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Explicit Estimates Involving the Primorial Integers and Applications

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Abstract

In this paper, we propose several explicit bounds for the function that counts the number of primorial integers less than or equal to a given positive real number. As applications, we obtain an effective version of Pósa's inequality, and a method to estimate the maximal value of the sum over prime divisors $\sum_{p|q} f(p)$ for a positive decreasing function f, when q ranges over all integers less than x. In particular, we improve the upper bound for the maximal value of $\sum_{p|q} \frac{\log p}{p-1}$.

1 Introduction and statement of results

As usual, let $(p_k)_{k\geq 1}$ denote the increasing sequence of prime numbers, and let N_k be the primorial (prime-factorial) integer of index k, the product of its k first terms. The integers N_k are the terms of the sequence A002110 in the *On-line Encyclopedia of Integer Sequences* (OEIS) [22], and play an important role in number theory from Euclid's proof of the infinity of primes to the remarkable equivalences of the Riemann hypothesis due to Nicolas [13] and Robin [19].

The approximations linked to N_k have found some unexpected applications in various areas of number theory; see the papers of Betts [5], Planat et al. [16] and Zhang [23]. However, the most frequently used results concern their logarithm $\theta(p_k)$ where θ denotes the Chebyshev function, here we cite the recent papers of Axler [1] and Dusart [8].

The principal object of this manuscript is to study the function $K(x)$ which counts the number of primorial integers less than or equal to x; for instance $K(1) = 0$, $K(2) = 1$, $K(3) = 1$, $K(5) = 1$, $K(6) = 2$,... and a table of some large values of x is given in Section 7.

The function $K(x)$ is very close to $\pi(x)$ (the usual prime counting function) since it will be shown in what follows that $K(x) \approx \pi(\log x)$. It often appears implicitly as an important key in the demonstration of several results; see, for instance, Balazard [2] and Hassani [11].

We let $\log_i$ denote the i-fold iterated logarithm. The asymptotic expansion of $K(x)$ is easy to evaluate. Let us show this using less precise results than those in Balazard [2]:

Theorem 1 (Balazard)**.** *For every real number $x \geq 2$ and every integer $m \geq 0$, we have*

$$K(x) = \frac{\log x}{\log_2 x}\left(\sum_{j=0}^{m} \frac{j!}{\log_2^j x} + O\left(\frac{1}{\log_2^{m+1} x}\right)\right).$$

Proof. Since for a real $x \geq 2$, the integer $K := K(x)$ is also given by the inequalities

$$\theta(p_K) \leq \log x < \theta(p_{K+1}),$$

using the prime number theorem in the forms

$$\theta(t) = t + O_m\left(\frac{t}{\log^{m+2} t}\right)$$

and

$$\pi(t) = \mathrm{li}(t) + O_m\left(\frac{t}{\log^{m+2} t}\right),$$

where $\mathrm{li}(x)$ indicates the usual logarithmic integral function, one gets successively

$$p_K = \log x + O_m\left(\frac{\log x}{\log_2^{m+2} x}\right),$$

$$K(x) = \pi(p_K) = \mathrm{li}(\log x) + O_m\left(\frac{\log x}{\log_2^{m+2} x}\right).$$

Finally, by substituting $\log x$ in the classic asymptotic expansion of $\mathrm{li}(x)$ below

$$\mathrm{li}(x) = \int_2^x \frac{dt}{\log t} = \frac{x}{\log x}\left(0! + \frac{1!}{\log x} + \cdots + \frac{m!}{\log^m x} + O\left(\frac{1}{\log^{m+1} x}\right)\right),$$

we establish the result. $\square$

Consequently, the asymptotic expansion of the function $xK(x)$ corresponds to a type of formula among a large class of expansions related to the iterated logarithms studied in Belbachir and Berkane [4] to which an asymptotic expansion for the sum of inverses is determined. The following proposition is a straightforward application of Belbachir and Berkane [4, Theorem 1]:

Proposition 2. *For every real number $x \geq 2$ and every integer $m \geq 2$, we have*

$$\sum_{2\leq n\leq x} \frac{1}{nK(n)} = \frac{1}{2}\log_2^2 x - \log_2 x - \log_3 x + C$$
$$+ \frac{\delta_2}{\log_2 x} + \cdots + \frac{\delta_m}{(m-1)\log_2^{m-1} x} + O\left(\frac{1}{\log_2^m x}\right),$$

where C is an absolute constant, and $\{\delta_j\}_{j\geq 0}$ is the sequence A233824 in the OEIS [22] given by the recurrence relation $\delta_n + 1!\delta_{n-1} + 2!\delta_{n-2} + \cdots + (n-1)!\delta_1 = n \cdot n!$.

In this article, by comparing $K(x)$ with $\pi(\log x)$ over large and special ranges, our first result (see Theorem 10) is an effective version of an inequality of Pósa [15], where he proved that for all $n > 1$ there is a k_n such that $N_k > p_{k+1}^n$ for all $n \geq k_n$. We describe an algorithm calculate k_n for every n.

