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Divisibility of Divisor Functions of Even Perfect Numbers

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Abstract

Let $k > 2$ be a prime such that $2^k - 1$ is a Mersenne prime. Let $n = 2^{\alpha-1}p$, where $\alpha > 1$ and $p < 3 \cdot 2^{\alpha-1} - 1$ is an odd prime. Define $\sigma_k(n)$ to be the sum of the kth powers of the positive divisors of n. Continuing the work of Cai et al. and Jiang, we prove that $n \mid \sigma_k(n)$ if and only if n is an even perfect number other than $2^{k-1}(2^k - 1)$. Furthermore, if $n = 2^{\alpha-1}p^{\beta-1}$ for some $\beta > 1$, then $n \mid \sigma_5(n)$ if and only if n is an even perfect number other than 496.

1 Introduction and main results

For a positive integer n, let $\sigma(n)$ be the sum of the positive divisors of n. We call n *perfect* if $\sigma(n) = 2n$ (sequence A000396 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [11]). Due to the work of Euclid and Euler, it is well-known that an even integer n is perfect if and only if $n = 2^{p-1}(2^p - 1)$, where both p and $2^p - 1$ are primes. A prime of the form $2^p - 1$ is called a *Mersenne* prime. Up to now, fewer than 60 Mersenne primes are known. Two questions are still open: whether there are infinitely many even perfect numbers and whether there exists an odd perfect number, though various progress has been made. For example, Pomerance [6] showed that an odd perfect number must have at least 7 distinct prime factors. Nielsen improved the result by proving that an odd perfect number must have at least 9 distinct prime factors. For related results, see [7, 8].

Meanwhile, mathematicians have generalized the concept of perfect numbers. Pollack and Shevelev [5] introduced k-*near-perfect* numbers. For $k \geq 1$, a k-near-perfect number n is the sum of all of its proper divisors with at most k exceptions. A positive integer n is called *near-perfect* if n is the sum of all but exactly one of its proper divisors (A181595). Pollack and Shevelev showed how to construct near-perfect numbers and established an upper bound of $x^{5/6+o(1)}$ for the number of near-perfect numbers in $[1, x]$ as $x \to \infty$. Li and Liao [4] gave two equivalent conditions of all even near-perfect numbers of the form $2^\alpha p_1 p_2$ and $2^\alpha p_1^2 p_2$, where $\alpha > 0$ and p_1, p_2 are distinct primes. In 2013, Ren and Chen [10] found all near-perfect numbers with two distinct prime factors. Continuing the work, Tang et al. [14] showed that there is no odd near-perfect number with three distinct prime divisors. For other beautiful results on near-perfect numbers and deficient-perfect numbers (A271816, A341475), see [12, 13].

The present paper focuses on another generalization of perfect numbers by connecting an even perfect number n with the divisibility of $\sigma_k(n)$, where $k \geq 1$ and

$$\sigma_k(n) \ := \ \sum_{d|n} d^k.$$

In 2006, Luca and Ferdinands [3] proved that for $k \geq 2$, there are infinitely many n such that $n \mid \sigma_k(n)$. In 2015, Cai et al. [1] proved the following theorem.

Theorem 1. *Let $n = 2^{\alpha-1}p$, where $\alpha > 1$ is an integer and p is an odd prime. If $n \mid \sigma_3(n)$, then n is an even perfect number. The converse is also true for $n \neq 28$.*

Three years later, Jiang [2] improved the theorem as follows.

Theorem 2. *Let $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta > 1$ are integers and p is an odd prime. Then $n \mid \sigma_3(n)$ if and only if n is an even perfect number $\neq 28$.*

These theorems show a beautiful relationship between an even perfect number n and $\sigma_3(n)$. A natural extension is to consider $\sigma_k(n)$ for some other values of k. Unfortunately, Theorem 1 does not hold when $k = 5$ or 7, for example. A quick computer search gives $\sigma_5(22) \equiv 0 \pmod{22}$ and $\sigma_7(86) \equiv 0 \pmod{86}$. However, if we add one more restriction on p, the following theorem holds.

Theorem 3. *Let $k > 2$ be a prime such that $2^k - 1$ is a Mersenne prime. If $n = 2^{\alpha-1}p$, where $\alpha > 1$ and $p < 3 \cdot 2^{\alpha-1} - 1$ is an odd prime. Then $n \mid \sigma_k(n)$ if and only if n is an even perfect number $\neq 2^{k-1}(2^k - 1)$.*

Theorem 3 can be considered a generalization of Theorem 1 as we have a wider range of k with the new restriction on p as a compensation. Interestingly, when $k = 5$, we can generalize Theorem 3 the same way as Jiang generalized Theorem 1.

Theorem 4. *If $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta > 1$ and $p < 3 \cdot 2^{\alpha-1} - 1$ is an odd prime. Then $n \mid \sigma_5(n)$ if and only if n is an even perfect number $\neq 496$.*

Unfortunately, our method is not applicable to other values of k even though computation supports the following conjecture.

Conjecture 5. Let $k > 2$ be a prime such that $2^k - 1$ is a Mersenne prime. If $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta > 1$ and $p < 3 \cdot 2^{\alpha-1} - 1$ is an odd prime. Then $n \mid \sigma_k(n)$ if and only if n is an even perfect number $\neq 2^{k-1}(2^k - 1)$.

