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Four Biquadrates Whose Sum is a Perfect Square

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Abstract

All known nontrivial parametric solutions of the diophantine equation

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = y^2$$

give the values of $x_i, i = 1, \dots, 4$, in terms of polynomials in two parameters. In the three simplest solutions, these polynomials are of degrees 2, 4 and 12 respectively. In this paper we obtain infinitely many solutions of the equation under consideration in terms of quadratic polynomials in two parameters, as well as a solution in terms of quartic polynomials in three parameters. We also show how more general multi-parameter solutions of the equation may be obtained.

1 Introduction

This paper is concerned with the problem of finding four biquadrates whose sum is a perfect square, that is, we are required to find four integers $x_i, i = 1, \dots, 4$, such that

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = y^2. \tag{1}$$

where y is an integer. If we take three of the integers $x_i, i = 1, \dots, 4$, as 0, we immediately get a trivial solution of Eq. (1). All other solutions of the diophantine equation (1) will be considered as nontrivial.

All known nontrivial parametric solutions of Eq. (1) are expressed in terms of polynomials in two parameters—the simplest being given by the identity,

$$(a^4+b^4)^2 = a^8+(ab)^4+(ab)^4+b^8.$$

Dickson [3, pp. 657–658]) has mentioned two other solutions—found by Fauquembergue—in which the values of $x_i, i=1,\ldots,4$, are given by polynomials of degrees 4 and 12 in terms of two parameters. Recently Alvarado and Delorme [1] have obtained two solutions in which $x_i, i=1,\ldots,4$, are given by polynomials of degrees 16 and 28, and they have shown how infinitely many such solutions may be obtained by using elliptic curves.

In this paper we will obtain infinitely many solutions of Eq. (1) in terms of quadratic polynomials in two parameters, as well as a solution in terms of quartic polynomials in three parameters. We also show how more general multi-parameter solutions of the equation may be obtained.

2 Four biquadrates whose sum is a square

2.1 Solutions in terms of quadratic polynomials in two parameters

In this section we will obtain solutions of the diophantine equation (1) in terms of polynomials of degree 2 in two parameters by using the well-known composition of forms identity,

$$(f^2+fg+g^2)(u^2+uv+v^2) = p^2+pq+q^2, \tag{2}$$

where f, g, u, v are arbitrary parameters and

$$p = fu-gv, \quad q = fv+gu+gv,$$

together with the identity,

$$r^4+s^4+(r+s)^4 = 2(r^2+rs+s^2)^2, \tag{3}$$

where r and s are arbitrary parameters.

It follows from (2) that

$$\begin{aligned}(f^2+fg+g^2)(u^2+uv+v^2)^2 &= (p^2+pq+q^2)(u^2+uv+v^2),\\ &= r^2+rs+s^2,\end{aligned} \tag{4}$$

where

$$r = fu^2-2guv-(f+g)v^2, \quad s = gu^2+(2f+2g)uv+fv^2. \tag{5}$$

Thus, when the values of r and s are given by (5), we get, on combining the two identities (3) and (4),

$$r^4+s^4+(r+s)^4 = 2(f^2+fg+g^2)^2(u^2+uv+v^2)^4, \tag{6}$$

and on adding $h^4(u^2+uv+v^2)^4$ to both sides of (6), we get the identity,

$$r^4+s^4+(r+s)^4+h^4(u^2+uv+v^2)^4=(2(f^2+fg+g^2)^2+h^4)(u^2+uv+v^2)^4. \quad (7)$$

If we choose integer values of f, g, and h such that

$$2(f^2+fg+g^2)^2+h^4=t^2, \quad (8)$$

where t is some integer, it follows from (5) and (7) that a solution of (1) is given by

$$\begin{aligned}(x_1,x_2,x_3,x_4,y)=(&fu^2-2guv-(f+g)v^2, gu^2+(2f+2g)uv+fv^2,\\ &(f+g)u^2+2fuv-gv^2, h(u^2+uv+v^2), t(u^2+uv+v^2)^2),\end{aligned}$$

where u and v are arbitrary parameters.