The second result (see Theorem 19) is a good improvement of an approximation of the greatest sum over prime divisors $\mathfrak{L}(q) = \sum_{p|q} \frac{\log p}{p-1}$ when q ranges over all integers not exceeding x. The method employed to prove this theorem can also be applied to evaluate the maximum of sum of type $\mathfrak{L}_f(q) = \sum_{p|q} f(p)$ for any positive decreasing function f on $(1, \infty)$. In Section 5, we show that the maximal value of $\mathfrak{L}_f(q)$ is $\mathfrak{L}_f(N_{K(x)})$.

These results are follow-ups of fully explicit bounds for the O-term in the formula of $K(x)$, mainly Theorems 14–18.

Finally, throughout this paper, e represents Napier's constant, p a prime number and the calculations are made by using `Maple 17`.

2 Technical lemmas

The sequence $\log p_k$ has the same asymptotic behavior as its Cesàro average $\frac{\log N_k}{k}$, where the first few terms are

$$\frac{\log N_k}{k} = \log k + \log_2 k - 1 + \frac{\log_2 k - 2}{\log k} - \frac{\log_2^2 k - 6\log_2 k + 11}{2\log^2 k} + O\left(\frac{\log_2^3 k}{\log^3 k}\right).$$

Robin [20] and Dusart [7] obtained fully explicit bounds for p_k and $\theta(p_k)$, we gathered the ones we need in the following lemma.

Lemma 3.

$$\theta(p_k) > k \log k,\ \ \forall k \geq 3, \tag{1}$$

$$\theta(p_k) \geq k(\log k + \log_2 k - a),\ \ \forall k \geq 2,\ \ \textit{and } a = 1.0769, \tag{2}$$

$$\theta(p_k) \geq k(\log k + \log_2 k - 1 + \frac{\log_2 k - 2.1454}{\log k}),\ \ \forall k \geq 3, \tag{3}$$

$$\theta(p_k) \leq k(\log k + \log_2 k - 0.9465),\ \ \forall k \geq 14, \tag{4}$$

$$p_k \leq k(\log k + \log_2 k - \frac{1}{2}),\ \ \forall k \geq 20, \tag{5}$$

$$p_k \leq k \log p_k,\ \ \forall k \geq 4. \tag{6}$$

Similarly, the following lemma directly gives an upper bound of $\theta(p_{k+1})$ in terms of k instead of $k+1$. Note that $c = 1 - \log 2$ is an absolute constant.

Lemma 4. *We have, when* $k \geq 2$*:*

$$\theta(p_{k+1}) \leq k\left(\log k + \log_2 k - c + \frac{\log_2 k + c}{\log k}\right). \tag{7}$$

Proof. From inequality (5), by taking the logarithm and using the fact that $\log(1+x) \leq x$ for all $x > 0$, we easily obtain

$$\log p_k \leq \log k + \log_2 k + \frac{\log_2 k - 0.5}{\log k}; \tag{8}$$

this is valid even for $k \geq 18$. In particular, replacing k by $2m-1$ and using the inequality $\log(2m-1) < \log(m) + \log(2)$, one gets that for all $m \geq 2$:

$$\log p_{2m-1} \leq \log m + \log_2 m + \log 2 + \frac{\log_2 m + \log 2 - 0.5}{\log m} + \frac{\log 2}{\log^2 m}.$$

Since the right-hand side is a strictly increasing function $M(m)$, the sum of all $\log p_{2m-1}$ until $k-1$ is bounded above by $\int_2^k M(t)dt$, and we have

$$\sum_{m=1}^{k-1} \log p_{2m-1} \leq k\left(\log k + \log_2 k - c + \frac{\log_2 k + c}{\log k}\right) - D(k),$$

where

$$D(k) = \left(2\log 2 - \frac{3}{2}\right)\mathrm{li}(k) + \left(\frac{3}{2} - 2\log 2\right)\mathrm{li}(2) - 3\log 2 - 2\log_2 2 + 4 - 2\frac{\log_2 2}{\log 2} - \frac{2}{\log 2}.$$

Finally, since the function $D(x)$ is negative when $x \geq 11$, by applying the logarithm to the inequality obtained in the lemma below, one gets the result for $k \geq 11$. A computer check handles the cases $2 \leq k \leq 11$. □

Lemma 5. *We have, when $k \geq 11$:*

$$N_{k+1} < \prod_{i=1}^{k-1} p_{2i-1}.$$

Proof. We proceed by induction on k. The inequality holds for $k = 11$. As $k \geq 3$ implies $2k - 1 \geq k + 2$ and then $p_{k+2} \leq p_{2k-1}$, the lemma follows from the fact that

$$N_{k+2} < p_{k+2} \prod_{i=1}^{k} p_{2i-1} \leq \prod_{i=1}^{k+1} p_{2i-1}.$$

□

Lemma 6. *For all $k \geq 1$, we have*

$$p_{k+1} \leq \frac{5}{3} p_k,$$

and the inequality is sharp at $k = 2$.