Our paper is structured as follows. Section 2 provides several preliminary results that are used repeatedly throughout the paper, Section 3 proves Theorem 3 and Section 4 proves Theorem 4. Since the proof of several claims made in Section 3 and Section 4 are quite technical, we move them to the Appendix for the ease of reading.

2 Preliminaries

Let $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta > 1$ are integers and $p < 3 \cdot 2^{\alpha-1} - 1$ is an odd prime. Let $k > 2$ be a prime such that $2^k - 1$ is a Mersenne prime. We will stick with these notation throughout the paper. If $n \mid \sigma_k(n)$, then

$$\begin{aligned} 2^{\alpha-1}p^{\beta-1} \mid \sigma_k(2^{\alpha-1})\sigma_k(p^{\beta-1}) &= (1 + 2^k + \cdots + 2^{(\alpha-1)k})(1 + p^k + \cdots + p^{(\beta-1)k}) \\ &= \frac{2^{\alpha k} - 1}{2^k - 1} \cdot \frac{p^{\beta k} - 1}{p^k - 1}. \end{aligned}$$

Because $(2, 2^{\alpha k} - 1) = 1$ and $(p, p^{\beta k} - 1) = 1$, it follows that

$$2^{\alpha-1} \text{ divides } \frac{p^{\beta k} - 1}{p^k - 1}, \text{ so } 2^{\alpha} \text{ divides } p^{\beta k} - 1, \tag{1}$$

$$p^{\beta-1} \text{ divides } \frac{2^{\alpha k} - 1}{2^k - 1}. \tag{2}$$

Furthermore, rewrite (1) as

$$2^{\alpha-1} \mid \frac{p^{\beta k} - 1}{p^k - 1} = \frac{(p^k - 1)(p^{k(\beta-1)} + p^{k(\beta-2)} + \cdots + 1)}{p^k - 1} = \sum_{i=0}^{\beta-1} p^{ki}.$$

Since each term is odd and the summation is divisible by 2, we know that $2 \mid \beta$. The following lemma is the key ingredient in the proof of Theorem 3.

Lemma 6. *Let $n = 2^{\alpha-1}(2^k - 1)^{\beta-1}$, where $\alpha, \beta > 1$ are integers. Then $n \nmid \sigma_k(n)$.*

Proof. We use proof by contradiction. Suppose $n \mid \sigma_k(n)$. By (1) and (2), we have

$$2^{\alpha} \mid (2^k - 1)^{\beta k} - 1, \tag{3}$$

$$(2^k - 1)^{\beta} \mid (2^{\alpha k} - 1) = (2^k - 1)((2^k)^{\alpha-1} + \cdots + 1). \tag{4}$$

Write $\alpha = (2^k - 1)^u \alpha_1$ and $\beta = 2^v \beta_1$, where $u \geq 0$, $v \geq 1$ and $(2^k - 1, \alpha_1) = (2, \beta_1) = 1$. By Lemma 13, $\alpha \leq v + k$.

If $u = 0$, we get $\alpha = \alpha_1$. From (4), $\beta = 1$, which contradicts the fact that $2 \mid \beta$.

If $u \geq 1$, Remark 15 implies that $\beta \leq u + 2^k - 1$. We have

$$2^{(2^k-1)^u - k} \;\leq\; 2^{\alpha - k}\beta_1 \;\leq\; 2^v \beta_1 \;=\; \beta \;\leq\; u + 2^k - 1.$$

Since for all $u \geq 1$ and $k \geq 3$,

$$2^{(2^k-1)^u - k} \;>\; u + 2^k - 1$$

by Lemma 11, we have a contradiction. This finishes our proof. □

3 Proof of Theorem 3

For the forward implication, we prove that if $n = 2^{\alpha-1}p$ and $n \mid \sigma_k(n)$, then α is prime and $p = 2^\alpha - 1$. By Lemma 6, $n \neq 2^{k-1}(2^k - 1)$. We have

$$\begin{aligned}
\sigma_k(n) \;=\; \sigma_k(2^{\alpha-1}p) \;&=\; \sigma_k(2^{\alpha-1})\sigma_k(p) \\
&=\; (1 + 2^k + \cdots + 2^{k(\alpha-1)})(1 + p^k) \\
&=\; (1 + 2^k + \cdots + 2^{k(\alpha-1)})(1 + p)\sum_{i=1}^{k} p^{k-i}(-1)^{i+1}.
\end{aligned}$$

So, $2^{\alpha-1}p \mid \sigma_k(n)$ implies that $2^{\alpha-1} \mid 1+p$ and $p \mid 1+2^k+\cdots+2^{k(\alpha-1)}$. There exist $k_1, k_2 \in \mathbb{N}$ such that $p = k_1 2^{\alpha-1} - 1$ and $1 + 2^k + \cdots + 2^{k(\alpha-1)} = \frac{2^{k\alpha}-1}{2^k-1} = k_2 p$. So,

$$(2^\alpha - 1)\sum_{i=0}^{k-1} 2^{i\alpha} \;=\; 2^{k\alpha} - 1 \;=\; k_3(k_1 2^{\alpha-1} - 1), \tag{5}$$

where $k_3 = (2^k - 1)k_2$.

