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New Solutions of the Tarry-Escott Problem of Degrees 2, 3 and 5

Ajai Choudhry
13/4 A Clay Square
Lucknow-226001
India
ajaic203@yahoo.com

Abstract

In this paper we obtain new parametric ideal solutions of the well-known Tarry-Escott problem of degrees 2, 3 and 5. While several mathematicians have obtained ideal solutions of the Tarry-Escott problem, the parametric solutions given in this paper have a remarkable symmetry that is not to be found in any of the already known solutions of the problem.

1 Introduction

The Tarry-Escott problem (written briefly as TEP) of degree k consists of finding two distinct sets of integers, $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$, such that

$$\sum_{i=1}^{n} x_i^j = \sum_{i=1}^{n} y_i^j, \quad j = 1, 2, \ldots, k, \tag{1}$$

where k is a given positive integer. It is well-known that for a non-trivial solution of (1) to exist, we must have $n \geq k+1$ [9, p. 616]. Solutions of (1) with $n = k+1$ are known as ideal solutions of the problem.

It would be recalled that simple solutions of the TEP of degree 2 were first noticed by Goldbach and by Euler in 1750–51. Ever since then, numerous authors have given parametric ideal solutions of the TEP when $2 \leq k \leq 7$ and numerical ideal solutions when $k = 8, 9$ or 11 [1; 3–5; 6, pp. 705–713; 8, pp. 33–57; 11]. Dickson [7, pp. 52, 55–58] has given a complete parametric ideal solution of the TEP of degree 2 as well as a method of generating all integer solutions of the TEP of degree 3. The complete solution of the TEP is not known for any value of $k > 3$.

In this paper, for the TEP of degrees 2 and 3, we obtain new parametric ideal solutions that have a remarkable symmetry that is not to be found in any of the known solutions of the TEP.

With reference to the diophantine system defined by (1), for each value of the exponent j, let σ_j denote the common sum of either side of (1). For the TEP of degree 2, i.e., for the diophantine system $\sum_{i=1}^{3} x_i^j = \sum_{i=1}^{3} y_i^j, j = 1, 2$, we will obtain the complete ideal solution in terms of polynomials in six parameters p, q, r, a, b, and c such that both σ_1 and σ_2 are nonzero symmetric functions of the three parameters p, q, r, and also symmetric functions of the three parameters a, b, c. Further, the values of $x_i, y_i, i = 1, 2, 3$, will all be expressible as polynomials $\phi(\xi_1, \xi_2, \ldots, \xi_6)$, where $\xi_1, \xi_2, \ldots, \xi_6$ is some permutation of the parameters p, q, r, a, b, and c.

For the TEP of degree 3, i.e., for the diophantine system $\sum_{i=1}^{4} x_i^j = \sum_{i=1}^{4} y_i^j, j = 1, 2, 3$, we will obtain an ideal solution in terms of polynomials in four parameters p, q, r, and s such that all the common sums $\sigma_j, j = 1, 2, 3$, are nonzero symmetric functions of all the four parameters p, q, r, and s, and the values of $x_i, y_i, i = 1, \ldots, 4$, are expressible as polynomials $\phi(\xi_1, \ldots, \xi_4)$, where $\xi_1, \ldots, \xi_4$ is some permutation of the parameters p, q, r, and s.

For the TEP of degree 5, we will derive an ideal solution in terms of six parameters p, q, r, a, b, and c using the solution already obtained for the TEP of degree 2. In this case, while the common sums $\sigma_j, j = 1, 3, 5$, are all zero, the nonzero sums σ_2 and σ_4 are symmetric functions of p, q, and r, as well as symmetric functions of a, b, and c.

2 Ideal solutions of the Tarry-Escott problem of degrees 2, 3 and 5

2.1 A parametric ideal solution of the TEP of degree 2

Theorem 1. *A parametric solution of the simultaneous diophantine equations,*

$$x_1^r + x_2^r + x_3^r = y_1^r + y_2^r + y_3^r, \quad r = 1, 2, \tag{2}$$

is given in terms of six arbitrary parameters p, q, r, a, b, and c by

$$\begin{aligned} x_1 &= \phi(p, q, r, a, b, c), \quad x_2 = \phi(p, q, r, b, c, a), \quad x_3 = \phi(p, q, r, c, a, b), \\ y_1 &= \phi(p, q, r, a, c, b), \quad y_2 = \phi(p, q, r, c, b, a), \quad y_3 = \phi(p, q, r, b, a, c), \end{aligned} \tag{3}$$

where $\phi(f, g, h, u, v, w) = fu + gv + hw$. All integer solutions of the simultaneous diophantine Eqs. (2) may be generated by taking scalar multiples of the solution given by (3).