Proof. According to Dusart [7], the interval $[x, \frac{1}{25\log^2(x)}]$ contains at least one prime for all $x \geq 396738$. As 396833 is prime, we have that, for $p_k \geq 396833 = p_{33609}$,

$$p_{k+1} \leq (1 + \frac{1}{25\log^2(p_k)})p_k < \frac{4001}{4000} p_k < \frac{5}{3} p_k.$$

Finally, by computer, the last inequality is shown to be also valid for $2 \leq p_k \leq 396832$. □

Lemma 7. *For every positive integer m, we have*

$$\prod_{k=1}^{m} N_k \leq N_{\frac{m(m+1)}{2}}.$$

Proof. By setting $s := \sum_{k=1}^{m} k$, we want to show that $\prod_{k=1}^{m} N_k$ is less than the primorial of index s. This is equivalent to

$$\sum_{k=1}^{m} \theta(p_k) \leq \theta(p_s).$$

According to the inequality(2), we have

$$\begin{aligned} \theta(p_s) &\geq s(\log s + \log_2 s - a) \\ &\geq s\left(\log m + \log\left(\frac{m+1}{2}\right) + \log_2 m - a\right), \end{aligned}$$

which implies for $m \geq 14$ that

$$\theta(p_s) \geq s\left(\log m + \log_2 m + \log\frac{15}{2} - a\right). \tag{9}$$

On the other hand, according to the inequality (4), it follows that for $m \geq 14$:

$$\sum_{k=1}^{m} \theta(p_k) \leq s(\log m + \log_2 m - 0.9465) + \sum_{k=1}^{13} \theta(p_k). \tag{10}$$

However, the left-hand side of the subtraction of the inequality (10) from (9) is an increasing function on m, which is already positive for $m \geq 14$. So, our inequality holds when $m \geq 14$ and, we checked $m \leq 13$ by computer to end the proof. □

3 Effective version of Pósa's inequality

Let us start by giving explicit formulations of the equivalence $K(x) \approx \pi(\log x)$.

Lemma 8. *For all $\varepsilon > 0$, there is a real number x_0 such that*

$$\pi\left(\frac{\log x}{1+\varepsilon}\right) \leq K(x)$$

for all $x \geq x_0$. In particular, $\forall x \geq 1$ we have $\pi\left(\frac{\log x}{1.00000075}\right) \leq K(x)$.

Proof. First, a form of the prime number theorem asserts that there exists a decreasing sequence to zero of positive real numbers $(\delta_n)_{n\geq 0}$ and a sequence $(u_n)_{n\geq 0}$ such that $|\theta(x)-x| < \delta_n x$ for all $x \geq u_n$. Hence, for all $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that $|\theta(x) - x| < \varepsilon x$ for all $x \geq u_{n_0}$. Then, for a given ε, $\theta(x) < (1+\varepsilon)x$ we have

$$\begin{aligned} K(x) = \max\{k \in \mathbb{N}^*,\ N_k \leq x\} &= \max\{k \in \mathbb{N}^*,\ \theta(p_k) \leq \log x\} \\ &\geq \max\{k \in \mathbb{N}^*,\ p_k \geq u_{n_0},\ (1+\varepsilon)p_k \leq \log x\} \\ &\geq \max\{k \in \mathbb{N}^*,\ (1+\varepsilon)p_k \leq \log x\}. \end{aligned}$$

However, the maximum in the last inequality is just $\pi(\log x/(1+\varepsilon))$.

In particular, according to Platt and Trudgian [17] we have $\theta(x) < 1.00000075x$ for all $x > 0$. Then, for all $x \geq 8$ and further by computer for $x \geq 1$ we deduce that

$$K(x) \geq \pi\left(\frac{\log x}{1.00000075}\right).$$

□

Lemma 9. *For all $\varepsilon > 0$, there is a real number x_0 such that*

$$\forall x \geq x_0,\ K(x) \leq \pi\left(\frac{\log x}{1-\varepsilon}\right).$$

In particular, for $x \geq x_0$ we have

$$K(x) \leq \pi\left(\frac{\log x}{\log \alpha}\right),$$

for the values in Table 1.

α	1.7	2	2.1	2.2	2.5
x_0	3	32	3503	N_{11}	N_{50}

Table 1: Some values of x_0.

Proof. Similarly to the first part of the above proof, for all $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that $|\theta(x) - x| < \varepsilon x$ for all $x \geq u_{n_0}$. Thus, for a given ε, $\theta(x) < (1-\varepsilon)x$ implies readily that

$$\begin{aligned} K(x) &\leq \max\left\{k \in \mathbb{N}^*,\ p_k \geq u_{n_0},\ (1-\varepsilon)p_k \leq \log x\right\} \\ &\leq \max\left\{k \in \mathbb{N}^*,\ (1-\varepsilon)p_k \leq \log x\right\}. \end{aligned}$$

However, the maximum in the last inequality is just $\pi(\log x/(1-\varepsilon))$.