Suppose that $k_1 = 1$. Then $p = 2^{\alpha-1} - 1$ and (5) implies that either $2^{\alpha-1} - 1 \mid (2^\alpha - 1)$ or $2^{\alpha-1} - 1 \mid \sum_{i=0}^{k-1} 2^{i\alpha}$. If the former, we write

$$1 \;=\; 2^\alpha - 1 - 2(2^{\alpha-1} - 1) \;\equiv\; 0 \pmod{2^{\alpha-1} - 1},$$

which is impossible. Suppose the latter. Because $2^\alpha \equiv 2 \pmod{p}$, we have

$$\sum_{i=0}^{k-1} 2^{i\alpha} \;\equiv\; \sum_{i=0}^{k-1} 2^i \;\equiv\; 2^k - 1 \pmod{p},$$

which implies that p divides $2^k - 1$. Hence, $p = 2^k - 1$. However, Lemma 6 implies that $n \nmid \sigma_k(n)$, which contradicts our assumption. So, $k_1 \geq 2$; however, $k_1 < 3$ by assumption.

So, $k_1 = 2$; we have $p = 2^\alpha - 1$ and α is a prime. Therefore, n is an even perfect number $\neq 2^{k-1}(2^k - 1)$.

For the backward implication, write $n = 2^{q-1}(2^q - 1)$, where $q \neq k$ and $2^q - 1$ are primes. We have

$$\begin{aligned} \sigma_k(n) &= (1 + 2^k + 2^{2k} + \cdots + 2^{(q-1)k})(1 + (2^q - 1)^k) \\ &= \frac{2^{qk} - 1}{2^k - 1}(1 + (2^q - 1)^k). \end{aligned}$$

Clearly, 2^{q-1} divides $1 + (2^q - 1)^k$. It suffices to show that $2^q - 1$ divides $\frac{2^{qk}-1}{2^k-1}$. The fact $n \neq 2^{k-1}(2^k - 1)$ implies that $2^q - 1$ and $2^k - 1$ are two distinct primes. So, $(2^q - 1, 2^k - 1) = 1$. Because $2^q - 1 \mid 2^{qk} - 1$, $2^q - 1$ divides $\frac{2^{qk}-1}{2^k-1}$. Therefore, $n \mid \sigma_k(n)$.

4 Proof of Theorem 4

4.1 Preliminary results

We provide lemmas that give useful bounds used in the proof of Theorem 4.

Lemma 7. *Let $n = 2^{\alpha-1}p^3$, where $\alpha > 1$, $p \equiv 3 \pmod 4$ and $p < 3 \cdot 2^{\alpha-1} - 1$. Then $n \nmid \sigma_5(n)$.*

Proof. We prove by contradiction. Suppose that $n \mid \sigma_5(n)$. We have

$$\begin{aligned} \sigma_5(2^{\alpha-1}p^3) &= (1 + 2^5 + \cdots + 2^{5(\alpha-1)})(1 + p^5 + p^{10} + p^{15}) \\ &= (1 + 2^5 + \cdots + 2^{5(\alpha-1)})(p^{10} + 1)(p + 1)(p^4 - p^3 + p^2 - p + 1). \end{aligned}$$

So,

$$2^{\alpha-1} \mid (p^{10} + 1)(p + 1) \tag{6}$$

$$p^3 \mid 1 + 2^5 + \cdots + 2^{5(\alpha-1)} = \frac{2^{5\alpha} - 1}{2^5 - 1}. \tag{7}$$

Because $p^{10} + 1 \equiv 2 \pmod 4$, we know that $2^{\alpha-2} \mid p + 1$. Hence, $p = k_1 2^{\alpha-2} - 1$ for some $k_1 \in \mathbb{N}$. Combining with $p < 3 \cdot 2^{\alpha-1} - 1$, we get $1 \leq k_1 \leq 5$. By (7), write $2^{5\alpha} - 1 = 31k_2p^3$ for some $k_2 \in \mathbb{N}$. Therefore,

$$31k_2(k_1 2^{\alpha-2} - 1)^3 = (2^\alpha - 1)(2^{4\alpha} + 2^{3\alpha} + 2^{2\alpha} + 2^\alpha + 1). \tag{8}$$

Suppose that p divides both $2^\alpha - 1$ and $\sum_{i=0}^4 2^{i\alpha}$. Then $2^\alpha \equiv 1 \pmod p$ and so, $\sum_{i=0}^4 2^{i\alpha} \equiv 5 \pmod p$. Hence, $p = 5$, which contradicts the congruence $p \equiv 3 \pmod 4$. It must be that either $p^3 \mid \sum_{i=0}^4 2^{i\alpha}$ or $p^3 \mid 2^\alpha - 1$. We consider two corresponding cases.

Case 1: $(k_1 2^{\alpha-2}-1)^3 \mid 2^\alpha - 1$. So, $(k_1 2^{\alpha-2}-1)^3 \leq 2^\alpha - 1$. In order that the inequality is true for some $\alpha \geq 2$, we must have $1 \leq k_1 \leq 2$. Otherwise,

$$(k_1 2^{\alpha-2}-1)^3 \;\geq\; (3\cdot 2^{\alpha-2}-1)^3 \;>\; 2^\alpha - 1,$$

for all $\alpha \geq 2$. We consider two cases.

(i) $k_1 = 1$. Then $2^{\alpha-2}-1 \mid 2^\alpha - 1$. Because

$$3 \;=\; (2^\alpha - 1) - 4(2^{\alpha-2}-1) \;\equiv\; 0 \pmod{2^{\alpha-2}-1},$$

$p = 2^{\alpha-2}-1 = 3$. So, $\alpha = 4$ and $n = 2^3 3^3$, a contradiction as $2^3 3^3 \nmid \sigma_5(2^3 3^3)$.

