Journal of Integer Sequences, Vol. 24 (2021),
Article 21.7.6

Combinatorial Proof of an Identity on Genocchi Numbers

Beáta Bényi
Faculty of Water Sciences
University of Public Service
H-1441 Budapest, P.O. Box 60
Hungary
benyi.beata@uni-nke.hu

Matthieu Josuat-Vergès
CNRS, IRIF (UMR 8243)
Université de Paris
FRANCE
matthieu.josuat-verges@u-pem.fr

Abstract

In this note we present a combinatorial proof of an identity involving poly-Bernoulli numbers and Genocchi numbers. We introduce combinatorial objects called m-barred Callan sequences and use them to show that the identity holds in a more general manner.

1 Introduction

In this paper we present a combinatorial proof of an identity that establishes an interesting relation between the C-poly-Bernoulli numbers and Genocchi numbers. The Genocchi numbers, G_n, can be defined by the generating function [13, Exercise 5.8]

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1}.$$

The first few values are

$$0, 1, -1, 0, 1, 0, -3, 0, 17, 0, -155, \ldots$$

(see A036968 in the OEIS *(On-Line Encyclopedia of Integer Sequences)* [12]).

The *poly-Bernoulli numbers* $B_n^{(k)}$ are defined for $k \in \mathbb{Z}$ by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \tag{1}$$

where $\mathrm{Li}_k(z)$ is the polylogarithm function given by

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (|z| < 1).$$

For negative k indices, the poly-Bernoulli numbers are integers (A099594 in [12]). Several combinatorial objects are enumerated by these numbers, for instance, lonesum matrices [7], matrices uniquely reconstructible from their row and column sum vectors. For further objects see [3, 4]. Arakawa and Kaneko [2] proved analytically while Bényi and Hajnal [3] combinatorially that

$$\sum_{j=0}^{n} (-1)^j B_{n-j}^{(-j)} = 0. \tag{2}$$

Arakawa and Kaneko introduced a C-version of poly-Bernoulli numbers in [1] as

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}. \tag{3}$$

In particular, $\{C_n^{(1)}\}_{n\in\mathbb{N}}$ are the ordinary Bernoulli numbers, B_n, defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

For negative k, the numbers $C_n^{(k)}$ are again positive integers. They are the array A136126 in [12]. Table 1 below shows the first few values. Bényi and Hajnal [4] introduced this integer array as the number of lonesum matrices without any all-zero columns.

Kaneko, Sakurai, and Tsumura [10] proved the C-version of Identity (2) using analytical methods.

$$\sum_{j=0}^{n} (-1)^j C_{n-j}^{(-j-1)} = -G_{n+2} \quad (n \geq 0). \tag{4}$$

In this note we present a combinatorial proof of Identity (4).

Recently, Matsusaka [11] showed that both (2) and (4) are special cases of a more general polynomial identity between symmetrized poly-Bernoulli numbers and Gandhi-polynomials.

2 Definitions

In this section we provide the necessary definitions and notation. We establish a connection between *barred Callan sequences* and *Dumont permutations of the first kind* defined in [8] which proves the identity.

Bényi and Nagy [6] defined *Callan sequences* that are essentially the same as the permutation class that Callan introduced in the entry A099594 of the OEIS [12]. Based on this notion, Bényi and Matsusaka [5] defined *barred Callan sequences*. We now recall the definition.

Let $k, n \geq 0$ and consider the sets $K = \{1, \dots, k\} \cup \{*\}$ (referred to as *blue elements*), and $N = \{1, \dots, n\} \cup \{*\}$ (referred to as *red elements*).

Definition 1. A *Callan sequence* of size $k \times n$ is a sequence

$$(B_1, R_1)(B_2, R_2) \cdots (B_m, R_m)(B^*, R^*),$$

for some m with $0 \leq m \leq n$, such that

- $\{B_1, \dots, B_m, B^*\}$ form a set partition of K into $m+1$ non-empty blocks,
- $\{R_1, \dots, R_m, R^*\}$ form a set partition of N into $m+1$ non-empty blocks,
- $* \in B^*$, and $* \in R^*$.

Note that a Callan sequence can be encoded in an obvious way by the following data:

- an integer m with $0 \leq m \leq \min(k, n)$;
- an ordered partition $(B_1, \dots, B_m)$ of some subset $K' \subset \{1, \dots, k\}$; and
- an ordered partition $(R_1, \dots, R_m)$ of some subset $N' \subset \{1, \dots, n\}$.