For the effective upper bounds, the idea is to look at the best practical values of $\alpha > 1$ and $k_0(\alpha)$ such as $N_{k_0} > \alpha^{p_{k_0}}$, since this would result in $K(x)$ is not being greater than $\pi(\log x/\log \alpha)$ for all $x \geq N_{k_0}$. Using inequality (3) combined with inequality (8), the problem is reduced to finding $\alpha > 1$ and k_0 such that

$$\left(\log k + \log_2 k + \frac{\log_2 k}{\log k}\right)(1 - \log \alpha) \geq 1 + \frac{2.1454}{\log k} - \frac{0.5 \log \alpha}{\log k}, \forall k \geq k_0.$$

Starting from a certain $k_0(\alpha)$, the function on the right is greater than the one on the left as long as $1 < \alpha < e$. After checking the small values, we obtain the results in Table 2.

α	1.7	2	2.1	2.2	2.5
k_0	2	4	6	11	50

Table 2: Some values of k_0.

The values of x_0 given in Table 1 are the possible minimums except for $\alpha = 2.2$ and $\alpha = 2.5$, where the calculations showed that $K(x)$ is smaller than $\pi(\log x/\log \alpha)$ for $x = N_{10}$ and $x = N_{49}$. ☐

Through an elementary method, this last proposition permits us to get an effective version of a result of Pósa [15] which proved that for all $n > 1$ there is a k_n such that $N_k > p_{k+1}^n$ for all $n \geq k_n$.

Theorem 10. *For all integer $n \geq 1$, there exists $x_0(n) \geq 1$ such that for all $x \geq x_0(n)$,*

$$\big(\forall k \in \mathbb{N}^*,\ K(x) < k < \pi(\sqrt[n]{x})\big) \Rightarrow N_k > p_{k+1}^n.$$

Furthermore, if k verifies $p_{k+1} \geq (\frac{5}{3})^n$:

$$\forall m \geq k,\ N_m > p_{m+1}^n.$$

Proof. For $n \geq 1$ and $x_0(n) = \frac{n^{2n}}{(\log 1.7)^{2n}}$, we can show easily that $\frac{\log x}{\log 1.7} \leq \sqrt[n]{x}$ as long as $x \geq x_0(n)$. Thus, we obtain that $N_k > x \geq p_{k+1}^n$ for all k satisfying the inequalities

$$K(x) < k < \pi(\sqrt[n]{x}),\ x \geq x_0(n).$$

Furthermore, if k fulfils previous inequalities together with $p_{k+1} \geq (5/3)^n$, it implies by induction on k that $N_m > p_{m+1}^n$ for all $m > k$. Indeed, according to Lemma 6, we have

$$p_{m+2}^n \leq (5/3)^n p_{m+1}^n < (5/3)^n N_m \leq p_{m+1} N_m = N_{m+1}.$$

In practice, using the Maple algorithm below, the calculations of the values of k_n listed in Table 3, show that the first choice $x_0(n)$ for $n \leq 30$ is sufficient.

```
// Algorithm1 Computation of K(x).

restart; with(numtheory);
K := proc (L) local s, k;
s := 2; for k from 2 do
if s <= L then s := s*ithprime(k)
else return k-2 end
if end do end proc
```

```
// Algorithm2 Computation of kn.

posa := proc (n) local x0, x, k, R, m, s, i, t;
x0 := floor((n/ln(1.7))^(4*n)); R := (5/3)^n;
for x from x0 to x0+5 do
for k from K(x)+1 to pi(floor(x^(1/n)))-1 do
if R <= ithprime(k+1) then m := k end if end do
end do; for s to m do
if ithprime(s+1)^n < product(ithprime(i), i = 1 .. s) then t := s;
return t end if
end do end proc
```

□

We thus regain the results of Euclid: $N_k > p_{k+1}$ for $k \geq 1$ and those of Bonse [18]: $N_k > p_{k+1}^2$ for $k \geq 4$ and $N_k > p_{k+1}^3$ for $k \geq 5$. The sequence $(k_n)_{n\geq 1}$ is sequence A056127 in the OEIS [22].

Pósa's inequality has been studied extensively. The reader can find similar forms in papers of Hassani [10] and Sándor [21].

Let us end this section by giving some special approximations for limited ranges.

Theorem 11. *(i) For $1 < x \leq \exp(10^{19})$, we have $\pi(\log x) \leq K(x)$.*

('ii) For $2310 \leq x \leq \exp(10^{19})$, we have $\frac{\log x}{\log_2 x}(1 + \frac{1}{4\log_2 x}) \leq K(x)$.

(iii) For $210 \leq x \leq \exp(10^{19})$, we have $\frac{\log x}{\log_2 x} \leq K(x)$.

Proof. Let $0 < \log x \leq 10^{19}$. According to Büthe [6], the primes p_k smaller than $\log x$ verify $\theta(p_k) < p_k$. So,

$$\begin{aligned}\pi(\log x) &= \max\{k \in \mathbb{N}^*,\ p_k \leq \log x\} \\ &\leq \max\{k \in \mathbb{N}^*,\ \theta(p_k) \leq \log x\} = K(x).\end{aligned}$$

After a direct computation of the small values, we derive the last two lower bounds from the first result together with the lower bounds for $\pi(x)$, namely (see Dusart [7]):

$$\frac{x}{\log x} \leq \pi(x), \text{ for } x \geq 17 \text{ and } \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right) \leq \pi(x), \text{ for } x \geq 599.$$

□

4 Explicit bounds for the primorial counting function

First, we give some upper bounds for $K(x)$ by using a classical method. We will use appropriate bounds for $\theta(p_k)$ (essentially formulas (1) and (2)) while taking advantage of the fact that $K(x)$ is constant over the intervals $[N_k, N_{k+1})$. Thankfully, with the following lemma, we rediscover the estimates given by Robin [20] of the large values of $\omega(n)$. Here $\omega(n)$ denotes the number of prime distinct divisors of n.