(ii) $k_1 = 2$. Then $2^{\alpha-1}-1 \mid 2^\alpha - 1$. Because

$$1 \;=\; (2^\alpha - 1) - 2(2^{\alpha-1}-1) \;\equiv\; 0 \pmod{2^{\alpha-1}-1},$$

$p = 2^{\alpha-1}-1 = 1$, a contradiction.

Case 2: $(k_1 2^{\alpha-2}-1)^3 \mid \sum_{i=0}^{4} 2^{i\alpha}$. Let $x_0 = 2^\alpha$. Let $f(x) = x^4 + x^3 + x^2 + x + 1$.

(i) If $k_1 = 1$, we have $2^\alpha \equiv 4 \pmod{p}$. So, $f(x_0) \equiv 341 \pmod{p}$. Because p divides $f(x_0)$, it follows that p divides 341 and so, $p = 11$ or 31. Since $p = 2^{\alpha-2}-1$, $p = 31$, and $\alpha = 7$. However, $n = 2^6 31^3 \nmid \sigma_5(n)$.

(ii) If $k_1 = 2$, we have $2^\alpha \equiv 2 \pmod{p}$. So, $f(x_0) \equiv 31 \pmod{p}$. Because p divides $f(x_0)$, it follows that $p = 31$ and $\alpha = 6$. However, $n = 2^5 31^3 \nmid \sigma_5(n)$.

(iii) If $k_1 = 3$, we have $3x_0 \equiv 4 \pmod{p}$. So, $3^4 f(x_0) \equiv 781 \pmod{p}$. Because p divides $f(x_0)$, it follows that $p \mid 781$ and so, $p \in \{11, 71\}$. Since $p = 3\cdot 2^{\alpha-2}-1$, we know $p = 11$, $\alpha = 4$, and $n = 2^3 11^3$. However, $n = 2^3 11^3 \nmid \sigma_5(n)$.

(iv) If $k_1 = 4$, we have $x_0 \equiv 1 \pmod{p}$. So, $f(x_0) \equiv 5 \pmod{p}$. It follows that $p = 5$, which contradicts the congruence $p \equiv 3 \pmod{4}$.

(v) If $k_1 = 5$, we have $5x_0 \equiv 4 \pmod{p}$. So, $5^4 f(x_0) \equiv 2101 \pmod{p}$. It follows that p divides 2101, so $p = 11$ or 191. Both cases are impossible.

This completes our proof. □

Lemma 8. *Let* $n = 2^{\alpha-1}p^{\beta-1}$, $p \equiv 1 \pmod{4}$ *and* $n \mid \sigma_k(n)$. *Write* $\beta = 2^v \beta_1$, *where* $v \geq 1$ *and* $(2, \beta_1) = 1$. *Then*

$$p^{2^v - 1} \;\leq\; \frac{2^{k(v+1)}-1}{2^k - 1}. \tag{9}$$

Proof. Let $p-1=2^t p_1$, where $t \geq 2$ and $2 \nmid p_1$. Because

$$p^k - 1 = (p-1)\sum_{i=1}^{k} p^{k-i} = 2^t p_1 \sum_{i=1}^{k} p^{k-i}, \tag{10}$$

we have $2^t \mid\mid (p^k-1)$. By Lemma 16, $2^{t+v} \mid\mid p^{k\beta}-1$. Hence,

$$2^v \mid\mid \frac{p^{k\beta}-1}{p^k-1}.$$

By (1),

$$\alpha \leq v+1. \tag{11}$$

Hence, we have

$$p^{2^v-1} \ \leq\ p^{\beta-1} \ \leq\ \frac{2^{k\alpha}-1}{2^k-1} \ \leq\ \frac{2^{k(v+1)}-1}{2^k-1}.$$

□

Lemma 9. *Let* $n = 2^{\alpha-1}p^{\beta-1}$, $p \equiv 3 \pmod 4$ *and* $n \mid \sigma_k(n)$. *Write* $\beta = 2^v\beta_1$, *where* $v \geq 1$ *and* $(2,\beta_1)=1$. *Then*

$$p^{2^v-2k-1} \ <\ \frac{2^{k(v-1)}}{2^k-1}. \tag{12}$$

Proof. Let $p^2-1=2^s p_2$, where $2 \nmid p_2$. Then $s \geq 3$. By (10), $2 \mid\mid p^k-1$ and by Lemma 17, $2^{v+s-1} \mid\mid p^{k\beta}-1$. Hence,

$$2^{v+s-2} \mid\mid \frac{p^{k\beta}-1}{p^k-1}.$$

By (1),

$$\alpha \ \leq\ v+s-1. \tag{13}$$

We have

$$\begin{aligned} p^{2^v-1} \ \leq\ p^{\beta-1} \ \leq\ \frac{2^{k\alpha}-1}{2^k-1} \ &\leq\ \frac{2^{k(v+s-1)}-1}{2^k-1} \\ =\ \frac{2^{ks}2^{k(v-1)}-1}{2^k-1} \ &<\ \frac{p^{2k}2^{k(v-1)}-1}{2^k-1} \text{ because } p^2 > 2^s. \end{aligned}$$

Therefore,

$$p^{2^v-2k-1} \ <\ \frac{2^{k(v-1)}-1/p^{2k}}{2^k-1} \ <\ \frac{2^{k(v-1)}}{2^k-1}.$$

□

Lemma 10. *Let* $n = 2^{\alpha-1}p^{\beta-1}$, $p \equiv 3 \pmod 4$ *and* $n \mid \sigma_k(n)$. *Write* $\beta = 2^v\beta_1$ *and* $p+1 = 2^\lambda p_1$, *where* $(2,\beta_1) = (2,p_1) = 1$. *Then one of the following must hold*