We readily found, by computer trials, the following four sets of values of (f,g,h,t) that satisfy Eq. (8):

$$(2,2,1,17), \quad (2,4,7,63), \quad (2,14,17,433), \quad (4,22,7,833).$$

These four solutions of Eq. (8) immediately yield four solutions of Eq. (1) in which the values of x_i are given by quadratic polynomials in the arbitrary parameters u and v. The first two solutions of Eq. (1) thus obtained may be written explicitly as follows:

$$\begin{aligned}(x_1,x_2,x_3,x_4,y)=(&2u^2-4uv-4v^2, 2u^2+8uv+2v^2, 4u^2+4uv-2v^2,\\ &u^2+uv+v^2, 17(u^2+uv+v^2)^2),\end{aligned} \quad (9)$$

and

$$\begin{aligned}(x_1,x_2,x_3,x_4,y)=(&2u^2-8uv-6v^2, 4u^2+12uv+2v^2, 6u^2+4uv-4v^2,\\ &7(u^2+uv+v^2), 63(u^2+uv+v^2)^2).\end{aligned} \quad (10)$$

We will now show how infinitely many solutions in integers of Eq. (8) may be obtained. For fixed rational numerical values of g and h, Eq. (8) is a quartic model of an elliptic curve. If this elliptic curve is of positive rank, we can obtain infinitely many rational points on this elliptic curve, and by appropriate scaling, we can obtain infinitely many integer solutions of Eq. (8). For instance, when $(g,h)=(2,1)$, we may write Eq. (8) as

$$2f^4+8f^3+24f^2+32f+33=t^2, \quad (11)$$

and one rational point on the curve (11) is known to be $(f,t)=(2,17)$.

The birational transformation defined by

$$\begin{aligned}f&=-2(145X+17Y+1208)/(144X-17Y+1176),\\ t&=(4913X^3+121584X^2+982600X-6864Y+2585088)\\ &\quad\times(144X-17Y+1176)^{-2},\end{aligned} \quad (12)$$

and

$$\begin{aligned} X &= 2(20f^2 + 64f + 17t + 81)/(f-2)^2, \\ Y &= 4(102f^3 + 408f^2 + 72ft + 816f + 145t + 833)/(f-2)^3, \end{aligned} \tag{13}$$

reduces the elliptic curve (11) to the Weierstrass form,

$$Y^2 = X^3 - 200X - 1088. \tag{14}$$

A reference to Cremona's elliptic curve tables [2] shows that the rank of the elliptic curve (14) is 1. In fact, using APECS (a package written in Maple for working with elliptic curves), we readily found the rational point $P = (217/9, 2431/27)$ on the curve (14). Since the point P does not have rational coordinates, it follows from the Nagell-Lutz theorem [4, p. 56] on elliptic curves that P is a point of infinite order. We can now find infinitely many rational points on the curve (14) using the group law, and using the relations (12), we can find infinitely many rational points on the curve (11). We can now obtain infinitely many integer solutions of Eq. (8) and thus generate infinitely many parametric solutions of Eq. (1).

As a numerical example, the point P on the curve (14) generates a solution that is equivalent to the solution (9) while the point $2P$ generates the following parametric solution of Eq. (1):

$$\begin{aligned} x_1 &= 162278u^2 + 667292uv + 171368v^2, \\ x_2 &= 333646u^2 + 342736uv - 162278v^2, \\ x_3 &= 171368u^2 - 324556uv - 333646v^2, \\ x_4 &= 166823(u^2 + uv + v^2), \\ y &= 121336214273(u^2 + uv + v^2)^2, \end{aligned}$$

where u and v are arbitrary parameters.

2.2 Solutions in terms of quartic polynomials in three parameters

We will now obtain a solution of (1) in which the values of $x_i, i = 1, \ldots, 4$, are given by quartic polynomials in three parameters.

We write

$$x_1 = t + a, \quad x_2 = t + b, \quad x_3 = t + c, \quad x_4 = t, \tag{15}$$

when the left-hand side of Eq. (1) becomes a quartic function of t in which the coefficient of t^4 is a perfect square, namely 4. Dickson [3, p. 639] has described a method, originally given by Fermat, for making such a quartic function a perfect square. Applying this method, we find that the left-hand side of Eq. (1) becomes a perfect square if we take

$$t = \phi(a, b, c)/(24(a + b - c)(a - b + c)(a - b - c)),$$

where

$$\begin{aligned} \phi(a, b, c) = {} & 9(a^4 + b^4 + c^4) - 20(a^3(b + c) + b^3(c + a) + c^3(a + b)) \\ & + 54(a^2b^2 + b^2c^2 + c^2a^2) - 12abc(a + b + c). \end{aligned}$$

Using the relations (15), we now get four biquadrates whose sum is a perfect square. On appropriate scaling, we may write these biquadrates as $x_i^4, i = 1, \ldots, 4$, with the values of x_i being given by the quartic polynomials,

$$x_1 = \psi(a, b, c), \quad x_2 = \psi(b, c, a), \quad x_3 = \psi(c, a, b), \quad x_4 = \phi(a, b, c), \tag{16}$$

where

$$\begin{aligned} \psi(a, b, c) = 33a^4 - 44a^3b - 44a^3c + 30a^2b^2 + 36a^2bc \\ + 30a^2c^2 + 4ab^3 - 36ab^2c - 36abc^2 + 4ac^3 \\ + 9b^4 - 20b^3c + 54b^2c^2 - 20bc^3 + 9c^4. \end{aligned}$$