Lemma 12. *For all real numbers $x \geq 1$, we have $K(x) = \max_{1\leq n\leq x} \omega(n)$. Furthermore, if $K(x) = K$, then for all integers $n \leq x$ with $\omega(n) = K$, we have $n \geq N_K$.*

In other words, $N_{K(x)}$ is the smallest integer less than x whose decomposition into prime numbers is the longest. Thus, $(K(n))_{n\geq 1}$ is the sequence A111972 in the OEIS [22].

Proof. As $N_k \leq n \leq x < N_{k+1}$ means that $\omega(n) \leq k$ and $K(n) = k$, one sees that $\omega(n) \leq K(n)$ in every interval $[N_k, N_{k+1})$, which implies that

$$\max_{1\leq n\leq x} \omega(n) = \max_{1\leq n< N_{K+1}} \omega(n) = K.$$

Let $q_1 q_2 \cdots q_K$ be an integer less than x such that $q_1 < q_2 < \ldots < q_K$ are prime numbers. For $K = 1$ it is obvious that $q_1 \geq p_1$. Now, assuming $q_i \geq p_i$ for $i < K$, it is necessary that $q_K \geq p_K$ otherwise $q_K < q_{K-1}$. □

For to prove our first upper bound, we need the following lemma.

Lemma 13. *For a large real number $A > 0$, if $\kappa(A)$ is a root of the equation $t \log t = A$, then*

$$\kappa(A) = \frac{A}{\log A}(1 + o(1)),$$

furthermore, for $A > e$:

$$\frac{A}{\log A} < \kappa(A) \leq \big(1 + \frac{1}{e}\big)\frac{A}{\log A}.$$

Proof. See Olver [14, Theorem 5. 1, Ex. 5. 7]. □

Theorem 14. *We have, when $x \geq N_{13}$:*

$$K(x) \leq \big(1 + \frac{1}{e}\big)\ \frac{\log x}{\log_2 x},$$

and

$$K(x) \leq 1.3841 \frac{\log x}{\log_2 x},\ \textit{for } x \geq 3.$$

Proof. From inequality (1), one easily deduces that, for $x \geq N_{13}$:

$$K(x) \leq \max\left\{k \in \mathbb{N}^*,\ k \log k \leq \log x\right\},$$

however this last set is only a part of the set of roots of inequality $t \log t \leq \log x$. So, according to Lemma 13, one gets

$$K(x) \leq \kappa(\log x) \leq (1 + \frac{1}{e})\ \frac{\log x}{\log_2 x},\ \forall x \geq N_{13}.$$

Now, as the function $F_0(x) = \frac{K(x)\log_2 x}{\log x}$ is decreasing over every interval $[N_k,\ N_{k+1})$ once $k \geq 3$, then, a computer verification can be done only on N_k where $3 \leq k \leq 12$. This verification shows that the maximum is reached on N_9 and it is also true for all real numbers $x < N_3$ with $F_0(N_9) \leq 1.3841$, which concludes the proof. □

Theorem 15. *We have, when $x \geq 3$, the inequality*

$$K(x) \leq \frac{\log x}{\log_2 x}\big(1 + \frac{1.4575}{\log_2 x}\big).$$

Proof. For $x \geq 2$, we consider the function

$$F_1(x) = \frac{K(x)(\log_2 x)^2}{\log x} - \log_2 x.$$

The function F_1 is decreasing over every interval $[N_k,\ N_{k+1})$ as long as $k \geq 5$ since $-\log_2 x$ is decreasing and the function $\frac{K(x)(\log_2 x)^2}{\log x}$ decreases when $x \geq 1619$. So, F_1 reaches its maximum at an integer N_{k_0} where $k_0 \geq 5$. On the other hand, for $k \geq 6$, using inequality (2), invoking the decrease of $\frac{(\log(x))^2}{x}$ (valid for $x \geq 8$) together with the fact that $\log(1+x) < x$ and $\frac{1}{1+x} \leq 1$, we obtain after a long expansion that

$$F_1(N_k) = \frac{k(\log\theta(p_k))^2}{\theta(k)} - \log\theta(p_k)) \leq G(k),$$

with

$$G(k) = a + \frac{\log_2 k - a}{\log k} + \frac{1}{\log k}\left(\frac{\log_2 k - a}{\log k}\right)^2 + \frac{2a}{\log k}\left(\frac{\log_2 k - a}{\log k}\right) + \frac{a^2}{\log k}.$$

The function G is decreasing and smaller than 1.3832444 for $k \geq \exp(\exp(a+1)) \simeq 2922$. Then, $F_1(N_k) \leq 1.3833$ for $k \geq 2922$.