(a)
$$p \;=\; k,$$

(b)
$$(2^\lambda - 1)^{\beta-1} \;\leq\; 2^{\lambda+v} - 1,$$

(c)
$$(2^\lambda - 1)^{\beta-1} \;\leq \sum_{i=0}^{k-1} 2^{i(\lambda+v)}.$$

Proof. From (1) and (2), we have

$$2^\alpha \mid p^\beta - 1 \text{ and } p^{\beta-1} \mid 2^{k\alpha} - 1 \;=\; (2^\alpha - 1)\sum_{i=0}^{k-1} 2^{i\alpha}.$$

By Lemma 18, $2^{\lambda+v} \mid\mid p^\beta - 1$. So, $\alpha \leq \lambda + v$.

Case 1: $p \mid 2^\alpha - 1$ and $p \mid \sum_{i=0}^{k-1} 2^{i\alpha}$. The fact that $2^\alpha \equiv 1 \pmod p$ implies that $\sum_{i=0}^{k-1} 2^{i\alpha} \equiv k \pmod p$. Because $p \mid \sum_{i=0}^{k-1} 2^{i\alpha}$ and k is prime, it must be that $p \;=\; k$. We have scenario (a).

Case 2: $p \mid 2^\alpha - 1$ and $p \nmid \sum_{i=0}^{k-1} 2^{i\alpha}$. So,

$$2^\alpha \mid p^\beta - 1 \text{ and } p^{\beta-1} \mid 2^\alpha - 1.$$

We have
$$(2^\lambda - 1)^{\beta-1} \;\leq\; p^{\beta-1} \;\leq\; 2^\alpha - 1 \;\leq\; 2^{\lambda+v} - 1.$$
We have scenario (b).

Case 3: $p \nmid 2^\alpha - 1$ and $p \mid \sum_{i=0}^{k-1} 2^{i\alpha}$. So,

$$2^\alpha \mid p^\beta - 1 \text{ and } p^{\beta-1} \mid \sum_{i=0}^{k-1} 2^{i\alpha}.$$

We have
$$(2^\lambda - 1)^{\beta-1} \;\leq\; p^{\beta-1} \;\leq\; \sum_{i=0}^{k-1} 2^{i\alpha} \;\leq\; \sum_{i=0}^{k-1} 2^{i(\lambda+v)}.$$
We have scenario (c). □

4.2 Proof of Theorem 4

We now bring together all preliminary results and prove Theorem 4 by case analysis.

Proof. The backward implication follows from Theorem 3. We prove the forward implication. Let $n = 2^{\alpha-1}p^{\beta-1}$, where $\alpha, \beta > 1$ and $p < 3\cdot 2^{\alpha-1}-1$ is an odd prime. Suppose that $n \mid \sigma_5(n)$. Computation shows that $n \neq 496$.

Case 1: $p \equiv 1 \pmod 4$. By (9),

$$5^{2^v-1} \;\leq\; p^{2^v-1} \;\leq\; \frac{2^{5(v+1)}-1}{2^5-1}, \tag{14}$$

which only holds if $1 \leq v \leq 2$ by Lemma 19.

(i) $v = 1$. By (11), $\alpha = 2$ then by (2), $p \mid 33$, which contradicts the congruence $p \equiv 1 \pmod 4$.

(ii) $v = 2$. By (14), $p \leq 10$ and so $p = 5$. By (11), $2 \leq \alpha \leq 3$. However, neither value of α satisfies (2).

Case 2: $p \equiv 3 \pmod 4$. Note that because $k = 5$, we can ignore scenario (a) of Lemma 10. By (12),

$$3^{2^v-11} \;\leq\; p^{2^v-11} \;<\; \frac{2^{5(v-1)}}{2^5-1}, \tag{15}$$

which implies $1 \leq v \leq 4$ by Lemma 20.

(i) $v = 4$. By (15), $p = 3$. So, in (13), $s = 3$ and $2 \leq \alpha \leq 6$. If $\alpha \leq 5$, (2) gives

$$3^{15} \mid 3^{16\beta_1-1} \;\leq\; \frac{2^{25}-1}{31}, \text{ a contradiction.}$$

If $\alpha = 6$, (2) does not hold.

(ii) $v = 3$. Then $\beta \geq 8$. By Lemma 10, either $(2^\lambda-1)^{\beta-1} \leq 2^{\lambda+3}-1$ or $(2^\lambda-1)^{\beta-1} \leq \sum_{i=0}^{4} 2^{i(\lambda+3)}$.

(a) If $(2^\lambda-1)^{\beta-1} \leq 2^{\lambda+3}-1$, then $\lambda < 2$ because $\beta \geq 8$, a contradiction.

(b) If $(2^\lambda-1)^{\beta-1} \leq \sum_{i=0}^{4} 2^{i(\lambda+3)}$, then $\beta \leq 15$ in order that $\lambda \geq 2$. Since $8 \mid \beta$, we know $\beta = 8$. Plugging $\beta = 8$ into $(2^\lambda-1)^{\beta-1} \leq \sum_{i=0}^{4} 2^{i(\lambda+3)}$, we have $2 \leq \lambda \leq 4$ and so $2 \leq s \leq 5$. By (13), $2 \leq \alpha \leq 7$ and by (2), we acquire

$$p^7 \mid \frac{2^{5\alpha}-1}{31} \;\leq\; \frac{2^{35}-1}{31}.$$

Hence, $p \in \{3, 7, 11, 19\}$. Computation shows that for each pair (α, p), (2) does not hold.