Proof. When $x_i, y_i, i = 1, 2, 3$, are defined by (3), direct computation shows that

$$\begin{aligned}\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i &= (p+q+r)(a+b+c),\\ \sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3} y_i^2 &= (p^2+q^2+r^2)(a^2+b^2+c^2) + 2(pq+qr+rp)(ab+bc+ca).\end{aligned} \tag{4}$$

The relations (4) prove that (3) gives a solution of the simultaneous Eqs. (2) such that the common sums σ_1 and σ_2 are symmetric functions of the parameters p, q, r, and also of the parameters a, b, and c.

To show that the solution given by (3) generates all integer solutions of the simultaneous Eqs. (2), we will use the following complete solution of these simultaneous equations given by Dickson [7, p. 52]:

$$\begin{array}{lll} x_1 = AD + C, & x_2 = AG + BD + C, & x_3 = BG + C,\\ y_1 = AD + BG + C, & y_2 = BD + C, & y_3 = AG + C,\end{array} \tag{5}$$

where A, B, C, D and G are arbitrary parameters.

In our solution of the simultaneous Eqs. (2) given by (3), we take $a = D, b = G, c = 0, p = AD + AG + C, q = C, r = BD + BG + C$, when our solution may be written as follows:

$$\begin{aligned} x_1 &= (D+G)(AD+C),\\ x_2 &= (D+G)(AG+BD+C),\\ x_3 &= (D+G)(BG+C),\\ y_1 &= (D+G)(AD+BG+C),\\ y_2 &= (D+G)(BD+C),\\ y_3 &= (D+G)(AG+C).\end{aligned} \tag{6}$$

The values of $x_i, y_i, i = 1, 2, 3$, given by (6) have a common factor $D + G$, and since both the Eqs. (2) are homogeneous, the common factor $D + G$ may be removed by appropriate scaling, and then the solution (6) coincides exactly with the complete solution (5) of the Eqs. (2) given by Dickson. It follows that the solution of the simultaneous Eqs. (2) given by (3) generates all integer solutions of these equations. □

As a numerical example, when $(a, b, c, p, q, r) = (0, 1, 2, 0, 1, 3)$, Theorem 1 yields the following ideal solution of the TEP of degree 2:

$$2^j + 3^j + 7^j = 1^j + 5^j + 6^j, \quad j = 1, 2.$$

2.2 A parametric ideal solution of the TEP of degree 3

We will first give a theorem that gives, in parametric terms, two triads of integers with equal sums and equal products. We note that two complete solutions of the problem of finding two triads of integers with equal sums and equal products have been given independently by Choudhry [2] and by Kelly [10]. While the solution of this problem given below in Theorem 2 is not complete, it is much simpler and is noteworthy for its symmetry, and we will use it to obtain a parametric solution of the TEP of degree 3.

Theorem 2. *A parametric solution of the diophantine system,*

$$X_1^2 + X_2^2 + X_3^2 = Y_1^2 + Y_2^2 + Y_3^2, \tag{7}$$

$$X_1X_2X_3 = Y_1Y_2Y_3, \tag{8}$$

is given in terms of four arbitrary parameters $p, q, r,$ *and* s *by*

$$\begin{aligned} X_1 &= \phi(p,q,r,s), \quad X_2 = \phi(p,r,s,q), \quad X_3 = \phi(p,s,q,r), \\ Y_1 &= \phi(p,q,s,r), \quad Y_2 = \phi(p,r,q,s), \quad Y_3 = \phi(p,s,r,q), \end{aligned} \tag{9}$$

where $\phi(a,b,c,d) = (ab+cd)(ac-bd)$.