Finally, for $5 \leq k \leq 2921$, we conclude by computer verification over intervals $[N_k,\ N_{k+1})$ that the maximum of $F_1(x)$ is reached at N_{47} with $F_1(N_{47}) \leq 1.4575$ and our upper bound is valid for $x < N_5$ as well. □

With the techniques used above, the inequality in the following theorem requires a wider estimate of $\theta(p_k)$. Nevertheless, for this last round, we will use a strong relation between $\pi(x)$ and $\theta(x)$ were given by Robin [20], namely

$$\forall x \geq 2, \quad \frac{\pi(x)\log\theta(x)}{\theta(x)} \leq 1 + \frac{1}{\log\theta(x)} + \frac{2.89726}{\log^2\theta(x)}. \tag{11}$$

Theorem 16. *We have, when $x \geq 3$, the inequality*

$$K(x) \leq \frac{\log x}{\log_2 x}\left(1 + \frac{1}{\log_2 x} + \frac{2.89726}{\log_2^2 x}\right).$$

Proof. Similarly, by studying $F_2(N_k)$ which corresponds to the function

$$F_2(x) = \frac{K(x)(\log_2 x)^3}{\log x} - (\log_2 x)^2 - \log_2 x, \text{ when } k \geq 10.$$

Since $\pi(p_k) = K(N_k) = k$, formula (11) guarantees that $F_2(N_k) \leq 2.89726$, $\forall k \geq 10$. Hence, we must now verify that our inequality is valid for $x < N_{10}$.

For $x \geq 863$, let M be the strictly increasing function that appears on the right side of our upper bound. A verification by computer over the intervals $[N_k,\ N_{k+1})$, with $5 \leq k \leq 9$, shows that $M(x)$ is always greater than $K(x)$, by a difference of at least 1.5. The calculations also show that the upper bound is also true for $x < N_5$, which concludes the proof. □

Theorem 17. *If the Riemann hypothesis holds, we have*

$$K(x) \le \mathrm{li}^{-1}(\log x) + 0.12\sqrt{\log x}, \ \forall x \ge 42, \tag{12}$$

where li^{-1} *is the inverse of the logarithmic integral function.*

Proof. According to Robin [20], for $k \ge 5$

$$\theta(p_k) > \mathrm{li}^{-1}(k) - 0.12\sqrt{k\log^3 k},$$

and as $\mathrm{li}(x)$ is an increasing function, one gets successively

$$\begin{aligned} k &\le \mathrm{li}\big(\theta(p_k)\big) + \int_{\theta(p_k)}^{\theta(p_k)+0.12\sqrt{k\log^3 k}} \frac{dx}{\log x} \\ &\le \mathrm{li}(\theta(p_k)) + 0.12\frac{\sqrt{k\log^3 k}}{\log\theta(p_k)}. \end{aligned}$$

Now, applying inequality (1), one gets

$$k \le \mathrm{li}(\theta(p_k)) + 0.12\frac{\sqrt{\theta(p_k)}\log k}{\log\theta(p_k)} \le \mathrm{li}(\theta(p_k)) + 0.12\sqrt{\theta(p_k)}, \ \forall k \ge 5,$$

which is equivalent to

$$K(N_k) \le \mathrm{li}(\log N_k) + 0.12\sqrt{\log N_k}, \ \forall k \ge 5.$$

Simply, the term on the right side is an increasing function of k, which yields that

$$K(x) \le \mathrm{li}^{-1}(\log x) + 0.12\sqrt{\log x}, \ \forall x \ge N_5.$$

Finally, with computer verifications, we extend the result to $x \ge 42$. □

Theorem 18. *We have, when* $x \ge 2310$

$$K(x) \ge \frac{\log x}{\log_2 x}\left(1 + \frac{1}{4\log_2 x}\right).$$

Moreover, for $x \ge 210$

$$K(x) \ge \frac{\log x}{\log_2 x}.$$

Proof. Let $\kappa(x) = \frac{\log x}{\log_2 x}(1 + \frac{1}{4\log_2 x})$ and $f_a(x) = k\big(\log k + \log_2 k - a\big)$. As the function $\frac{\log_2 x - c}{\log x}$ is decreasing for $x \ge 8$, inequality (7) gives that

$$\theta(p_{k+1}) \le f(k) = k\big(\log k + \log_2 k - c + 0.006389\big), \text{ when } x \ge 10^{500}.$$

Hence, for $x \geq 10^{500}$, the function $K(x)$ is greater than the minimum of the set of positive integer solutions of the inequality $f_a(k) > \log x$, where $a = c - 0.006389 \leq 0.30046382$.