(iii) $v = 2$. Then $4 \mid \beta$. By Lemma 10, either $(2^\lambda - 1)^{\beta-1} \leq 2^{\lambda+2} - 1$ or $(2^\lambda - 1)^{\beta-1} \leq \sum_{i=0}^{4} 2^{i(\lambda+2)}$. Since $\beta \geq 4$ and $\lambda \geq 2$, the former does not hold. If the later, since $\lambda \geq 2$, it must be that $\beta < 12$ and so $\beta \in \{4, 8\}$.

  (a) $\beta = 4$. Lemma 7 rejects this case.

  (b) $\beta = 8$. Plugging $\beta = 8$ into $(2^\lambda - 1)^{\beta-1} \leq \sum_{i=0}^{4} 2^{i(\lambda+2)}$, we have $2 \leq \lambda \leq 3$ and so $2 \leq s \leq 4$. By (13), $2 \leq \alpha \leq 5$. We are back to item (ii) part (b).

(iv) $v = 1$. By Lemma 10, either $(2^\lambda - 1)^{\beta-1} \leq 2^{\lambda+1} - 1$ or $(2^\lambda - 1)^{\beta-1} \leq \sum_{i=0}^{4} 2^{i(\lambda+1)}$. If the former, $\beta = 2$ and $n = 2^{\alpha-1}p$. By Theorem 3, n is an even perfect number. If the latter, since $\lambda \geq 2$, it must be that $\beta \leq 9$ and so $\beta \in \{2, 6\}$.

  (a) If $\beta = 2$, Theorem 3 guarantees that n is an even perfect number.

  (b) If $\beta = 6$, then $2 \leq \lambda \leq 4$ and so $2 \leq s \leq 5$. By (13), $2 \leq \alpha \leq 5$ and by (2), we acquire
  $$p^5 \mid \frac{2^{5\alpha} - 1}{31} \leq \frac{2^{25} - 1}{31}.$$
  Hence, $p \in \{3, 7, 11\}$. Computation shows that for each pair (α, p), (2) does not hold.

We have finished the proof. □

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A Technical proofs used for Lemma 6

We provide proofs of claim(s) made in the proof of Lemma 6. Notation from Lemma 6 is retained here.

Lemma 11. *For all $u \geq 1$ and $k \geq 3$, we have*

$$2^{(2^k-1)^u - k} > u + 2^k - 1.$$

Proof. We prove by induction on u. For $u = 1$, it is clear that $2^{2^k-1-k} > 2^k$ for all $k \geq 3$. Assume that the inequality holds for $u = n \geq 1$ and for all $k \geq 3$. We want to show that the inequality holds for $u = n + 1$ and for all $k \geq 3$. Fixing $k \geq 3$, we have

$$2^{(2^k-1)^{n+1}-k} = 2^{-k}(2^{(2^k-1)^n})^{2^k-1} > 2^{-k}(2^k(n + 2^k - 1))^{2^k-1}$$

by the inductive hypothesis. Hence, it suffices to show that $2^{-k}(2^k(n+2^k-1))^{2^k-1} \geq n+2^k$; equivalently, $2^{k(2^k-1)}(n+2^k-1)^{2^k-1} \geq 2^k(n+2^k)$. Because $2^{k(2^k-1)} \geq 2^k$, it remains to show

$$(n+2^k-1)^{2^k-1} \;\geq\; n+2^k.$$

Let $\ell = 2^k - 1$. The above inequality becomes

$$(n+\ell)^\ell - 1 \;\geq\; n+\ell.$$

Equivalently,

$$(n+\ell)^{\ell-1} - \frac{1}{n+\ell} \;\geq\; 1,$$

which is true because $(n+\ell)^{\ell-1} \geq 2$. □

Lemma 12. *For all odd $k \geq 3$, we have $2^{k+1} \mid\mid (2^k-1)^{2k} - 1$.*

Proof. Write

$$(2^k-1)^{2k} - 1 \;=\; \sum_{i=0}^{2k}\binom{2k}{i}(2^k)^{2k-i}(-1)^i - 1 \;=\; \sum_{i=0}^{2k-1}\binom{2k}{i}(2^k)^{2k-i}(-1)^i.$$

When $i = 2k-1$, we have the term $-2k \cdot 2^k = -k2^{k+1}$. Because k is odd, $2^{k+1} \mid\mid k2^{k+1}$. This finishes our proof. □

Lemma 13. *The following holds*

$$2^{v+k} \mid\mid (2^k-1)^{\beta k} - 1.$$

Proof. We prove by induction on v. When $v = 1$, write

$$(2^k-1)^{\beta k} - 1 \;=\; (2^k-1)^{2k\beta_1} - 1 \;=\; ((2^k-1)^{2k} - 1)\sum_{i=1}^{\beta_1}(2^k-1)^{2k(\beta_1 - i)}.$$

Because the summation is 1 mod 2 and by Lemma 12, $2^{k+1} \mid\mid (2^k-1)^{2k} - 1$, our claim holds for $v = 1$. Inductive hypothesis: suppose that there exists $z \geq 1$ such that the claim holds for all $v \in [1, z]$. We show that the claim holds for $v = z+1$. We have

$$(2^k-1)^{2^{z+1}\beta_1 k} - 1 \;=\; ((2^k-1)^{2^z\beta_1 k} - 1)((2^k-1)^{2^z\beta_1 k} + 1).$$

