s(2n+1)}\right\},\quad (n\text{ even}). \tag{37}$$

To complete the proof, we determine which one of the two elements is larger. First, when $n\equiv 0 \pmod 4$, we set $n=4k$ and with (1) we obtain

$$\begin{aligned} \frac{s(n)}{s(n-1)} &= \frac{s(4k)}{s(4k-1)}=\frac{s(2k)}{s(2k-1)+s(2k)}<\frac{1}{2}, \\ \frac{s(2n+2)}{s(2n+1)} &= \frac{s(8k+2)}{s(8k+1)}=\frac{s(4k+1)}{s(4k)+s(4k+1)}=\frac{s(2k)+s(2k+1)}{2s(2k)+s(2k+1)}>\frac{1}{2}, \end{aligned}$$

where the inequalities on the right are easy to verify, using the fact that odd-index Stern numbers are larger than their neighbors. These inequalities prove (29).

When $n \equiv 2 \pmod 4$, we set $n = 4k+2$, obtaining

$$\frac{s(n)}{s(n-1)} = \frac{s(2k+1)}{s(2k)+s(2k+1)} = \frac{s(k)+s(k+1)}{2s(k)+s(k+1)},$$
$$\frac{s(2n+2)}{s(2n+1)} = \frac{s(4k+3)}{s(4k+2)+s(4k+3)} = \frac{s(k)+2s(k+1)}{2s(2k)+3s(2k+1)}.$$

It is now easy to verify that

$$\frac{s(k)+s(k+1)}{2s(k)+s(k+1)} > \frac{s(k)+2s(k+1)}{2s(2k)+3s(2k+1)},$$

for instance by multiplying both sides by the denominators and then simplifying. This shows that the first element in (37) is maximal, which proves (30); the proof is now complete. □

*Remark* 17. Based on Table 2 and the second part of Proposition 15(a), it may be tempting to conjecture that $r_n > -1/2$ for all $n \equiv 3 \pmod 4$, $n > 3$, and $r_n < -1/2$ for all $n \equiv 1 \pmod 4$. However, this is not true, as the counterexamples in Table 3 show. In the latter case, all counterexamples with $n < 2000$ are shown, while in the former case, there are 8 more up to $n = 2000$.

| $n \equiv 3 \pmod 4$ | $r_n < -1/2$ | $n \equiv 1 \pmod 4$ | $r_n > -1/2$ |
|---|---|---|---|
| 67 | −.5002864226 | 509 | −.4954752823 |
| 131 | −.5067977789 | 765 | −.4975876362 |
| 259 | −.5115134818 | 1021 | −.4920186458 |
| 387 | −.5012417771 | 1277 | −.4983631397 |
| 515 | −.5150841917 | 1533 | −.4936355414 |
| 643 | −.5031636440 | 1789 | −.4968495572 |

Table 3: The first 6 counterexamples related to Remark 17

# 5 Algebraic properties

In this section we will deal with some algebraic properties of the polynomials $P_n(z)$, mainly factorizations involving cyclotomic polynomials, and an identity involving resultants.

## 5.1 Factors of certain differences

Computations show that for all $n \leq 2000$ the polynomials $P_n(z)$ are irreducible, with two exceptions which we will address later in Subsection 5.3. We also saw in the previous two sections that in spite of some general results, the zero distribution of $P_n(z)$ tends to be quite irregular.

It is therefore somewhat surprising that certain infinite classes of differences of the polynomials $P_n(z)$ have the greatest possible regularity, by being products of cyclotomic polynomials. Computations show, for instance,

$$\begin{aligned} P_{30}(z)-P_{14}(z) &= \big(z^8-z^4+1\big)\big(z^4-z^2+1\big)^2\big(z^2-z+1\big)^3\big(z^2+z+1\big)^4 \\ &= \Phi_{24}(z)\cdot\Phi_{12}(z)^2\cdot\Phi_6(z)^3\cdot\Phi_3(z)^4, \end{aligned} \tag{38}$$

where $\Phi_n(z)$ is the $n$th cyclotomic polynomial, a monic polynomial with integer coefficients that can be defined by

$$\Phi_n(z) = \prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}} \big(z-e^{2\pi ik/n}\big). \tag{39}$$

Among numerous other properties, the cyclotomic polynomials satisfy

$$\Phi_p(z)=\sum_{j=0}^{p-1} z^j, \qquad \Phi_{2p}(z)=\sum_{j=0}^{p-1}(-z)^j \quad (p\ge 3 \text{ prime}), \tag{40}$$

and if $p$ is an arbitrary prime with $p\nmid r$, then

$$\Phi_{rp^m}(z)=\Phi_{rp}\big(z^{p^{m-1}}\big). \tag{41}$$

We are now ready to state the first result of this section.

