Journal of Integer Sequences, Vol. 24 (2021),
Article 21.4.2

A Note on Sumsets and Restricted Sumsets

Jagannath Bhanja
Harish-Chandra Research Institute
Chhatnag Road, Jhunsi
Prayagraj 211019
India
jagannathbhanja@hri.res.in

Abstract

In this note we find the optimal lower bound for the size of the sumsets HA and $H\hat{}A$ over finite sets H, A of nonnegative integers, where $HA = \bigcup_{h \in H} hA$ and $H\hat{}A = \bigcup_{h \in H} h\hat{}A$. We also find the underlying algebraic structure of the sets A and H for which the size of the sumsets HA and $H\hat{}A$ is minimum.

1 Introduction

For a given finite set A of integers and for a positive integer h, the *sumset* hA and the *restricted sumset* $h\hat{}A$ are fundamental objects in the field of *additive number theory*. The sumset hA is the set of integers that can be written as the sum of h elements of A, whereas the sumset $h\hat{}A$ is the set of integers that can be written as the sum of h *pairwise distinct* elements of A. In this regard, two of the important problems in additive number theory are to find the best possible lower bounds for the size of the sumsets hA and $h\hat{}A$, and to find the structure of the finite set A for which the sumsets hA and $h\hat{}A$ contain the minimum number of elements. These two problems have been well established in the group of integers [4, 5].

Theorem 1. *[5, Theorem 1.4, Theorem 1.6] Let A be a finite set of k integers. Let h be a positive integer. Then*

$$|hA| \geq h(k-1)+1.$$

Moreover, if this lower bound is exact with $h \geq 2$, then A is an arithmetic progression.

Theorem 2. *[4, Theorem 1, Theorem 2] Let A be a finite set of k integers. Let $h \leq k$ be a positive integer. Then*

$$|h\hat{}A| \geq h(k-h)+1.$$

Moreover, if this lower bound is exact with $k \geq 5$ and $2 \leq h \leq k-2$, then A is an arithmetic progression.

Now let H be a given finite set of nonnegative integers. Define the sumset [1, p. 175]

$$HA := \bigcup_{h \in H} hA,$$

and the restricted sumset

$$H\hat{}A := \bigcup_{h \in H} h\hat{}A.$$

Here we are assuming that $0A = 0\hat{}A = \{0\}$.

For a set A and for an integer c, we let $c \cdot A = \{ca : a \in A\}$. For integers a, b with $a \leq b$, we also let $[a, b] = \{a, a+1, \ldots, b\}$.

The case $H = [0, r]$ is more interesting and has been studied before (for recent papers, see [1, 2, 3]). Bajnok [1] defined the sumsets for an arbitrary finite set H of nonnegative integers, and asked to study similar problems like the sumsets hA and $h\hat{}A$ over finite abelian groups. That is, find the optimal lower bound for the size of the sumsets HA and $H\hat{}A$, and the structure of the sets H, A for which the sumsets HA, $H\hat{}A$ contain the minimum number of elements.

In this note, we study these two problems for the sumset HA in Section 2, and the sumset $H\hat{}A$ in Section 3, for finite sets A of nonnegative integers (or nonpositive integers) and H of nonnegative integers. We consider two separate cases, namely

(i) the set A consists of positive integers and

(ii) the set A consists of nonnegative integers.

The cases

(iii) the set A consists of negative integers and

(iv) the set A consists of nonpositive integers,

follow from the cases (i) and (ii), respectively, as $H(c \cdot A) = c \cdot HA$ and $H\hat{}(c \cdot A) = c \cdot H\hat{}A$ for arbitrary integers c. As consequences of our results we obtain some recent results in this direction.

In Section 2 and Section 3, we use the following notation: for a set $S = \{s_1, s_2, \ldots, s_{k-1}, s_k\}$ with $k \geq 2$ and $s_1 < s_2 < \cdots < s_{k-1} < s_k$, we write $\min(S) = s_1$, $\min_+(S) = s_2$, $\max(S) = s_k$, and $\max_-(S) = s_{k-1}$.

2 Regular sumset

Theorem 3. *Let A be a set of k positive integers. Let H be a set of r positive integers with $\max(H) = h_r$. Then*

$$|HA| \geq h_r(k-1) + r. \tag{1}$$

This lower bound is optimal.