By the inductive hypothesis, $2^{z+k} \mid\mid (2^k-1)^{2^z\beta_1 k} - 1$, so it suffices to show that $2 \mid\mid (2^k - 1)^{2^z\beta_1 k} + 1$. Observe that

$$(2^k-1)^{2^z\beta_1 k} + 1 \;=\; (4^k - 2^{k+1} + 1)^{2^{z-1}\beta_1 k} + 1 \equiv 2 \pmod 4.$$

Hence, $2 \mid\mid (2^k-1)^{2^z\beta_1 k} + 1$, as desired. This completes our proof. □

Lemma 14. *Let m be chosen such that $(2^k-1)^m \mid\mid 2^{(2^k-1)k}-1$. Then for all $u \geq 0$,*

$$(2^k-1)^{u+m} \mid\mid 2^{(2^k-1)^{u+1}k\alpha_1}-1.$$

Proof. First, we claim that $m \geq 2$. To prove this, write

$$2^{(2^k-1)k}-1 \;=\; (2^k-1)\sum_{i=2}^{2^k}(2^k)^{(2^k-i)}.$$

Since each term in the summation is congruent to 1 mod 2^k-1 and there are 2^k-1 terms, the summation is divisible by 2^k-1. Therefore, $(2^k-1)^2 \mid 2^{(2^k-1)k}-1$.

We are ready to prove the lemma. We proceed by induction. Recall that in the proof of Lemma 6, we define $\alpha_1 := \alpha/(2^k-1)^u$. For $u=0$, write

$$\begin{aligned}2^{(2^k-1)k\alpha_1}-1 &= (2^{(2^k-1)k}-1)(2^{(2^k-1)k(\alpha_1-1)}+2^{(2^k-1)k(\alpha_1-2)}+\cdots+1)\\ &= (2^{(2^k-1)k}-1)\sum_{i=1}^{\alpha_1}(2^k)^{(2^k-1)(\alpha_1-i)}.\end{aligned}$$

By assumption, $(2^k-1)^m \mid\mid 2^{(2^k-1)k}-1$. Each term in the summation $\sum_{i=1}^{\alpha_1}(2^k)^{(2^k-1)(\alpha_1-i)}$ is congruent to 1 mod 2^k-1, so the summation is congruent to α_1 mod 2^k-1. Hence, our lemma holds for $u=0$. Inductive hypothesis: suppose that there exists $z \geq 0$ such that our lemma holds for all $u \leq z$. We show that our lemma holds for $u=z+1$. Write

$$\begin{aligned}2^{(2^k-1)^{z+2}k\alpha_1}-1 &= (2^{(2^k-1)^{z+1}k\alpha_1}-1)\cdot\\ &\quad (2^{(2^k-1)^{z+1}k\alpha_1(2^k-2)}+2^{(2^k-1)^{z+1}k\alpha_1(2^k-3)}+\cdots+1)\\ &= (2^{(2^k-1)^{z+1}k\alpha_1}-1)\sum_{i=2}^{2^k}2^{(2^k-1)^{z+1}k\alpha_1(2^k-i)}.\end{aligned}$$

By the inductive hypothesis, $(2^k-1)^{z+m} \mid\mid 2^{(2^k-1)^{z+1}k\alpha_1}-1$. Each term in the summation is congruent to 1 mod $(2^k-1)^m$. Since there are 2^k-1 terms, the summation is congruent to (2^k-1) mod $(2^k-1)^m$. Because $m \geq 2$, (2^k-1) exactly divides the summation. So,

$$(2^k-1)^{z+m+1} \text{ exactly divides } 2^{(2^k-1)^{z+2}k\alpha_1}-1,$$

as desired. This completes our proof. □

Remark 15. Note that for all $k \geq 3$, in order that $(2^k-1)^m \leq 2^{(2^k-1)k}-1$, we must have $m < 2^k$. By Lemma 14, $(2^k-1)^{u+2^k}$ does not divide $2^{(2^k-1)^{u+1}k\alpha_1}-1$ for all $u \geq 0$.

B Technical proofs used for Lemma 8

We provide proofs of claim(s) made in the proof of Lemma 8. Notation from Lemma 8 is retained here.

Lemma 16. *With notation as in Lemma 8, the following holds*

$$2^{t+v} \mid\mid p^{2^v \beta_1 k} - 1.$$

Proof. We prove by induction on v. When $v = 1$, write

$$\begin{aligned} p^{2k\beta_1} - 1 &= (p^{2k} - 1)(p^{2k(\beta_1 - 1)} + p^{2k(\beta_1 - 2)} + \cdots + 1) \\ &= (p^k - 1)(p^k + 1) \sum_{i=1}^{\beta_1} p^{2k(\beta_1 - i)} \\ &= (p^k - 1)(p + 1) \left(\sum_{i=1}^{k} p^{k-i} (-1)^{i+1} \right) \sum_{i=1}^{\beta_1} p^{2k(\beta_1 - i)}. \end{aligned} \tag{16}$$

Since $p + 1 \equiv 2 \pmod 4$, $2 \mid\mid (p + 1)$. We showed that $2^t \mid\mid (p^k - 1)$ in the proof of Lemma 8. Also, the two summations are odd. Therefore, $2^{t+1} \mid\mid p^{2k\beta_1} - 1$.