This is done by letting $K' = K \setminus B^*$ and $N' = N \setminus R^*$.

To deal with these objects, it is be convenient to use the following terminology

- B^* and R^* are called *extra blocks*, while B_i and R_i are called *ordinary blocks*,
- each pair (B_i, R_i) or (B^*, R^*) is a *Callan pair*; moreover, the former are called *ordinary pairs* and the latter is called the *extra pair*.

A Callan sequence is thus a linear arrangement of Callan pairs with the extra pair at the end.

Example 2. The Callan sequences with $k = 2$ and $n = 2$ are listed below.

$(12*, 12*)$, $(12, 12)(*, *)$, $(1, 12)(2*, *)$, $(2, 12)(1*, *)$, $(12, 1)(*, 2*)$,
$(12, 2)(*, 1*)$, $(1, 1)(2*, 2*)$, $(2, 1)(1*, 2*)$, $(1, 2)(2*, 1*)$, $(2, 2)(1*, 1*)$,
$(1, 1)(2, 2)(*, *)$, $(1, 2)(2, 1)(*, *)$, $(2, 1)(1, 2)(*, *)$, $(2, 2)(1, 1)(*, *)$.

Definition 3. *A barred Callan sequence* of size $k \times n$ is a sequence obtained by inserting a bar (denoted $|$) into a Callan sequence of size $k \times n$ with the restriction that the bar cannot be at the end of the sequence. We denote by $\mathcal{C}_n^k$ the set of barred Callan sequences with k blue elements and n red elements.

Example 4. We list here the 7 barred Callan sequences for $k = 2$ and $n = 1$.

$$|(12*, 1*); \quad |(12, 1)(*, *); \quad |(2, 1)(1*, *); \quad |(1, 1)(2*, *);$$
$$(1, 1)|(2*, *); \quad (2, 1)|(1*, *); \quad (12, 1)|(*, *).$$

Bényi and Matsusaka [5] showed that the number of barred Callan sequences of size $k \times n$ is the C-poly-Bernoulli number $C_n^{(-k-1)}$. To avoid negative indices in what follows, we use the notation C_n^k instead of $C_n^{(-k-1)}$. In particular, we have $C_n^k = |\mathcal{C}_n^k|$.

Table 1 gives the first few values of the numbers C_n^k. Note that the symmetry $C_n^k = C_k^n$ is clear from the combinatorial interpretation in terms of barred Callan sequences, as one can exchange red and blue blocks to get a $n \times k$ Callan sequence from a $k \times n$ Callan sequence.

	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	3	7	15	31	63
2	1	7	31	115	391	1267
3	1	15	115	675	3451	16275
4	1	31	391	3451	25231	164731
5	1	63	1267	16275	164731	1441923

Table 1: C-poly-Bernoulli numbers C_n^k.

Remark 5. We note here that Callan sequences are equivalent to the special class of partial permutations where elements greater than n are excedances and the elements smaller than or equal to n are deficiencies. The number C_n^k also counts alternative tableaux of rectangular shape $k \times n$ and other related tableaux of rectangular shapes that are in bijection with alternative tableaux such as permutation tableaux or tree-like tableaux. See [3, 4] for details.

The other number sequence that is involved in the identity we want to prove is the sequence of Genocchi numbers. Genocchi numbers can be defined in different ways, here we focus on the combinatorial point of view. There are several combinatorial objects that are enumerated by the Genocchi numbers. The first known interpretation was Dumont permutations of the first kind.

Definition 6 ([8])**.** A *Dumont permutation of the first kind* is a permutation $\pi \in S_{2n}$ such that each even entry begins a descent and each odd entry begins an ascent or ends the string, i.e., for every $i = 1, 2, \ldots, 2n$

$$\begin{aligned} \pi(i) \text{ even} &\implies i < 2n \text{ and } \pi(i) > \pi(i+1), \\ \pi(i) \text{ odd} &\implies i < 2n \text{ and } \pi(i) < \pi(i+1), \text{ or } i = 2n. \end{aligned}$$

Let $\mathcal{D}_{2n}$ denote the set of Dumont permutations on $2n$ elements.

Example 7. For $2n = 4$ we have $\mathcal{D}_4 = \{2143, 3421, 4213\}$. For $2n = 6$ we have

$$\begin{aligned} \mathcal{D}_6 = \{&642135, 634215, 621435421365, 342165, 214365, 564213, 563421, 562143, \\ &216435, 435621, 215643, 436215, 364215, 421563, 356421, 421635.\} \end{aligned}$$

In general, we have Theorem 8.