On making the invertible linear transformation,

$$a = (v + w)/2, \quad b = (u + w)/2, \quad c = (u + v)/2,$$

the values of x_i given by (16) may be written as

$$\begin{aligned} x_1 &= f(u, -v, -w), \qquad x_2 = (-u, v, -w), \\ x_3 &= f(-u, -v, w), \qquad x_4 = f(u, v, w), \end{aligned} \tag{17}$$

where $f(u, v, w)$ is defined in terms of arbitrary parameters u, v, and w as follows:

$$f(u, v, w) = 2(u^4 + v^4 + w^4) + 3(u^2v^2 + v^2w^2 + w^2u^2) + 6uvw(u + v + w).$$

With the values of $x_i, i = 1, \ldots, 4$, given by (17), it is readily verified that $\sum_{i=1}^{4} x_i^4 = y^2$ where

$$\begin{aligned} y = 8(u^8 + v^8 + w^8) + 24(u^6(v^2 + w^2) + v^6(w^2 + u^2) + w^6(u^2 + v^2)) \\ + 34(u^4v^4 + v^4w^4 + w^4u^4) + 276u^2v^2w^2(u^2 + v^2 + w^2). \end{aligned} \tag{18}$$

Thus a solution of Eq. (1) is given by (17) and (18) in terms of the arbitrary parameters u, v, and w.

2.3 Multi-parameter solutions

We will now describe a method of obtaining more general solutions of Eq. (1) in several parameters by using the known solutions of Eq. (1).

Let a solution of Eq. (1) be $(x_1, x_2, x_3, x_4, y) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta)$. Instead of choosing the values of $x_i, i = 1, \ldots, 4$, given by (15), we now write,

$$\begin{aligned} x_1 &= \alpha_1 t_1 + \gamma_1 t_2, \quad x_2 = \alpha_2 t_1 + \gamma_2 t_2, \\ x_3 &= \alpha_3 t_1 + \gamma_3 t_2, \quad x_4 = \alpha_4 t_1 + \gamma_4 t_2, \end{aligned} \tag{19}$$

where $\gamma_i, i = 1, \ldots, 4$, are arbitrary parameters, and proceeding as before, we again observe that the left-hand side of Eq. (1) becomes a quartic function of t_1 in which the coefficient of t_1^4 is a perfect square, namely β^2 and hence, as before, we can obtain a value of t_1 that yields four biquadrates whose sum is a perfect square. We thus obtain more parametric solutions of Eq. (1).

We could also choose the parameters γ_i in (19) such the sum of the four biquadrates γ_i^4 is a perfect square, say δ^2, and now the left-hand side of Eq. (1) becomes a binary quartic form in t_1 and t_2 such that the coefficients of t_1^4 and t_2^4 are β^2 and δ^2, respectively. If we now write

$$y = \beta t_1^2 + 2\left(\sum_{i=1}^{4} \alpha_i^3 \gamma_i\right) t_1 t_2/\beta + \delta t_2^2, \tag{20}$$

and substitute the values of $x_i, i = 1, \ldots, 4$, and y given by (19) and (20), respectively, in Eq. (1), we get after suitable transposition and removal of common factors, a linear equation in t_1 and t_2 whose solution is as follows:

$$\begin{aligned} t_1 &= 2\left(\sum_{i=1}^{4} \alpha_i \gamma_i^3\right) \beta^2 - 2\left(\sum_{i=1}^{4} \alpha_i^3 \gamma_i\right) \beta\delta, \\ t_2 &= \beta^3\delta - 3\left(\sum_{i=1}^{4} \alpha_i^2 \gamma_i^2\right) \beta^2 + 2\left(\sum_{i=1}^{4} \alpha_i^3 \gamma_i\right)^2. \end{aligned} \tag{21}$$

With the values of t_1 and t_2 given by (21), a solution of Eq. (1) is given by (19) and (20). Since we can choose the values of $\alpha_i, \gamma_i, i = 1, \ldots, 4$, in terms of 2 or 3 parameters, the solutions of Eq. (1) thus obtained are in terms of 4, 5 or 6 parameters. All the multi-parameter solutions of Eq. (1) obtained in this manner are too cumbersome to write and accordingly we do not give any of them explicitly.

It would of considerable interest to find a parametric solution of Eq. (1) in which the parameters can be so chosen that the value of y becomes a perfect square. If that can be achieved, we will obtain four biquadrates whose sum is also a biquadrate. There seems to be no simple way of finding such a parametric solution of Eq. (1).

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