Proof. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $H = \{h_1, h_2, \ldots, h_r\}$, where $0 < a_1 < a_2 < \cdots < a_k$ and $0 < h_1 < h_2 < \cdots < h_r$. Set

$$S_1 := h_1 A \tag{2}$$

and

$$S_i := (h_i - h_{i-1})A + h_{i-1}a_k, \tag{3}$$

for $i = 2, 3, \ldots, r$. Clearly $S_i \subseteq h_i A$ for $i = 1, 2, \ldots, r$, and $\max(S_i) < \min(S_{i+1})$ for $i = 1, 2, \ldots, r-1$. Therefore $S_1, S_2, \ldots, S_r$ are pairwise disjoint subsets of HA. Hence, by Theorem 1, we have

$$\begin{aligned} |HA| &\geq \sum_{i=1}^{r} |S_i| \\ &= |h_1 A| + \sum_{i=2}^{r} |S_i| \\ &\geq h_1(k-1) + 1 + \sum_{i=2}^{r} [(h_i - h_{i-1})(k-1) + 1] \\ &= h_r(k-1) + r. \end{aligned} \tag{4}$$

This proves (1).

To see that the lower bound in (1) is optimal, let $A = [1, k]$ and $H = [1, r]$. Then $HA = [1, rk]$, and hence $|HA| = rk$. This completes the proof of the theorem. □

Remark 4. If A contains nonnegative integers with $0 \in A$, then $HA = h_r A$, as $h_i A \subseteq h_r A$ for $i = 1, 2, \ldots, r-1$. Therefore, by Theorem 1, we have $|HA| \geq h_r(k-1)+1$. Furthermore, this bound is optimal, and it can be seen by taking $A = [0, k-1]$ and $H = [1, r]$.

Now we prove the inverse result of Theorem 3.

Theorem 5. *Let A be a set of $k \geq 2$ positive integers and H be a set of $r \geq 2$ positive integers with $\max(H) = h_r$. If $|HA| = h_r(k-1) + r$, then H is an arithmetic progression of difference d and A is an arithmetic progression of difference $d \cdot \min(A)$.*

Proof. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $H = \{h_1, h_2, \ldots, h_r\}$, where $0 < a_1 < a_2 < \cdots < a_k$ and $0 < h_1 < h_2 < \cdots < h_r$. Let $|HA| = h_r(k-1) + r$. Then the sumset HA contains precisely the elements of the sets S_i for $i = 1, \ldots, r$, which are defined in (2), (3).

First, we show that A is an arithmetic progression. Observe that the assumption $|HA| = h_r(k-1)+r$ together with (4) implies $|h_1A| = h_1(k-1)+1$. If $h_1 > 1$, then from Theorem 1, it follows that the set A is an arithmetic progression. So, let $h_1 = 1$. Then

$$S_1 = h_1A = A = \{a_1, a_2, \ldots, a_k\}.$$

Set

$$S := \{a_1, h_2a_1, (h_2-1)a_1 + a_2, \ldots, (h_2-1)a_1 + a_{k-1}\}.$$

Clearly $S \subseteq HA$ and $\max(S) = (h_2-1)a_1 + a_{k-1} < (h_2-1)a_1 + a_k = \min(S_2)$. Thus $S = S_1$. In other words, $(h_2-1)a_1 + a_{i-1} = a_i$ for $i = 2, 3, \ldots, k$. Equivalently, $a_i - a_{i-1} = (h_2-1)a_1$ for $i = 2, 3, \ldots, k$. Hence, A is an arithmetic progression.

Next we show that H is an arithmetic progression. For $i = 1, 2, \ldots, r-1$, consider the integers $(h_{i+1} - h_i)a_1 + a_{k-1} + (h_i - 1)a_k$. Clearly

$$\begin{aligned}\max\nolimits_{-}(S_i) = a_{k-1} + (h_i-1)a_k &< (h_{i+1}-h_i)a_1 + a_{k-1} + (h_i-1)a_k \\ &< (h_{i+1}-h_i)a_1 + h_ia_k = \min(S_{i+1}).\end{aligned}$$

But we already have

$$\max\nolimits_{-}(S_i) = a_{k-1} + (h_i-1)a_k < h_ia_k = \max(S_i) < (h_{i+1}-h_i)a_1 + h_ia_k = \min(S_{i+1}).$$

Thus

$$(h_{i+1}-h_i)a_1 + a_{k-1} + (h_i-1)a_k = h_ia_k \text{ for } i = 1, 2, \ldots, r-1.$$