Theorem 8 (Dumont [8])**.** *The cardinality of* $\mathcal{D}_{2n}$ *is* $(-1)^{n+1}G_{2n+2}$.

3 Main result

Identity (4) can be reformulated as in Theorem 9 using our notation.

Theorem 9 ([10])**.** *The identity*

$$\sum_{j=0}^{n} (-1)^j C_{n-j}^{j} = -G_{n+2} \tag{5}$$

holds for $n \geq 0$.

We introduce some combinatorial objects that play crucial role in our combinatorial proof of a generalization of the identity above. They can be seen as barred Callan sequences with extra structure. We could say that they interpolate in a certain sense between barred Callan sequences and Dumont permutations.

Definition 10. Let $\mathcal{C}_n^k(m)$ denote the set of sequences defined by the following conditions. First, the elements in the sequence are

- m blue bars labelled with $1, \ldots, m$, and $m+1$ red bars labelled with $0, \ldots, m$,
- Callan pairs of a $k \times n$ Callan sequence shifted up by m, i.e., with base sets $K = \{m+1, \ldots, m+k\} \cup \{*\}$ and $N = \{m+1, \ldots, m+n\} \cup \{*\}$.

Second, these elements (labelled bars and Callan pairs) are ordered in a way such that

- a blue bar with label i is followed by a bar with label strictly smaller than i and
- a red bar with label i is followed either by a Callan pair or by a bar with label strictly greater than i.

The elements of $\mathcal{C}_n^k(m)$ are called *m-barred Callan sequences* and $C_n^k(m)$ denotes the cardinality of the set $\mathcal{C}_n^k(m)$.

Remark 11. Bényi and Matsusaka [5] defined m-barred Callan sequences otherwise in, we call our special objects m-barred Callan sequence for the sake of convenience.

Remark 12. The numbers $C_n^k(1)$ are the so-called *poly-Bernoulli D-relatives*. Bényi and Hajnal [4] introduced poly-Bernoulli D-relatives as the number of lonesum matrices that do not contain any all-zero column and any all-zero row. See A272644 in [12].

Note that a bar is always followed by another element in the sequence, so the last element of the sequence is a Callan pair. More precisely, the last element is necessarily the extra pair of the underlying Callan sequence. For example, an element of $\mathcal{C}_4^6(3)$ is given as

$$|_3|_2|_1|_2(5,45)|_3(79,7)(4,6)|_1|_0(68*,*).$$

Note 13. Sometimes it is convenient to think that each Callan pair is associated with the (possibly empty) sequence of consecutive bars preceding it. In the previous example, we have the four "groups"

$$|_3|_2|_1|_2(5,45) \quad |_3(79,7) \quad (4,6) \quad |_1|_0(68*,*).$$

Moreover, the last bar of each group is a red bar.

Let us first record some properties that are simple consequences of the definition.

Proposition 14. *We have the following identities*

$$C_n^k(m) = C_k^n(m), \tag{6}$$
$$C_n^k(0) = C_n^k, \tag{7}$$
$$C_0^0(m) = (-1)^{m+1}G_{2m+2}. \tag{8}$$

Proof. The simple bijection that exchanges blue blocks with red blocks in each Callan pair establish a bijection between the sets $\mathcal{C}_n^k(m)$ and $\mathcal{C}_k^n(m)$ and implies the first identity.

The elements in $\mathcal{C}_k^n(0)$ contain a unique bar, $|_0$. Thus, we have a simple bijection between $\mathcal{C}_k^n(0)$ and $\mathcal{C}_k^n$ by changing $|_0$ into $|$. Hence, we get the second identity.

The elements in $\mathcal{C}_0^0(m)$ contain a unique Callan pair at the last position, $(*,*)$. We also know that $|_m$ preceds $(*,*)$ since there is no label strictly greater than m. Regarding this property, it is easy to see that there is a simple bijection between $\mathcal{C}_0^0(m)$ and $\mathcal{D}_{2m}$. Remove $|_m(*,*)$ and transform $|_i$ into $2i$ and $|_i$ into $2i+1$. □

Our main result is that the numbers $C_n^k(m)$ also have an alternating sum equal to Genocchi numbers, as Theorem 15 states.