Inductive hypothesis: suppose that there exists $z \geq 1$ such that our claim holds for all $v \in [1, z]$. We show that our claim holds for $v = z + 1$. We have

$$p^{2^{z+1} k\beta_1} - 1 = p^{(2^z k\beta_1)\cdot 2} - 1 = (p^{2^z k\beta_1} + 1)(p^{2^z k\beta_1} - 1).$$

By the inductive hypothesis, $2^{z+t} \mid\mid p^{2^z k\beta_1} - 1$. Also, $p \equiv 1 \pmod 4$ implies that $p^{2^z k\beta_1} + 1 \equiv 2 \pmod 4$. So, $2 \mid\mid p^{2^z k\beta_1} + 1$. Therefore, $2^{z+t+1} \mid\mid p^{2^{z+1} k\beta_1} - 1$. We have finished our proof. □

C Technical proofs used for Lemma 9

We provide proofs of claim(s) made in the proof of Lemma 9. Notation from Lemma 9 is retained here.

Lemma 17. *With notation as in Lemma 9, the following holds*

$$2^{v+s-1} \mid\mid p^{k 2^v \beta_1} - 1.$$

Proof. We prove by induction on v. When $v = 1$, by (16), we only consider $(p + 1)(p^k - 1)$. We showed that $2 \mid\mid p^k - 1$ in the proof of Lemma 9. Since $2^s \mid\mid (p - 1)(p + 1)$ and $2 \mid\mid p - 1$, it follows that $2^{s-1} \mid\mid p + 1$. Therefore, $2^s \mid\mid p^{k2\beta_1} - 1$.

Inductive hypothesis: suppose that there exists $z \geq 1$ such that for all $v \in [1, z]$, our claim holds. We show that our claim also holds for $v = z + 1$. We have

$$p^{2^{z+1} k\beta_1} - 1 = p^{(2^z k\beta_1)\cdot 2} - 1 = (p^{2^z k\beta_1} + 1)(p^{2^z k\beta_1} - 1).$$

By the inductive hypothesis, $2^{z+s-1} \mid\mid p^{2^z k\beta_1} - 1$. Also, $p^2 \equiv 1 \pmod 4$ implies that $p^{2^z k\beta_1} + 1 \equiv 2 \pmod 4$. So, $2 \mid\mid p^{2^z k\beta_1} + 1$. Therefore, $2^{z+s} \mid\mid p^{2^{z+1} k\beta_1} - 1$. We have finished our proof. □

D Technical proofs used for Lemma 10

We provide proofs of claim(s) made in the proof of Lemma 10. Notation from Lemma 10 is retained here.

Lemma 18. *With notation as in Lemma 10, the following holds*

$$2^{\lambda+v} \mid\mid (2^\lambda p_1 - 1)^{2^v \beta_1} - 1.$$

Proof. We prove by induction on v. Observe that

$$\begin{aligned}(2^\lambda p_1 - 1)^{2\beta_1} - 1 &= \sum_{i=0}^{2\beta_1} \binom{2\beta_1}{i} (2^\lambda p_1)^{2\beta_1 - i}(-1)^i - 1 \\ &= \sum_{i=0}^{2\beta_1 - 1} \binom{2\beta_1}{i} (2^\lambda p_1)^{2\beta_1 - i}(-1)^i,\end{aligned}$$

which clearly indicates that $2^{\lambda+1} \mid\mid (2^\lambda p_1 - 1)^{2\beta_1} - 1$. So, the claim holds for $v = 1$. Inductive hypothesis: suppose that there exists $z \geq 1$ such that for all $v \in [1, z]$, the claim holds. We prove that the claim holds for $v = z + 1$. We have

$$(2^\lambda p_1 - 1)^{2^{z+1}\beta_1} - 1 = ((2^\lambda p_1 - 1)^{2^z \beta_1} - 1)((2^\lambda p_1 - 1)^{2^z \beta_1} + 1).$$

By the inductive hypothesis, $2^{\lambda+z} \mid\mid (2^\lambda p_1 - 1)^{2^z\beta_1} - 1$. Also, $(2^\lambda p_1 - 1)^{2^z \beta_1} + 1 \equiv 2 \pmod 4$ since $\lambda \geq 2$. Hence, $2^{\lambda+z+1} \mid\mid (2^\lambda p_1 - 1)^{2^{z+1}\beta_1} - 1$, as desired. □

E Technical proofs used for Theorem 4

Lemma 19. *If $v \geq 1$ and $5^{2^{v-1}} \leq \frac{2^{5(v+1)}-1}{31}$, then $1 \leq v \leq 2$.*

Proof. The inequality $5^{2^{v-1}} \leq \frac{2^{5(v+1)}-1}{31}$ implies that

$$5^{2^v - 1} \leq 5^{5(v+1)},$$

which is equivalent to $2^v - 1 \leq 5(v+1)$. Clearly, we have $1 \leq v \leq 4$. However, the inequality $5^{2^{v-1}} \leq \frac{2^{5(v+1)}-1}{31}$ does not hold when $v \in \{3, 4\}$. We conclude that $1 \leq v \leq 2$. □

Lemma 20. *If $v \geq 1$ and $3^{2^v - 11} < \frac{2^{5(v-1)}}{31}$, then $1 \leq v \leq 4$.*

Proof. The inequality $3^{2^v - 11} < \frac{2^{5(v-1)}}{31}$ implies that $3^{2^v - 11} < 3^{5(v-1)}$. Hence, $2^v < 5v + 6$, which holds only if $1 \leq v \leq 4$. □

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