**Proposition 18.** *For any integer $m\ge 1$ we have*

$$P_{2^{m+1}-2}(z)-P_{2^m-2}(z)=\prod_{j=0}^{m-1}\Phi_{3\cdot 2^j}(z)^{m-j}, \tag{42}$$

$$P_{2^{m+1}-1}(z)-P_{2^m-1}(z)=z\prod_{j=0}^{m-1}\Phi_{3\cdot 2^j}(z)^{m-j}. \tag{43}$$

Clearly, (38) is the case $m=4$ of (42). Proposition 18, in turn, is a special case of a more general class of identities. In order to simplify notation we set, for integers $\nu\ge 0$ and $m\ge 1$,

$$a(\nu,m) := (3\cdot 2^\nu+1)\cdot 2^{m-1}-2, \qquad b(\nu,m) := (3\cdot 2^\nu-1)\cdot 2^{m-1}-2. \tag{44}$$

**Proposition 19.** *For any pair of integers $\nu\ge 0$ and $m\ge 1$ we have*

$$P_{a(\nu,m)}(z)-P_{b(\nu,m)}(z)=\Big(P_{a(\nu,1)}\big(z^{2^{m-1}}\big)-P_{b(\nu,1)}\big(z^{2^{m-1}}\big)\Big)\prod_{j=0}^{m-2}\Phi_{3\cdot 2^j}(z)^{m-1-j}, \tag{45}$$

*and*

$$P_{a(\nu,m)+1}(z)-P_{b(\nu,m)+1}(z)=z\left(P_{a(\nu,m)}(z)-P_{b(\nu,m)}(z)\right). \tag{46}$$

*Proof.* We fix an integer $\nu \geq 0$ and proceed by induction on $m$. When $m = 1$, the two sides of (45) are identical since the empty product is 1 by convention. For the induction hypothesis, suppose that (45) holds for a certain $m$. From the definitions in (44) we obtain

$$a(\nu, m+1) = 2\big(a(\nu, m) + 1\big), \qquad b(\nu, m+1) = 2\big(b(\nu, m) + 1\big). \tag{47}$$

We recall (11), namely

$$P_{2n}(z) = (1+z) \cdot P_{n-1}(z^2) + P_n(z^2),$$

and set $n = a(\nu, m) + 1$ and $n = b(\nu, m) + 1$, respectively. Using (47) and subtracting the two identities thus obtained, we get

$$\begin{aligned} P_{a(\nu,m+1)}(z) - P_{b(\nu,m+1)}(z) = (1+z)\left(P_{a(\nu,m)}(z^2) - P_{b(\nu,m)}(z^2)\right) \\ + P_{a(\nu,m)+1}(z^2) - P_{b(\nu,m)+1}(z^2). \end{aligned} \tag{48}$$

With (10) we obtain

$$\begin{aligned} P_{a(\nu,m)+1}(z^2) - P_{b(\nu,m)+1}(z^2) = z^2\left(P_{a(\nu,m)}(z^2) - P_{b(\nu,m)}(z^2)\right) \\ + s(a(\nu, m) + 2) - s(b(\nu, m) + 2). \end{aligned} \tag{49}$$

Since

$$a(\nu, m) + 2 = (3 \cdot 2^\nu + 1) \cdot 2^{m-1}, \qquad b(\nu, m) + 2 = (3 \cdot 2^\nu - 1) \cdot 2^{m-1},$$

with (1) and (7) we see that the two Stern numbers at the right of (49) are the same. Hence (49) and (48) give

$$P_{a(\nu,m+1)}(z) - P_{b(\nu,m+1)}(z) = \left(1 + z + z^2\right)\left(P_{a(\nu,m)}(z^2) - P_{b(\nu,m)}(z^2)\right). \tag{50}$$

Now, using (45) as induction hypothesis, we see that one factor on the right of (50) will be

$$P_{a(\nu,1)}\big(z^{2^m}\big) - P_{b(\nu,1)}\big(z^{2^m}\big),$$

and the remaining factors are

$$\left(1 + z + z^2\right) \cdot \prod_{j=0}^{m-2} \Phi_{3\cdot 2^j}(z^2)^{m-1-j}. \tag{51}$$

By (41) with $r = 3$ and $p = 2$ we have

$$\Phi_{3\cdot 2^j}(z^2) = \Phi_{3\cdot 2}\big((z^2)^{2^{j-1}}\big) = \Phi_{3\cdot 2}\big(z^{2^j}\big) = \Phi_{3\cdot 2^{j+1}}(z),$$

and since $1 + z + z^2 = \Phi_3(z)$, the product (51) becomes

$$\Phi_3(z) \cdot \prod_{j=0}^{m-2} \Phi_{3\cdot 2^{j+1}}(z)^{m-1-j} = \prod_{j=0}^{m-1} \Phi_{3\cdot 2^j}(z)^{m-j}.$$

Thus we have shown that (45) holds for $m + 1$ in place of $m$, which completes the proof of (45) by induction.