This implies

$$a_k - a_{k-1} = (h_{i+1}-h_i)a_1 \text{ for } i = 1, 2, \ldots, r-1. \tag{5}$$

Therefore

$$h_2 - h_1 = h_3 - h_2 = \cdots = h_r - h_{r-1},$$

and hence the set H is an arithmetic progression. Furthermore, by (5), the set A is an arithmetic progression of difference $(h_{i+1}-h_i)a_1$. This completes the proof of the theorem. □

3 Restricted sumset

Theorem 6. *Let A be a set of k positive integers and $H = \{h_1, h_2, \ldots, h_r\}$ be a set of positive integers with $h_1 < h_2 < \cdots < h_r \le k$. Set $h_0 = 0$. Then*

$$|H\hat{}A| \ge \sum_{i=1}^{r}(h_i - h_{i-1})(k - h_i) + r. \tag{6}$$

This lower bound is optimal.

Proof. Let $A = \{a_1, a_2, \dots, a_k\}$, where $a_1 < a_2 < \cdots < a_k$. Set

$$S_1 := h_1 \hat{} A \tag{7}$$

and

$$S_i := (h_i - h_{i-1}) \hat{} A_i + \max(h_{i-1} \hat{} A), \tag{8}$$

for $i = 2, 3, \dots, r$, where $A_i = \{a_1, a_2, \dots, a_{k-h_{i-1}}\}$. Clearly $S_i \subseteq h_i \hat{} A$ for $i = 1, 2, \dots, r$, and $\max(S_i) < \min(S_{i+1})$ for $i = 1, 2, \dots, r-1$. Therefore $S_1, S_2, \dots, S_r$ are pairwise disjoint subsets of $H \hat{} A$. Hence, by Theorem 2, we have

$$\begin{aligned} |H \hat{} A| &\geq \sum_{i=1}^{r} |S_i| \\ &= |h_1 \hat{} A| + \sum_{i=2}^{r} |S_i| \\ &\geq h_1(k - h_1) + 1 + \sum_{i=2}^{r} [(h_i - h_{i-1})(k - h_i) + 1] \\ &= \sum_{i=1}^{r} (h_i - h_{i-1})(k - h_i) + r. \end{aligned} \tag{9}$$

This proves (6).

Next to see that the lower bound in (6) is optimal, let $A = [1, k]$ and $H = [1, r]$ with $r \leq k$. Then $H \hat{} A \subseteq [1, k + (k-1) + \cdots + (k - r + 1)]$. Therefore $|H \hat{} A| \leq rk - \frac{r(r-1)}{2}$. This together with (6) implies $|H \hat{} A| = rk - \frac{r(r-1)}{2}$, and hence completes the proof of the theorem. □

As a consequence of Theorem 6, we obtain the following corollary.

Corollary 7. *Let A be a set of k nonnegative integers with $0 \in A$. Let $H = \{h_1, h_2, \dots, h_r\}$ be a set of positive integers with $h_1 < h_2 < \cdots < h_r \leq k - 1$. Set $h_0 = 0$. Then*

$$|H \hat{} A| \geq \sum_{i=1}^{r} (h_i - h_{i-1})(k - h_i - 1) + r + 1. \tag{10}$$

This lower bound is optimal.

Proof. Let $A' = A \setminus \{0\}$. Then $H \hat{} A' \cup \{0\} \subseteq H \hat{} A$. This implies

$$|H \hat{} A| \geq |H \hat{} A'| + 1 \geq \sum_{i=1}^{r} (h_i - h_{i-1})(k - h_i - 1) + r + 1.$$

Furthermore, the optimality of the lower bound in (10) can be checked by taking $A = [0, k-1]$ and $H = [1, r]$, where k, r are positive integers with $r \leq k - 1$. □

The following result (which has recently been proved) is a particular case of Theorem 6 and Corollary 7.

Corollary 8. *[3, Theorem 2.1, Corollary 2.1] Let A be a set of k nonnegative integers and $H = [0, r]$ with $r \leq k$. If $0 \notin A$, then*

$$|H\hat{}A| \geq rk - \frac{r(r-1)}{2} + 1.$$

If $0 \in A$ and $r \leq k-1$, then

$$|H\hat{}A| \geq rk - \frac{r(r+1)}{2} + 1.$$

These lower bounds are optimal.

Now we prove the inverse theorem of Theorem 6.

Theorem 9. *Let A be a set of $k \geq 6$ positive integers. Let $H = \{h_1, h_2, \ldots, h_r\}$ be a set of $r \geq 2$ positive integers with $h_1 < h_2 < \cdots < h_r \leq k-1$. Set $h_0 = 0$. If*

$$|H\hat{}A| = \sum_{i=1}^{r} (h_i - h_{i-1})(k - h_i) + r,$$

then $H = h_1 + [0, r-1]$ and $A = \min(A) \cdot [1, k]$.