Now, let us show that the function $\kappa(x) = \frac{\log x}{\log_2 x}(1+\frac{1}{4\log_2 x})$ is not in the above set. Indeed, for $x \geq 10^{500}$

$$\log \kappa(x) \leq \log_2 x - \log_3 x + \frac{0.25}{\log_2 x},\ \log_2 \kappa(x) \leq \log_3 x,$$

$$\log \kappa(x) + \log_2 \kappa(x) \leq \log_2 x + \frac{0.25}{\log_2 x} \leq \log_2 x + \frac{0.25}{\log_2 x},$$

so, we obtain the following

$$f_a(\kappa(x)) \leq \log x(1 + \frac{1}{4\log_2 x})(1 + \frac{1}{4\log_2^2 x} - \frac{a}{\log_2 x}) < \log x.$$

Now suppose that $f(k) > \log x$. Then $x \to \infty$ implies $k \to \infty$, since f is bounded on every bounded subset of the set of all positive integers. Therefore, for $x \geq 10^{500}$, we have $f(k) \leq f_a(k)$ and, since $f_a(k)$ is increasing in k, we also have $f_a(k) \leq f_a(\kappa(x))$ if $k < \kappa(x)$, so that

$$f(k) \leq f_a(k) \leq f_a(\kappa(x)) < \log x,$$

which means that, if $x \geq 10^{500}$ and $f(k) \geq \log x$, then $k \geq \kappa(x)$ and so $K(x) \geq \kappa(x)$.

For the values of $x < 10^{500}$, Theorem 11 is used as $10^{500} < \exp 10^{19}$. The last lower bound is partly due to the fact that $\frac{\log x}{4\log_2^2 x} \geq 1$ when $x \geq 1.7 \cdot 10^{30}$ and again to Theorem 11-(iii). □

5 Maximal value of sums over prime divisors

Sums over primes and their evaluations are one of the main subjects of multiplicative number theory. Through a special case, we are concerned in this section to give an idea allowing to estimate the maximal value of sum of type $\mathfrak{L}_f(q) = \sum_{p|q} f(p)$ for a positive decreasing function f on $(1, \infty)$, when q ranges over all integers not exceeding x.

This type of sum appears without estimates in several papers, as in the work of Gordon and Rogers [9] where they derive refined forms of the sums of the divisor function; and in the work of Lehmer [12] where he studied a generalization of the Euler constant.

Our idea can be summarized as follows. As f is a positive decreasing function, then $\max_{1<q\leq x} \mathfrak{L}_f(q)$ is reached on the smallest integer $q(x)$ less than x among those having the longest decomposition in prime numbers. However, according to Lemma 12, we can clearly specify that $q(x)$ is nothing other than $N_{K(x)}$.

Finally, for a given real $x \geq 1$, if $K(x) = K$, we deduce the following for all positive decreasing functions f on $(1, \infty)$:

$$\max_{1<q\leq x} \mathfrak{L}_f(q) = \mathfrak{L}_f(N_K) = \sum_{p \leq p_K} f(p).$$

In particular, there is an approximation of the maximal value of $\mathfrak{L}(q) = \sum_{p|q} \frac{\log p}{p-1}$. For a constant C, Hassani [11] proved the following result

$$\max_{1<q\leq x} \mathfrak{L}(q) \leq \log_2 x + C \text{ and } \mathfrak{L}(q) \geq \frac{\log q}{q-1} \text{ for } q \geq 2.$$

We propose the following improvements:

Theorem 19. *We have, when $x \geq N_7$, the inequality*

$$\max_{1<q\leq x} \mathfrak{L}(q) \leq \log K(x) + \log_2 K(x).$$

Proof. As in Hassani [11], we start the estimation of the maximum as follows:

$$\begin{aligned}\max_{1<q\leq x} \mathfrak{L}(q) &= \sum_{p\leq p_K} \frac{\log p}{p} + \sum_{p\leq p_K} \frac{\log p}{p(p-1)} \\ &= \sum_{p\leq p_K} \frac{\log p}{p} + \sum_{p>1} \frac{\log p}{p(p-1)} - \sum_{p>p_K} \frac{\log p}{p(p-1)}.\end{aligned}$$

However, Rosser and Shoenfeld [3] proved the following:

$$\forall t \geq 32, \sum_{p\leq t} \frac{\log p}{p} \leq \log t + E + \frac{1}{\log t} \text{ and } \sum_{p>1} \frac{\log p}{p(p-1)} + E \approx -0.58.$$

So, recalling that

$$p_k \leq k(\log k + \log_2 k),$$

once $k \geq 6$, we have for $K \geq 331$ ($p_K \geq 32$ and $\log_2 K/\log K < 0.3$) then after checking by hand for $K \geq 7$ that

$$\begin{aligned}\sum_{p\leq p_K} \frac{\log p}{p} + \sum_{p>1} \frac{\log p}{p(p-1)} &\leq \log K + \log_2 K + \frac{\log_2 K}{\log K} + \frac{1}{\log p_{12}} - 0.58 \\ &\leq \log K + \log_2 K.\end{aligned}$$

Hence, for $x \geq N_7$ we obtain

$$\max_{1<q\leq x} \mathfrak{L}(q) \leq \log K(x) + \log_2 K(x). \tag{13}$$

Finally, an computer verifications show that inequality 13 is not true for integers smaller than N_7. □