Finally, (46) is just (49) with $z$ in place of $z^2$, if we recall that the Stern numbers on the right cancel each other. This completes the proof of Proposition 19. □

To make (45) more explicit for small values of $\nu$, we set

$$f_\nu(z) := P_{a(\nu,1)}(z) - P_{b(\nu,1)}(z), \qquad \nu \geq 0, \tag{52}$$

and display the first three of these polynomials in Table 4. The case $\nu = 0$ leads to Proposition 18.

| $\nu$ | $f_\nu(z)$ |
|---|---|
| 0 | $z^2 + z + 1$ |
| 1 | $z^5 + z^4 + z^3 + z + 1$ |
| 2 | $z^{11} + z^{10} + z^9 + z^7 + z^6 - z^4 + z^3 + 2z^2 + z - 1$ |

Table 4: $f_\nu(z)$ for $\nu = 0, 1, 2$

*Proof of Proposition 18.* According to (45) and (52) we need to evaluate

$$f_0\big(z^{2^{m-1}}\big) = \Phi_3\big(z^{2^{m-1}}\big).$$

We claim that

$$\Phi_3\big(z^{2^{m-1}}\big) = \prod_{j=0}^{m-1} \Phi_{3\cdot 2^j}(z), \qquad m \geq 1. \tag{53}$$

This can be shown either by an easy induction based on the factorization $z^4 + z^2 + 1 = (z^2 + z + 1)(z^2 - z + 1)$, or we can write (53) explicitly as

$$z^{2^m} + z^{2^{m-1}} + 1 = \big(z^2 + z + 1\big)\big(z^2 - z + 1\big)\big(z^4 - z^2 + 1\big) \cdots \big(z^{2^{m-1}} + z^{2^{m-2}} + 1\big)$$

and verify it by "telescoping" the right-hand side.

The identity (42) now immediately follows from multiplying the product on the right of (45) with that on the right of (53). Finally, the identity (43) is an immediate consequence of (46) and (42). □

## 5.2 A resultant identity

The resultant of two polynomials is an important expression, with numerous applications in algebra and number theory. Given two polynomials

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, \quad g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0 \tag{54}$$

$(a_n \neq 0, b_m \neq 0)$, the resultant of $f$ and $g$ can be defined by

$$\mathrm{Res}(f, g) = a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} \big(\alpha_i - \beta_j\big),$$

where $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_m$ are the zeros of $f$ and $g$, respectively. An equivalent definition is by way of the Sylvester determinant, the determinant of a certain $(m+n) \times (m+n)$ matrix which has the coefficients of $f$ and $g$ as entries. The resultant $\mathrm{Res}(f, g)$ is therefore a rational integer if $f$ and $g$ have integer coefficients.

We can now state and prove the following result.

**Proposition 20.** *For any integer $n \geq 0$ we have*

$$\mathrm{Res}\big(P_n(x), P_{n+1}(x)\big) = s(n+2)^n. \tag{55}$$

*Proof.* From among the numerous known properties of the resultant, we require just two. First if $a$ is a constant and $g$ is a monic polynomial, then

$$\mathrm{Res}(a, g) = \mathrm{Res}(g, a) = a^{\deg g}. \tag{56}$$

Second, if $f$ and $g$ are polynomials as given in (53), and if

$$f(x) = q(x) \cdot g(x) + r(x), \tag{57}$$

with polynomials $q$, $r$, and with $\nu := \deg r$, then

$$\mathrm{Res}(g, f) = b_m^{n-\nu}\mathrm{Res}(g, r). \tag{58}$$

A proof of this property can be found, for instance, in [18, p. 58]. We apply this to (10), namely $P_{n+1}(x) = xP_n(x) + s(n+2)$; so by (57) and (58), followed by (56), we get

$$\mathrm{Res}\big(P_n(x), P_{n+1}\big) = \mathrm{Res}\big(P_n(x), s(n+2)\big) = s(n+2)^n,$$

where we have used the fact that $P_n(x)$ is a monic polynomial of degree $n$. □

We were unable to identify or prove general identities between other pairs of polynomials $P_n(x)$, or identities for the discriminant of $P_n(x)$.

## 5.3 Factors of $P_n(z)$

We close this section with some remarks and observations on possible cyclotomic factors of the polynomials $P_n(z)$ themselves. Computations show that $P_{62}(z)$ is divisible by $\Phi_7(z)$ and $P_{1022}(z)$ is divisible by $\Phi_{11}(z)$. Noticing the close relationship between the corresponding subscripts, we were able to find a third factor; see Table 5.