Proof. We begin by writing

$$\begin{aligned} X_1 &= f(pq+rs), \quad X_2 = g(pr+qs), \quad X_3 = h(ps+qr), \\ Y_1 &= g(pq+rs), \quad Y_2 = h(pr+qs), \quad Y_3 = f(ps+qr), \end{aligned} \tag{10}$$

where $f, g, h, p, q, r,$ and s are arbitrary parameters. Now (8) is identically satisfied, while (7) reduces to

$$(pq+rs)^2(f^2-g^2) + (pr+qs)^2(g^2-h^2) + (ps+rq)^2(h^2-f^2) = 0,$$

or, equivalently,

$$((pq+rs)^2 - (ps+rq)^2)(f^2-h^2) + ((pr+qs)^2 - (pq+rs)^2)(g^2-h^2) = 0,$$

and hence we may take,

$$\begin{aligned} f^2 - h^2 &= (pr+qs)^2 - (pq+rs)^2, \\ g^2 - h^2 &= -((pq+rs)^2 - (ps+rq)^2). \end{aligned} \tag{11}$$

If we now take $h = pq - rs$, it follows from (11) that

$$f = pr - qs, \quad g = ps - qr,$$

and, on substituting the values of f, g, h in (10), we get the solution (9) stated in the theorem. □

Theorem 3. *A parametric solution of the simultaneous diophantine equations,*

$$x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4, \tag{12}$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2, \tag{13}$$
$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3, \tag{14}$$

is given in terms of four arbitrary parameters $p, q, r,$ *and* s *by*

$$\begin{aligned} x_1 &= \phi(p, q, r, s), & x_2 &= \phi(p, r, s, q), \\ x_3 &= \phi(p, s, q, r), & x_4 &= \phi(q, r, p, s), \\ y_1 &= \phi(p, q, s, r), & y_2 &= \phi(p, r, q, s), \\ y_3 &= \phi(p, s, r, q), & y_4 &= \phi(q, s, p, r), \end{aligned} \tag{15}$$

where

$$\phi(a, b, c, d) = a^2bc + abc^2 + ac^2d + acd^2 + b^2cd + bc^2d.$$

Proof. To solve the simultaneous Eqs. (12), (13) and (14), we write,

$$\begin{aligned} x_1 &= X_1 - X_2 - X_3, & x_2 &= -X_1 + X_2 - X_3, \\ x_3 &= -X_1 - X_2 + X_3, & x_4 &= X_1 + X_2 + X_3, \\ y_1 &= Y_1 - Y_2 - Y_3, & y_2 &= -Y_1 + Y_2 - Y_3, \\ y_3 &= -Y_1 - Y_2 + Y_3, & y_4 &= Y_1 + Y_2 + Y_3, \end{aligned}$$

when Eq. (12) is identically satisfied while Eqs. (13) and (14) reduce to Eqs. (7) and (8) respectively. Using the solution (9) of the simultaneous Eqs. (7) and (8) given by Theorem 2, we obtain the following solution of the simultaneous Eqs. (12), (13) and (14) in terms of four arbitrary parameters $p, q, r,$ and s:

$$\begin{aligned} x_1 &= \psi(p, q, r, s), & x_2 &= \psi(p, r, s, q), \\ x_3 &= \psi(p, s, q, r), & x_4 &= \psi(q, r, p, s), \\ y_1 &= \psi(p, q, s, r), & y_2 &= \psi(p, r, q, s), \\ y_3 &= \psi(p, s, r, q), & y_4 &= \psi(q, s, p, r), \end{aligned} \tag{16}$$

where

$$\begin{aligned} \psi(a, b, c, d) = {} & (bc - bd - cd)a^2 - (ac + ad - cd)b^2 \\ & + (ab + ad + bd)c^2 - (ab - ac + bc)d^2. \end{aligned}$$

We will now use a well-known theorem [7, Thm. 46, p. 50] according to which if $x_i, y_i, i = 1, \ldots, n$, is a solution of the diophantine system (1), then for arbitrary rational numbers M and K, another solution of the diophantine system (1) is given by $Mx_i + K, My_i + K, i = 1, \ldots, n$.