Proof. Let $A = \{a_1, a_2, \ldots, a_k\}$, where $0 < a_1 < a_2 < \cdots < a_k$. Let $|H\hat{}A| = \sum_{i=1}^{r}(h_i - h_{i-1})(k - h_i) + r$. Then the sumset $H\hat{}A$ contains precisely the elements of the sets S_i for $i = 1, \ldots, r$, which are defined in (7), (8).

First, we show that A is an arithmetic progression. Since $|H\hat{}A| = \sum_{i=1}^{r}(h_i - h_{i-1})(k - h_i) + r$, from (9), it follows that $|h_1\hat{}A| = h_1(k - h_1) + 1$. If $h_1 \geq 2$, then by Theorem 2, the set A is an arithmetic progression. Therefore, let $h_1 = 1$.

If $h_2 \geq 3$, then $h_2 - h_1 \geq 2$. By (9), we get $|S_2| = |(h_2 - h_1)\hat{}A_2| = (h_2 - h_1)(k - h_2) + 1$, where $A_2 = \{a_1, a_2, \ldots, a_{k-1}\}$. Therefore, by Theorem 2, the set A_2 is an arithmetic progression. To show that A is an arithmetic progression, it is left to show that $a_k - a_{k-1} = a_{k-1} - a_{k-2}$. Consider the following integers:

$$a_{k-2} < a_1 + \cdots + a_{h_2-1} + a_{k-2} < a_1 + \cdots + a_{h_2-1} + a_{k-1} < a_1 + \cdots + a_{h_2-1} + a_k = \min(S_2).$$

But we already have

$$a_{k-2} < a_{k-1} < a_k < a_1 + \cdots + a_{h_2-1} + a_k = \min(S_2),$$

where $\{a_{k-2}, a_{k-1}, a_k\} \subseteq S_1$. Thus

$$a_1 + \cdots + a_{h_2-1} + a_{k-2} = a_{k-1} \text{ and } a_1 + \cdots + a_{h_2-1} + a_{k-1} = a_k.$$

This implies

$$a_k - a_{k-1} = a_1 + \cdots + a_{h_2-1} = a_{k-1} - a_{k-2},$$

and we are done.

Now let $h_2 = 2$; i.e., $h_2 - h_1 = 1$. Set $T_1 := \{a_1, a_2, a_1 + a_2, a_1 + a_3, \ldots, a_1 + a_{k-1}\}$. Clearly $T_1 \subseteq h_1\hat{}A \cup h_2\hat{}A \subseteq H\hat{}A$ and $\max(T_1) = a_1 + a_{k-1} < a_1 + a_k = \min(S_2)$. Therefore $T_1 = S_1$. That is

$$\{a_1, a_2, a_1 + a_2, a_1 + a_3, \ldots, a_1 + a_{k-1}\} = \{a_1, a_2, a_3, \ldots, a_k\}.$$

Thus $a_i = a_1 + a_{i-1}$ for $i = 3, 4, \ldots, k$. Equivalently, $a_i - a_{i-1} = a_1$ for $i = 3, 4, \ldots, k$. To show that A is an arithmetic progression it is enough to show $a_2 - a_1 = a_k - a_{k-1}$. Consider the integer $a_2 + a_{k-1}$. Since $\max(S_1) = \max(T_1) = a_1 + a_{k-1} < a_2 + a_{k-1} < a_2 + a_k = \min^+(S_2)$ and $\max(S_1) = \max(T_1) = a_1 + a_{k-1} < a_1 + a_k = \min(S_2) < a_2 + a_k = \min^+(S_2)$, we must have $a_2 + a_{k-1} = a_1 + a_k$. This proves A is an arithmetic progression.