To analyze this situation, we let $m = p-1$, where $p$ is an odd prime. Then $\Phi_p(z)$ dividing $P_n(z)$ is equivalent to $P_n(z) \equiv 0 \pmod{z^p - 1}$, where $n = 2^{p-1} - 2$. In other words, if we reduce all exponents of $z$ in $P_n(z)$ modulo $p$, then we should get a multiple of $\Phi_p(z)$. In the case $p = 7$, $n = 62$, the reduced polynomial is then

$$\begin{aligned} P_{62}(z) &\equiv \sum_{j=1}^{7} \big(s(j) + s(7+j) + \cdots + s(56+j)\big) z^{7-j} \pmod{z^p - 1} \\ &= 52z^6 + 52z^5 + \cdots + 52z + 52, \end{aligned} \tag{59}$$

| $m$ | $n = 2^m - 2$ | Factor of $P_n(z)$ |
|---|---|---|
| 6 | 62 | $\Phi_7(z) = (z^7 - 1)/(z - 1)$ |
| 10 | 1022 | $\Phi_{11}(z) = (z^{11} - 1)/(z - 1)$ |
| 12 | 4094 | $\Phi_{13}(z) = (z^{13} - 1)/(z - 1)$ |

Table 5: All known factors of $P_n(z)$

which is indeed a multiple of $\Phi_7(z)$. This approach works because all seven sums of Stern numbers in arithmetic progressions in the first row of (59) are identical. It is not difficult to generalize this to arbitrary odd primes $p$.

**Proposition 21.** *For an odd prime $p$, let $n = 2^{p-1} - 2$. Then $\Phi_p(z)$ divides $P_n(z)$ if and only if the $p$ identities*

$$\sum_{k=0}^{q(p)-1} s(kp + j) = \frac{3^{p-1} - 1}{2p}, \qquad j = 1, 2, \ldots, p, \tag{60}$$

*all hold, where $q(p) := (2^{p-1} - 1)/p$ is the base-2 Fermat quotient.*

Before we prove this proposition, we note that by Fermat's little theorem both the Fermat quotient $q(p)$ and the right-hand side of (60) are integers. This last quotient is in fact 1/2 the base-3 Fermat quotient. When $p = 7$, we have $q(7) = 9$ and $(3^{p-1} - 1)/2p = 52$, consistent with (59).

It is rather surprising that the very strong condition (60) holds for the primes $p = 7$, 11, and 13. For $p = 5$, on the other hand, the five sums are 8, 6, 8, 8, and 10. Hence (60) fails for $j = 2$ and $j = 5$ and accordingly, $P_{14}(z)$ is not divisible by $\Phi_5(z)$.

We have used (60) as a criterion to exclude the cases $p = 17$, 19, 23, and 29. Obviously it suffices to find just one $j$ for which (60) fails.

*Proof of Proposition 21 (Sketch).* By reducing the exponents of $z$ modulo $p$ in the polynomial $P_n(z)$, we see that each sum on the left of (60) is a coefficient of $z^{p-j}$, $1 \leq j \leq p$. There are a total of $n + 1 = 2^{p-1} - 1$ coefficients, broken into $p$ sums, each with $(n + 1)/p = q(p)$ summands. By (12), the sum of all coefficients of $P_n(z)$ is

$$P_{2^{p-1}-2}(1) = \frac{1}{2}\left(3^{p-1} + 1\right) - 1.$$

So for all the sums in (60) to be equal, the right-hand side has to be $(3^{p-1} - 1)/2p$, as claimed. □

# 6 The polynomials $P_n(z)$ modulo 2

In this section we consider the polynomials $\overline{P}_n(z)$ that are formed from $P_n(z)$ by taking the smallest nonnegative residues modulo 2 of their coefficients. Strictly speaking, this does not rely on the deeper properties of the Stern sequence since we have the well-known congruence

$$s(k) \equiv \begin{cases} 0 \pmod 2, & \text{if } 3 \mid k; \\ 1 \pmod 2, & \text{if } 3 \nmid k. \end{cases} \tag{61}$$

This can be obtained by an easy induction, using (1).

## 6.1 Divisibility and irreducibility

In spite of the easy coefficient sequence, the polynomials $\overline{P}_n(z)$ have some interesting properties, and they behave very differently according as $3 \mid n$ or $3 \nmid n$. This difference is illustrated in Table 6.

| $n$ | $\overline{P}_n(z)$ factored |
|---|---|
| 9 | $z^9 + z^8 + z^6 + z^5 + z^3 + z^2 + 1$ |
| 10 | $\Phi_{12}(z) \cdot \Phi_6(z) \cdot \Phi_4(z) \cdot \Phi_2(z)^2$ |
| 11 | $\Phi_{12}(z) \cdot \Phi_6(z) \cdot \Phi_4(z) \cdot \Phi_2(z)^2 \cdot z$ |

Table 6: $\overline{P}_n(z)$ for $n = 9, 10, 11$

We begin with the two cases that are quite straightforward.

**Proposition 22.** *For $k \geq 2$ we have*

$$\overline{P}_{3k+1}(z) = (z+1)^2 \prod_{\substack{d \mid 3k+3 \\ d>3}} \Phi_d(z), \quad \overline{P}_{3k+2}(z) = z(z+1)^2 \prod_{\substack{d \mid 3k+3 \\ d>3}} \Phi_d(z). \tag{62}$$

*Proof.* By (3) and (61) we have

$$\begin{aligned} \overline{P}_{3k+1}(z) &= (z+1)(z^{3k} + z^{3(k-1)} + \cdots z^3 + 1) \\ &= (z+1) \cdot \frac{z^{3k+3} - 1}{z^3 - 1}. \end{aligned}$$

Now we use the well-known fact that

$$z^{3k+3} - 1 = \prod_{d \mid 3k+3} \Phi_d(z). \tag{63}$$

Dividing both sides by $z^3 - 1 = \Phi_1(z)\Phi_3(z)$ removes the cases $d = 1, 3$ from (63), and we also take out the factor $\Phi_2(z) = z + 1$ and obtain the first identity in (62). The case $3k+2$ is completely analogous. Alternatively, we could also use (9). □

In contrast to Proposition 22, when $3 \mid n$ we have the following result.