On taking $M = 1/2$ and

$$K = (p^2(qr + qs + rs) + q^2(pr + ps + rs) + r^2(pq + ps + qs) + s^2(pq + pr + qr))/2,$$

and applying the aforesaid theorem to the solution (16), we get the solution of the simultaneous Eqs. (12), (13) and (14) stated in Theorem 3. □

For the solution of the simultaneous Eqs. (12), (13) and (14) given by (15), we note that the values of the common sums $\sigma_j, j = 1, 2, 3$, may be written as,

$$\begin{aligned}
\sigma_1 &= 2e_1e_3 - 8e_4,\\
\sigma_2 &= -2e_1^2e_2e_4 + 2e_1^2e_3^2 - 8e_1e_3e_4 + 4e_2^2e_4 - 2e_2e_3^2,\\
\sigma_3 &= 3e_1^4e_4^2 - 3e_1^3e_2e_3e_4 + 2e_1^3e_3^3 - 12e_1^2e_3^2e_4 + 6e_1e_2^2e_3e_4\\
&\quad - 3e_1e_2e_3^3 - 24e_2^2e_4^2 + 24e_2e_3^2e_4 - 3e_3^4 + 64e_4^3,
\end{aligned} \tag{17}$$

where $e_i, i = 1, \ldots, 4$, are the elementary symmetric functions defined by

$$\begin{aligned}
e_1 &= p + q + r + s, & e_2 &= pq + pr + ps + qr + qs + rs,\\
e_3 &= pqr + pqs + prs + qrs, & e_4 &= pqrs.
\end{aligned}$$

The relations (17) show that the common sums $\sigma_j, j = 1, 2, 3$, are symmetric functions of the parameters p, q, r, and s.

As a numerical example of an ideal solution of the TEP of degree 3, when $(p, q, r, s) = (1, 2, 3, 4)$, Theorem 3 yields a solution which, on removing the common factor 2, may be written as follows:

$$53^j + 99^j + 114^j + 138^j = 54^j + 93^j + 123^j + 134^j, \quad j = 1, 2, 3.$$

2.3 A parametric ideal solution of the TEP of degree 5

We will first give, in Theorem 4, a parametric solution of the diophantine system $\sum_{i=1}^3 x_i^j = \sum_{i=1}^3 y_i^j, j = 1,\ 2,\ 4$, and we will use it in Theorem 5 to obtain ideal solutions of the TEP of degree 5.

Theorem 4. *A parametric solution of the simultaneous diophantine equations,*

$$\sum_{i=1}^{3} x_i^j = \sum_{i=1}^{3} y_i^j, \quad j = 1, 2, 4, \tag{18}$$

is given in terms of six arbitrary parameters $p, q, r, a, b,$ and c by

$$\begin{aligned}
x_1 &= \psi(p, q, r, a, b, c), & x_2 &= \psi(p, q, r, b, c, a), & x_3 &= \psi(p, q, r, c, a, b),\\
y_1 &= \psi(p, q, r, a, c, b), & y_2 &= \psi(p, q, r, c, b, a), & y_3 &= \psi(p, q, r, b, a, c),
\end{aligned} \tag{19}$$

where $\psi(f, g, h, u, v, w) = f(v - w) + g(w - u) + h(u - v)$.

Proof. In the solution (3) of the simultaneous Eqs. (2), if we replace p, q, r, by $r - q,\ p - r,\ q - p$, respectively, the new values of $x_i, y_i, i = 1, 2, 3$, are given by (19), and with these values of x_i, y_i, direct computation gives,

$$\begin{aligned}
\sum_{i=1}^{3} x_i &= \sum_{i=1}^{3} y_i = 0,\\
\sum_{i=1}^{3} x_i^2 &= \sum_{i=1}^{3} y_i^2 = 2(p^2+q^2+r^2-pq-qr-rp)(a^2+b^2+c^2-ab-bc-ca),\\
\sum_{i=1}^{3} x_i^4 &= \sum_{i=1}^{3} y_i^4 = 2(p^2+q^2+r^2-pq-qr-rp)^2(a^2+b^2+c^2-ab-bc-ca)^2.
\end{aligned} \tag{20}$$

It now follows that (19) gives a parametric solution of the simultaneous diophantine Eqs. (18). □

Theorem 5. *A parametric solution of the simultaneous diophantine equations,*

$$\sum_{i=1}^{6} x_i^j = \sum_{i=1}^{6} y_i^j, \quad j = 1, 2, \ldots, 5, \tag{21}$$

in terms of six arbitrary parameters $p, q, r, a, b,$ and c is given by

$$x_4 = -x_1, \;\; x_5 = -x_2, \;\; x_6 = -x_3, \;\; y_4 = -y_1, \;\; y_5 = -y_2, \;\; y_6 = -y_3, \tag{22}$$

and the values of $x_i, y_i, i = 1, 2, 3,$ are defined by (19).