Theorem 15. *The identity*

$$\sum_{j=0}^{n}(-1)^j C_{n-j}^j(m) = (-1)^{m+1}G_{n+2m+2} \tag{9}$$

holds for $n, m \geq 0$.

Note that when n is odd both sides of (9) are 0. The sum has an even number of terms and they can be paired using the symmetry $C_n^k(m) = C_k^n(m)$. So only the case where n is even is of interest. Note also that the case $m = 0$ in Theorem 15 is precisely Theorem 9.

The core of our bijective proof of Theorem 15 is the following identity.

Proposition 16. *The identity*

$$\sum_{j=0}^{n}(-1)^j C_{n-j}^{j}(m) = -\sum_{j=0}^{n-2}(-1)^j C_{n-2-j}^{j}(m+1) \tag{10}$$

holds for $n \geq 2$ and $m \geq 0$.

The combinatorial proof of Proposition 16 will be done in the next two sections. We show now that Theorem 15 and Theorem 9 are consequences of Proposition 16.

Proof of Theorem 15. As noted above, we can assume that n is even, $n = 2n'$ with $n' \in \mathbb{N}$. We can successively iterate the identity in Proposition 16 with (n, m), then $(n-2, m+1)$, etc. Each iteration gives a sum with one less term. Finally, we get the identity

$$\sum_{j=0}^{n}(-1)^j C_{n-j}^{j}(m) = (-1)^{n'} C_0^0(m+n').$$

The third equation in Proposition 14 implies that this is equal to $(-1)^{n'+n'+m+1}G_{2m+2n'+2}$ which simplifies to $(-1)^{m+1}G_{n+2m+2}$. $\square$

4 The bijection ϕ

We define in this section a one-to-one correspondence between two particular subsets of m-barred Callan sequences. The bijection is a slight modification of the map introduced in [3] in order to prove (2).

Definition 17. Let $\mathcal{C}_n^k(m,*)$ denote the subset of $\mathcal{C}_n^k(m)$ containing elements α such that the extra red block is empty (by convention, this means it contains only $*$). Let $\mathcal{C}_n^k(m,R*) = \mathcal{C}_n^k(m)\backslash\mathcal{C}_n^k(m,*)$ denote the complementary subset.

Following the previous convention, let $C_n^k(m,*)$ (respectively, $C_n^k(m,R*)$) denote the cardinality of $\mathcal{C}_n^k(m,*)$ (respectively, $\mathcal{C}_n^k(m,R*)$).

Suppose that $n \geq 1$ and $k \geq 0$ are fixed throughout this section. We define the map

$$\phi : \mathcal{C}_n^k(m,R*) \to \mathcal{C}_{n-1}^{k+1}(m,*)$$

as follows. Let α be an m-barred Callan sequence with a non-empty extra red block, $\alpha \in \mathcal{C}_n^k(m,R*)$. The general idea is to remove the maximal red element $m+n$ and add a new blue element $m' = m+k+1$. We distinguish four cases according to the location of the maximal red element $m+n$.

A1) $m+n$ is in the extra block as a singleton, i.e., $R^* = \{m+n, *\}$.

A2) $m+n$ is in the extra block with other red elements, i.e., $R^* \setminus \{m+n\} = R' \neq \{*\}$.

B1) $m+n$ is in an ordinary block as a singleton, i.e., there is an ordinary pair (B_i, R_i) with $R_i = \{m+n\}$.

B2) $m+n$ is in an ordinary block with other red elements, i.e., there is an ordinary pair (B_i, R_i) with $m+n \in R_i$ and $R_i \setminus \{m+n\} = R' \neq \varnothing$.

Accordingly, $\phi(\alpha)$ is defined as the result of the following procedure.

A1) Delete $m+n$ and add the new element m' to the first Callan pair.

A2) Replace the extra red block R^* with $\{*\}$, and place $(\{m'\}, R')$ in front of the Callan sequence as the first element.

B1) The successive steps are as follows:

- replace the extra red block R^* with $\{*\}$,
- replace $R_i = \{m+n\}$ with $R^* \backslash \{*\}$,
- move the new pair $(B_i, R^* \backslash \{*\})$ in front of the Callan sequence together with the group of bars to its left (as in Note 13),
- add m' to the $(i+1)$st blue block (this means either B_{i+1} or the extra blue block B^* if B_i was in the last ordinary pair).