**Proposition 23.** *For any $k \geq 1$, the polynomial $\overline{P}_{3k}(z)$ is irreducible over $\mathbb{Q}$.*

We will prove this by applying a recent theorem of Sawin, Shusterman, and Stoll [21]. To set the stage, we recall that the reciprocal of a polynomial $h(z)$ is defined by $h^*(z) = z^{\deg h} h(1/z)$. Then a polynomial $h$ is called *reciprocal* if $h(z) = \pm h^*(z)$. Furthermore, if $h \in \mathbb{R}[z]$ is given by $h(z) = a_n z^n + \cdots + a_1 z + a_0$, we define the norm $\|h\|$ as the $\ell_2$-norm of its coefficient sequence, namely

$$\|h\| = \left(a_n^2 + \cdots + a_1^2 + a_0^2\right)^{1/2}.$$

Finally, we require the rather technical condition of a robust pair of polynomials $(f(z), g(z)) \in \mathbb{Z}[z]^2$ with $f(0)g(0) \neq 0$. Such a pair is called *robust* if for every factorization

$$f_0(z)g_0(z) = f(z)g(z),$$

one of several conditions is satisfied; see [12, 21]. In what follows, we need only one of these conditions, namely

$$\|f_0\|^2 + \|g_0\|^2 > \|f\|^2 + \|g\|^2. \tag{64}$$

We can now state the relevant result from [21], as quoted in [12].

**Theorem 24** (Sawin, Shusterman, and Stoll)**.** *Let $f(z), g(z)$ be polynomials with integer coefficients such that $f(0)g(0) \neq 0$, and suppose that $(f^*, g)$ is a robust pair, $f^*(z) \neq \pm g(z)$ and $\gcd_{\mathbb{Z}[z]}(f, g) = 1$. If*

$$n > (1 + \deg f + \deg g) \cdot 2^{\|f\|^2 + \|g\|^2} - \deg f,$$

*then the non-reciprocal part of $f(z)z^n + g(z)$ is irreducible over $\mathbb{Z}[z]$.*

This theorem is just one of a class of similar irreducibility results, going back to Schinzel in the 1960s; see [12, 21] for further remarks and references.

*Proof of Proposition 23.* If $3 \mid n$, then by (3) and (61) we have

$$\overline{P}_n(z) = z^n + z^{n-1} + z^{n-3} + \cdots + z^3 + z^2 + 1,$$

and it is easy to verify that

$$\left(z^3 - 1\right)\overline{P}_n(z) = (z+1)z^{n+2} - \left(z^2 + 1\right). \tag{65}$$

Therefore the reciprocal part of the quadrinomial on the right of (65) is $z^3 - 1$, while the non-reciprocal part is $\overline{P}_n(z)$.

We now apply Theorem 24 with $f(z) = z + 1$ and $g(z) = -z^2 - 1$. Then $f^*(z) = f(z)$, $f(0)g(0) \neq 0$, $f^*(z) \neq \pm g(z)$, and $\gcd_{\mathbb{Z}[z]}(f, g) = 1$. Furthermore, $\|f\|^2 = \|g\|^2 = 2$, and the only other essential factorization of $f(z)g(z)$ is

$$f_0(z)g_0(z) = 1 \cdot \left(-z^3 - z^2 - z - 1\right),$$

and we have

$$\|f_0\|^2 + \|g_0\|^2 = 1 + 4 > 2 + 2 = \|f\|^2 + \|g\|^2.$$

Hence by Theorem 24 the non-reciprocal part of the polynomial in (65), namely $\overline{P}_n(z)$, is irreducible whenever

$$n + 2 > (1 + 1 + 2) \cdot 2^{2+2} - 1 = 4 \cdot 2^4 - 1 = 63,$$

which proves our proposition for $n \geq 63$ since in our case we have $3 \mid n$. For smaller $n$ we used computer algebra to show that $\overline{P}_n(z)$ is irreducible. □

## 6.2 Zero distribution

When $n \equiv 1, 2 \pmod 3$, by (62) and the definition (39) of cyclotomic polynomials, the zeros of $\overline{P}_n(z)$ are completely determined. The case $3|n$ is more challenging. We begin with a first rough estimate of the moduli of all the zeros, which will be required to obtain sharper results.