Corollary 20. *We have, when $x \geq 3$, the inequality*

$$\max_{1<q\leq x} \mathfrak{L}(q) \leq \log_2 x + \frac{1.4575}{\log_2 x}.$$

Proof. Using the upper bound of Theorem 15, we obtain successively

$$\log K(x) \leq \log_2 x - \log_3 x + \frac{1.4575}{\log_2 x}$$

$$\log_2 K(x) \leq \log_3 x + \log(1 - \frac{\log_3 x}{\log_2 x} + \frac{1.4575}{\log_2^2 x})$$

$$\log K(x) + \log_2 K(x) \leq \log_2 x + \frac{1.4575}{\log_2 x}, \ \ x \geq 2219.$$

Combining this with inequality (13), this gives the result for $x \geq N_7$. We conclude the proof using computer verifications for the small values. □

Corollary 21. *We have, when $x \geq 43$, the inequality*

$$\max_{1<q\leq x} \mathfrak{L}(q) \leq \log_2 x + \log(1.3841).$$

Proof. Similarly to the previous proof, but using the second upper bound of Theorem 14, we obtain that, for $x \geq 3$

$$\log K(x) \leq \log_2 x - \log_3 x + \log(1.3841).$$

And as $K(x) \leq \log x$ by definition, we get that $\log_2 K(x) \leq \log_3 x$. Therefore, using inequality (13) gives the result for $x \geq N_7$; then computer verification gives the result for $x \geq 43$. □

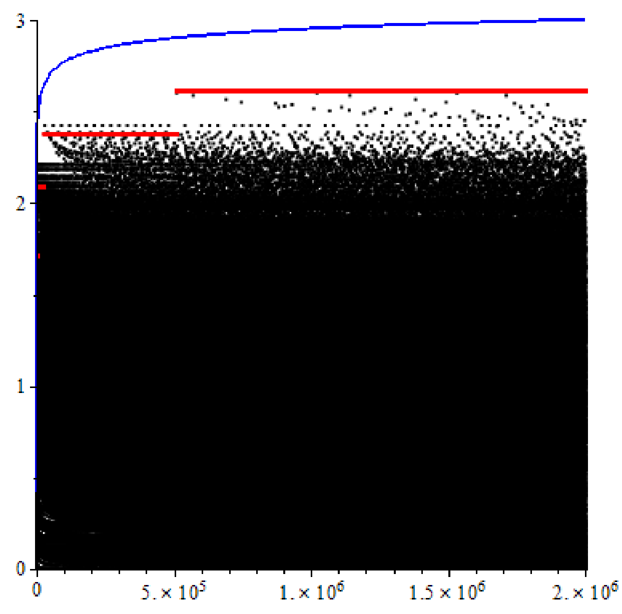

Figure 1: Graph of points $(q, \mathfrak{L}(q))$ for $1 \leq q \leq 2 \cdot 10^6$. The upper bound of Theorem 19 (resp., Corollary 21) is shown in red (resp., in blue).

6 Minimal value of sums over prime divisors

Concerning $\min\limits_{1<q\leq x} \mathfrak{L}(q)$, we obtain the following optimal inequality

$$\forall x \geq 2, \ \min_{1<q\leq x} \mathfrak{L}(q) \geq l(x).$$

Indeed, as the minimal value of $\mathfrak{L}(q)$ is reached on the largest prime number less than x, we can deduce that if $\pi(x) = r$, then

$$\min_{1<q\leq x} \mathfrak{L}(q) = l(p_r) \geq l(x),$$

since l is a decreasing function. According to the proof of Lemma 6, this lower bound is the best since we already have that, for x not large enough (≥ 396833)

$$x < p_{r+1} \Leftrightarrow \frac{p_r}{x} > \frac{p_r}{p_{r+1}} \geq \frac{4000}{4001},$$

which implies

$$l(x) \leq \min_{1<q\leq x} \mathfrak{L}(q) < l(\frac{4000}{4001}x).$$

7 Appendix

In Table 3 we give the values of $K(10^n)$ and the terms of the sequence

$$k_n = \min\{k, p_1 \cdots p_k > p_{k+1}^n\}$$

for $n \leq 30$.

n	$K(10^n)$	k_n	n	$K(10^n)$	k_n	n	$K(10^n)$	k_n
1	2	2	11	10	16	21	16	29
2	3	4	12	11	18	22	17	30
3	4	5	13	12	19	23	17	32
4	5	7	14	12	20	24	18	33
5	6	8	15	13	21	25	19	34
6	7	10	16	13	23	26	19	35
7	8	11	17	14	24	27	20	36
8	8	13	18	15	25	28	20	38
9	9	14	19	15	26	29	21	39
10	10	15	20	16	28	30	21	40

Table 3: Some values of $K(10^n)$ and $k_n = \min\{k, p_1 \cdots p_k > p_{k+1}^n\}$.

We can also give $K(10^{40}) = 26$, $K(10^{50}) = 31$, $K(10^{60}) = 36$, $K(10^{100}) = 53$, $K(10^{200}) = 92$, $K(10^{300}) = 128$, $K(10^{10^3}) = 350$ and, $K(10^{10^4}) = 2584$.

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