Proof. If we take the values of $x_i, y_i, i = 4, 5, 6,$ as given by (22), the relations (21) are identically satisfied for $k = 1, 3$ and 5, and the diophantine system (21) reduces to the simultaneous diophantine equations

$$\sum_{i=1}^{3} x_i^j = \sum_{i=1}^{3} y_i^j, \quad j = 2,\, 4. \tag{23}$$

Since a solution of the simultaneous Eqs. (23) is given by (19), it follows that a solution of the simultaneous diophantine Eqs. (21) is as stated in the theorem. □

It follows from the relations (22) that for the solution of the TEP of degree 5 given by Theorem 5, the sums $\sigma_j, j = 1, 3, 5,$ are all zero. Further, it follows from the relations (20) that the common sums σ_2 and σ_4 are symmetric functions of $p, q,$ and r, as well as symmetric functions of $a, b,$ and c.

As a numerical example, when $(a, b, c, p, q, r) = (0, 1, 3, 0, 1, 4)$, Theorem 5 yields the following ideal solution of the TEP of degree 5:

$$1^j+9^j+10^j+(-1)^j+(-9)^j+(-10)^j = 5^j+6^j+11^j+(-5)^j+(-6)^j+(-11)^j, \quad j = 1, 2, \ldots, 5.$$

3 Concluding remarks

In view of the simplicity and symmetry of the ideal solutions given by Theorems 1, 3, and 5, it seems that these solutions could be used to obtain new results on equal sums of like powers.

Further, the symmetry of the ideal solutions obtained in this paper suggests that there may exist similar solutions of the TEP of degree k when $k \neq 2, 3$ or 5. It would be interesting to find new parametric solutions of the TEP of degrees k where $k > 3$ such that all the common sums $\sigma_j, j = 1, \ldots, k$, are nonzero symmetric functions of the parameters. Such solutions may be of interest even if the solutions are not ideal solutions.

More generally, it would be of interest to find parametric solutions, with similar symmetric properties, of other symmetric diophantine systems of the type $\sum_{i=1}^{n} x_i^j = \sum_{i=1}^{n} y_i^j$, where the equality holds for certain values of the exponent j that are not the first k consecutive positive integers as in the case of the TEP.

References

[1] J. Chernick, Ideal solutions of the Tarry-Escott problem, *Amer. Math. Monthly* **44** (1937), 626–633.

[2] A. Choudhry, On triads of squares with equal sums and equal products, *Ganita* **49** (1998), 101–106.

[3] A. Choudhry, Ideal solutions of the Tarry-Escott problem of degrees four and five and related diophantine systems, *Enseign. Math.* **49** (2003), 101–108.

[4] A. Choudhry, A new approach to the Tarry-Escott problem, *Int. J. Number Theory* **13** (2017), 393–417.

[5] A. Choudhry and J. Wróblewski, Ideal solutions of the Tarry-Escott problem of degree eleven with applications to sums of thirteenth powers, *Hardy-Ramanujan J.* **31** (2008), 1–13.

[6] L. E. Dickson, *History of the Theory of Numbers*, Chelsea Publishing Company, 1952, reprint.

[7] L. E. Dickson, *Introduction to the Theory of Numbers*, Dover Publications, 1957, reprint.

[8] A. Gloden, *Mehrgradige Gleichungen*, Noordhoff, Groningen, 1944.

[9] H. L. Dorwart and O. E. Brown, The Tarry-Escott problem, *Amer. Math. Monthly* **44** (1937), 613–626.

[10] J. B. Kelly, Two equal sums of three squares with equal products, *Amer. Math. Monthly* **98** (1991), 527–529.

[11] S. Raghavendran and V. Narayanan, Novel parametric solutions for the ideal and non-ideal Prouhet Tarry Escott problem, *Mathematics* **8** (2020), Article 1775.

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