**Lemma 25.** *For all $k \geq 1$, the zeros of $\overline{P}_{3k}(z)$ lie in the annulus $1/2 \leq |z| \leq 2$.*

*Proof.* The upper bound follows immediately from Cauchy's Theorem 7 since all coefficients are 0 or 1. The lower bound follows similarly by considering the reciprocal $z^{3k}\overline{P}_{3k}(1/z)$. □

For the next results it is convenient to consider the polynomial on the right of (65), and we set

$$f_n(z) := (z + 1)z^{n+2} - (z^2 + 1). \tag{66}$$

Although only the case $3|n$ is relevant with regards to $\overline{P}_n(z)$, we may as well consider the polynomials $f_n(z)$ for all integers $n$. The conclusion of Lemma 25 also holds for all these polynomials.

**Proposition 26.** *For all $n \geq 1$, the zeros of $f_n(z)$ satisfy*

$$1 - \frac{\log(n + 2)}{n + 2} < |z| < 1 + \frac{\log(n + 2)}{n + 2}, \tag{67}$$

*where for odd $n$ the upper bound holds only for $n \geq 17$.*

*Proof.* (1) When $r := |z| < 1$, then by (66) we get

$$|f_n(z)| \geq \left|z^2 + 1\right| - |z + 1| \cdot r^{n+2}. \tag{68}$$

Using the triangle inequality again, we get $|z^2 + 1| \geq 1 - r^2$ and $|z + 1| \leq 1 + r$. If we make the assumption that $r \leq 1 - \log(n + 2)/(n + 2)$, then (68) gives

$$\begin{aligned} |f_n(z)| \geq 1 - &\left(1 - 2\frac{\log(n + 2)}{n + 2} + \left(\frac{\log(n + 2)}{n + 2}\right)^2\right) \\ &- \left(2 + \frac{\log(n + 2)}{n + 2}\right)\left(1 - \frac{\log(n + 2)}{n + 2}\right)^{n+2}. \end{aligned}$$

The power in the last term is less than $\exp(-\log(n+2)) = 1/(n+2)$, and thus

$$\begin{aligned}|f_n(z)| &> 2\frac{\log(n+2)}{n+2} - \left(\frac{\log(n+2)}{n+2}\right)^2 - \frac{2}{n+2} - \frac{\log(n+2)}{(n+2)^2} \\ &= \frac{\log(n+2)-2}{n+2} + \frac{\log(n+2)}{n+2}\left(1 - \frac{\log(n+2)}{n+2} - \frac{1}{n+2}\right).\end{aligned}$$

Now $\log(n+2) - 2 > 0$ for $n \geq 6$, and it is easy to verify that the expression in large parentheses on the right is positive for all $n \geq 1$. Hence $|f_n(z)| > 0$ whenever $r \leq 1 - \log(n+2)/(n+2)$, which proves the lower bound in (67) for $n \geq 6$. The cases $1 \leq n \leq 5$ can be verified by computations.

(2) When $r = |z| > 1$, then from (66) we get

$$|f_n(z)| \geq |z+1| \cdot r^{n+2} - \left|z^2+1\right|. \tag{69}$$

We also have $|z+1| \geq r-1$ and $|z^2+1| \leq r^2+1 \leq 5$ since by Lemma 25 we may restrict our attention to $r \leq 2$. Now, if $r \geq 1 + \log(n+2)/(n+2)$, then by (69) we get

$$|f_n(z)| \geq \frac{\log(n+2)}{n+2} \cdot \left(1 + \frac{\log(n+2)}{n+2}\right)^{n+2} - 5. \tag{70}$$

It can be shown by analytical methods, complemented by some computations for smaller cases, that the sequence

$$c_n := \frac{1}{n+2} \cdot \left(1 + \frac{\log(n+2)}{n+2}\right)^{n+2}$$

is increasing for $n \geq 7$. Furthermore, for $n \geq 200$ we have $c_n \geq c_{200} > 0.9337$. So, if we assume that $r \geq 1 + \log(n+2)/(n+2)$, then by (70) we have

$$|f_n(z)| \geq \log(n+2) \cdot 0.9337 - 5 > 0$$

when $n \geq 210$. The cases $1 \leq n \leq 209$ can again be checked by direct computation. □

*Remark* 27. With additional effort, Proposition 26 can be improved as follows:

(a) When $0 \leq |\arg z| \leq 2\pi/3$, then the upper bound in (67) can be decreased to 1, with equality only for $z = 1$ and for $z = e^{\pm 2\pi i/3}$ when $3|n$.

(b) When $2\pi/3 \leq |\arg z| \leq \pi$, then the lower bound in (67) can be increased to 1.

We conclude this section with a result on real zeros.

**Proposition 28.** *For any $n \geq 0$, the polynomial $f_n(z)$ has*

*(a) exactly one positive real zero, $z_1 = 1$;*

*(b) exactly one negative real zero* $-1 - \frac{\log n+1}{n+1} < z_2 < -1$ *when* $n$ *is odd, where the left inequality holds for* $n \geq 17$*;*

*(c) no negative real zeros when* $n$ *is even.*

*As a consequence, when* $k$ *is odd,* $\overline{P}_{3k}(z)$ *has exactly the one real zero given by (b) with* $n = 3k$*, and when* $k$ *is even, it has no real zeros.*

*Proof.* (a) $z_1 = 1$ is obviously a zero. But since the sequence of coefficients of $f_n(z) = z^{n+3} + z^{n+2} - z^2 - 1$ has exactly one sign change, by Descartes's Rule of Signs (see, e.g., [13, pp. 439–443]) there can be only this one positive zero.