Next we show that H is an arithmetic progression. For $i = 1, 2, \ldots, r-1$, consider the following integers:

$$\begin{aligned}\max_-(S_i) = a_{k-h_i} + a_{k-h_i+2} + \cdots + a_k &< a_{k-h_i+1} + \cdots + a_k = \max(S_i) \\ &< a_1 + \cdots + a_{h_{i+1}-h_i} + a_{k-h_i+1} + \cdots + a_k = \min(S_{i+1})\end{aligned}$$

and

$$\begin{aligned}\max_-(S_i) = a_{k-h_i} + a_{k-h_i+2} + \cdots + a_k &< a_1 + \cdots + a_{h_{i+1}-h_i} + a_{k-h_i} + a_{k-h_i+2} + \cdots + a_k \\ &< a_1 + \cdots + a_{h_{i+1}-h_i} + a_{k-h_i+1} + \cdots + a_k = \min(S_{i+1}).\end{aligned}$$

Therefore

$$a_{k-h_i+1} + \cdots + a_k = a_1 + \cdots + a_{h_{i+1}-h_i} + a_{k-h_i} + a_{k-h_i+2} + \cdots + a_k$$

for $i = 1, 2, \ldots, r-1$. This implies

$$a_{k-h_i+1} - a_{k-h_i} = a_1 + \cdots + a_{h_{i+1}-h_i} \text{ for } i = 1, 2, \ldots, r-1. \tag{11}$$

Since A is an arithmetic progression, the difference between any two consecutive elements in A is same. Therefore

$$a_2 - a_1 = a_{k-h_i+1} - a_{k-h_i} = a_1 + a_2 + \cdots + a_{h_{i+1}-h_i} \text{ for } i = 1, 2, \ldots, r-1.$$

This holds, only if $h_{i+1} - h_i = 1$ for $i = 1, 2, \ldots, r-1$. Hence, $H = h_1 + [0, r-1]$ and $A = a_1 \cdot [1, k]$. This completes the proof of the theorem.

□

Corollary 10. *Let A be a set of $k \geq 7$ nonnegative integers with $0 \in A$. Let $H = \{h_1, h_2, \ldots, h_r\}$ be a set of $r \geq 2$ positive integers with $h_1 < h_2 < \cdots < h_r \leq k-1$. Set $h_0 = 0$. If*

$$|H\hat{}A| = \sum_{i=1}^{r} (h_i - h_{i-1})(k - h_i - 1) + r + 1,$$

then $H = h_1 + [0, r-1]$ and $A = \min(A \setminus \{0\}) \cdot [0, k-1]$.

Proof. Let $A' := A\setminus\{0\}$. Then the equality $|H\hat{}A| = \sum_{i=1}^{r}(h_i - h_{i-1})(k - h_i - 1) + r + 1$ implies $|H\hat{}A'| = \sum_{i=1}^{r}(h_i - h_{i-1})(k - 1 - h_i) + r$. Thus by Theorem 9, we get $H = h_1 + [0, r-1]$ and $A' = \min(A') \cdot [1, k-1]$. Hence, $H = h_1 + [0, r-1]$ and $A = \min(A') \cdot [0, k-1]$. This proves the theorem. □

The following inverse result (which has recently been proved) is a particular case of Theorem 9 and Corollary 10.

Corollary 11. *[3, Theorem 2.2, Corollary 2.3] Let A be a set of $k \geq 7$ nonnegative integers and $H = [0, r]$ with $2 \leq r \leq k-1$. If $0 \notin A$ and $|H\hat{}A| = rk - \frac{r(r-1)}{2} + 1$, then $A = d \cdot [1, k]$ for some positive integer d.*

If $0 \in A$ and $|H\hat{}A| = rk - \frac{r(r+1)}{2} + 1$, then $A = d \cdot [0, k-1]$ for some positive integer d.

4 Acknowledgement

We would like to thank Professor R. Thangadurai for carefully going through this manuscript. We would also like to thank the editor-in-chief and the anonymous referee for the helpful comments and suggestions which helped improve the article. We also acknowledge the support of the PDF scheme of Harish-Chandra Research Institute, Prayagraj.

References

[1] B. Bajnok, *Additive Combinatorics: A Menu of Research Problems*, Discrete Mathematics and its Applications, CRC Press, 2018.

[2] É. Balandraud, Addition theorems in $\mathbb{F}_p$ via the polynomial method, preprint, 2017. Available at https://arxiv.org/abs/1702.06419.

[3] J. Bhanja and R. K. Pandey, Inverse problems for certain subsequence sums in integers, *Discrete Math.* **343** (2020), 112148.

[4] M. B. Nathanson, Inverse theorems for subset sums, *Trans. Amer. Math. Soc.* **347** (1995), 1409–1418.

[5] M. B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, 1996.

2020 *Mathematics Subject Classification*: Primary 11P70; Secondary 11B75, 11B13.
Keywords: sumset, restricted sumset.

Received August 22 2020; revised versions received August 23 2020; February 1 2021; March 27 2021; March 28 2021. Published in *Journal of Integer Sequences*, March 28 2021.

Return to Journal of Integer Sequences home page.