(b) For odd $n$, $f_n(z)$ is positive for sufficiently large negative $z$, while $f_n(-1) = -1$. Hence there is a zero $z_2 < -1$, and by (67) it must lie to the right of $-1-\log(n+1)/(n+1)$, for $n \geq 17$. Since $f_n(-z) = z^{n+3} - z^{n+2} - z^2 - 1$, there is just one sign change in the coefficient sequence of $f_n(-z)$, and again by Descartes's Rule of Signs there can be no further negative real zeros.

(c) When $n$ is even, we consider the derivative

$$f_n'(z) = z \cdot \left((n+3)z^{n+1} + (n+2)z^n - 2\right), \tag{71}$$

and we set $g_n(z) := (n+3)z^{n+1} + (n+2)z^n$. It is easy to see that, at least for negative $z$, this polynomial has a unique maximum $z_0 := -n(n+2)/((n+1)(n+3))$, and

$$g_n(z_0) = z_0^n \left(-\frac{n(n+2)}{n+1} + n + 2\right) = z_0^n \cdot \frac{n+2}{n+1}.$$

It is easy to verify that

$$\frac{n(n+2)}{(n+1)(n+3)} \cdot \frac{n+2}{n+1} < 1 \quad \text{for all} \quad n \geq 0,$$

and so we have $g_n(z_0) < 1$ and thus $g_n(z_0) - 2 < 0$ for all even $n$. Therefore by (71) we have $f_n'(z) > 0$ for all $z < 0$. Since $f_n(0) = -1$, this means that $f_n(z) < 0$ for all $z \leq 0$, which implies part (c).

Finally, the last statement of the proposition follows from (65) and the fact that $\overline{P}_{3k}(z)$ cannot have positive zeros since it has only nonnegative coefficients. □

# 7 Overpartitions and colored partitions

In this final section we consider the polynomials $P_n(z)$, as defined in (3), in the framework of binary and $b$-ary partitions [4, 8], where $b \geq 2$ is an integer base, and more generally in connection with overpartitions and colored partitions.

### 7.1 Binary and $b$-ary overpartitions

Rødseth and Sellers [20] introduced and studied $b$-ary overpartitions, in analogy to ordinary overpartitions that had been introduced a little earlier by Corteel and Lovejoy [3].

A *$b$-ary overpartition* of an integer $n \geq 1$ is a non-increasing sequence of nonnegative integer powers of $b$ whose sum is $n$, and where the first occurrence of a power $b$ may be overlined. We denote the number of $b$-ary overpartitions by $\overline{S}_b(n)$, which differs from the notation in [20].

**Example 29.** [20, p. 346]. The binary overpartitions of $n = 4$ are $4,\ \overline{4},\ 2+2,\ \overline{2}+2,\ 2+1+1,\ 2+\overline{1}+1,\ \overline{2}+1+1,\ \overline{2}+\overline{1}+1,\ 1+1+1+1,\ \overline{1}+1+1+1.$ Thus $\overline{S}_2(4) = 10$.

As seen in this example, the overlined parts form a $b$-ary partition into distinct parts, while the non-overlined parts form an ordinary $b$-ary partition.

The concept of a $b$-ary overpartition can be restricted in different ways, one of which is to restrict by an integer $\lambda$ the number of times a non-overlined power of $b$ may occur. Then the generating function is

$$\sum_{n=0}^{\infty} \overline{S}_b^{\lambda}(n) q^n = \prod_{j=0}^{\infty} \left(1 + q^{b^j}\right) \left(1 + q^{b^j} + q^{2 \cdot b^j} + \cdots + q^{\lambda \cdot b^j}\right), \tag{72}$$

where $\overline{S}_b^{\lambda}(n)$ is the number of $b$-ary overpartitions of $n$ in which each non-overlined power of $b$ may occur at most $\lambda$ times.

**Example 30.** Let $b = \lambda = 2$. Then (72) becomes

$$1 + 2q + 4q^2 + 5q^3 + 8q^4 + 10q^5 + 13q^6 + 14q^7 + 18q^8 + 21q^9 + 26q^{10} + \cdots \tag{73}$$

Thus, in particular, $\overline{S}_2^2(4) = 8$, which is consistent with Example 29, where all but the last two binary overpartitions are counted by $\overline{S}_2^2(4)$.

In the recent paper [9] we defined the concept of restricted multicolor $b$-ary partitions as a generalization of restricted $b$-ary overpartitions, and further defined polynomial analogues of the relevant partition functions. These polynomials then allowed us to not just count the partitions in question, but to characterize them.

In the special case of restricted $b$-ary overpartitions with $\lambda = 2$, these polynomials specialize as follows. Let $Z = (x, y, z)$ be a triple of variables, and $T = (r, s, t)$ a triple of positive integers. Then, in the notation of [9, Def. 2.4], we define

$$\sum_{n=0}^{\infty} \Omega_{b,T}^{(1,2)}(n; Z) q^n = \prod_{j=0}^{\infty} \left(1 + x^{r^j} q^{b^j}\right) \left(1 + y^{s^j} q^{b^j} + z^{t^j} q^{2 \cdot b^j}\right). \tag{74}$$

Comparing this with (72), we immediately get, for any base $b \geq 2$,

$$\overline{S}_b^2(n) = \Omega_{b,T}^{(1,2)}(n;1,1,1), \qquad n = 0,1,2,\ldots, \tag{75}$$

where the triple $T$ is arbitrary. The following result establishes a connection to the polynomials $P_n(x)$ defined in (3), and gives a combinatorial interpretation.

**Proposition 31.** *For any $n \geq 0$ we have*

$$P_n(x) = \Omega_{2,T}^{(1,2)}(n;x,1,1), \tag{76}$$

*where $T = (2,s,t)$, with $s$ and $t$ arbitrary. Furthermore,*

*(a) $P_n(1)$ is the number of restricted binary overpartitions of $n$, with $\lambda = 2$.*

*(b) The coefficient of $x^j$ in $P_n(x)$ is the number of overlined parts that sum to $j$.*

*Proof.* With $b = r = 2$ and $y = z = 1$, the right-hand side of (74) becomes

$$\prod_{j=0}^{\infty}\left(1 + x^{2^j}q^{2^j}\right)\prod_{j=0}^{\infty}\left(1 + q^{2^j} + q^{2^{j+1}}\right).$$

By the uniqueness of the binary expansion, the first product gives

$$\prod_{j=0}^{\infty}\left(1 + (xq)^{2^j}\right) = \sum_{k=0}^{\infty}(xq)^k = \frac{1}{1-xq}.$$

The identity (76) now follows from Proposition 5 by equating coefficients of $q^n$.

(a) This follows from combining (76) with (75).

(b) When we remove the overlined parts that sum to $j$ from a binary overpartition of $n$, we are left with a restricted binary partition of $n-j$ that can have at most $\lambda = 2$ equal parts. But this is a hyperbinary expansion of $n-j$, and we know (see, e.g., [19, p. 470]) that this number is $s(n-j+1)$. This fact, together with the definition (3), proves part (b). □

## 7.2 Generalizations and colored partitions

By considering (76) in conjunction with (74), it becomes clear that the polynomials $P_n(x)$ can be generalized in several different directions. We could take different bases $b$, or different values of $\lambda$, or both. For instance, when $b = \lambda \geq 2$ and $r = 2$, we get (in a simplified notation)

$$\sum_{n=0}^{\infty} P_n^{(b)}(x)q^n = \prod_{j=0}^{\infty}\left(1 + x^{2^j}q^{b^j}\right)\left(1 + q^{b^j} + q^{2\cdot b^j} + \cdots + q^{b\cdot b^j}\right). \tag{77}$$

Then the polynomials $P_n^{(b)}(x)$ are analogues of $P_n(x)$, with the $b$-generalized Stern numbers (as defined in [5]; see also [8]) in place of the Stern numbers $s(n)$.

Another level of generalization would be to consider polynomial analogues of restricted colored $b$-ary partitions, as investigated in [9]. As a specific example we mention

$$\sum_{n=0}^{\infty} \widetilde{P}_n(x)q^n = \prod_{j=0}^{\infty} \left(1 + x^{2^j} q^{2^j}\right) \left(1 + q^{2^j}\right) \left(1 + q^{2^j}\right) \left(1 + q^{2^j} + q^{2^{j+1}}\right). \tag{78}$$

In analogy to Proposition 31(b), $\widetilde{P}_n(1)$ is the number of ways $n$ can be written as a sum of powers of 2, where each part can be in one of four colors, with the added conditions that each part in the first three colors can occur only once, while a part in the fourth color can occur at most twice.

The generating function (78) also shows that the coefficients of $\widetilde{P}_n(x)$ can be obtained by starting with the Stern sequence $s(n)$ and forming the sequence of partial sums twice:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 3 | 2 | 3 | 1 | 4 | 3 | ... |
| 1 | 2 | 4 | 5 | 8 | 10 | 13 | 14 | 18 | 21 | ... |
| 1 | 3 | 7 | 12 | 20 | 30 | 43 | 57 | 75 | 96 | ... |

so that

$$\widetilde{P}_n(x) = x^n + 3x^{n-1} + 7x^{n-2} + 12x^{n-3} + 20x^{n-4} + 30x^{n-5} + 43x^{n-6} + \cdots,$$

where we leave the details to the reader.

Each of the sequences of polynomials $P_n^{(b)}(x)$, $\widetilde{P}_n(x)$, or others of this type, would lead to questions and results that are similar in nature to the contents of this paper.

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