B2) The successive steps are as follows:

- replace the extra red block R^* with $\{*\}$,
- replace (B_i, R_i) with two pairs $(B_i, R^* \backslash \{*\})$ and $(\{m'\}, R')$,
- move the new pair $(B_i, R^* \backslash \{*\})$ in front of the sequence together with the group of consecutive bars preceding it (as in Note 13).

Example 18. Suppose that m, n, k are such that we remove 9 and add 8. The four cases can be illustrated as given in the list below.

$$
\begin{array}{lll}
A1) & (5,35)|_1(47,247)|_1|_0(23,68)(6*,9*) & \rightarrow & (58,35)|_1(47,247)|_1|_0(23,68)(6*,*) \\
A2) & (5,35)|_1(47,2)|_1|_0(23,68)(6*,479*) & \rightarrow & (8,47)(5,3,5)|_1(47,2)|_1|_0(23,68)(6*,*) \\
B1) & (5,358)|_1(47,24)|_1|_0(23,9)(6*,67*) & \rightarrow & |_1|_0(23,67)(5,358)|_1(47,24)(68*,*) \\
B2) & (5,38)|_1(47,4)|_1|_0(23,259)(6*,67*) & \rightarrow & |_1|_0(23,67)(5,38)|_1(47,4)(8,25)(6*,*)
\end{array}
$$

Definition 19. For $n \geq 0$ and $k \geq 1$, let $\mathcal{C}_n^k(m, *, |m')$ denote the subset of $\mathcal{C}_n^k(m, *)$ containing elements α such that the maximal blue element $m' = m + k$ is in an ordinary Callan pair as a singleton and there is at least one bar preceding this pair. Following the previous convention, let $C_n^k(m, *, |m')$ denote the cardinality of $\mathcal{C}_n^k(m, *, |m')$.

Note that when $k = 0$ and $n > 0$, we have $\mathcal{C}_n^0(m, *) = \varnothing$. Indeed, the only $0 \times n$ Callan sequence is $(*, 123 \cdots n*)$, which has a nonempty extra red block. Accordingly, we also take the convention $\mathcal{C}_n^0(m, *, |m') = \varnothing$.

Note also that $\mathcal{C}_0^k(m, *, |m') = \varnothing$. Indeed, the only $k \times 0$-Callan pair is $(123 \cdots k*, *)$, so it is not possible to have m' as a singleton in an ordinary pair.

Lemma 20. *The map ϕ defined as above is a bijection between the sets $\mathcal{C}_n^k(m, R*)$ and $\mathcal{C}_{n-1}^{k+1}(m, *) \setminus \mathcal{C}_{n-1}^{k+1}(m, *, |m')$.*

Proof. Let us first check that $\phi(\alpha) \in \mathcal{C}_{n-1}^{k+1}(m, *)$. Note that each group of consecutive bars is unchanged (though these groups are possibly permuted in case B1 and B2), so we only need to check that the pairs form a valid Callan sequence. This is straightforward.

Then, let us check that $\phi(\alpha) \notin \mathcal{C}_{n-1}^{k+1}(m, *, |m')$. In cases A1 and B1, the blue element m' is added to a blue block (which is nonempty by definition) so m' is not a single element in its block. In cases A2 and B2, m' is in a singleton block, so we need to check that there is no bar preceding the pair containing m'. In case A2, this is because this pair is at the beginning of the sequence. In case B2, the new pair $(\{m'\}, R')$ is created with another pair to its left, and since the pair to its left is moved together with the associated group of bars, there is again another pair to the left of $(\{m'\}, R')$ in $\phi(\alpha)$.

In the previous paragraph, we observed the following properties in $\phi(\alpha)$:

- m' is in the first pair iff the sequence was obtained from the cases A1 or A2.
- m' is alone in its block iff the sequence was obtained from the cases A2 or B2.

So, the four cases can be distinguished based on which of the above properties they satisfy. In each case, based on the position of m' it is straightforward to describe the inverse procedure, so ϕ is injective. □

The bijection ϕ can be used to obtain pairwise cancellations in the alternating sums.

Lemma 21. *The identity*

$$\sum_{j=0}^{n} (-1)^j C_{n-j}^j(m) = \sum_{j=0}^{n} (-1)^j C_{n-j}^j(m, *, |m').$$

holds for $n, m \geq 0$.

Proof. The set $\mathcal{C}^j_{n-j}(m)$ can be partitioned as

$$\mathcal{C}^j_{n-j}(m) = \mathcal{C}^j_{n-j}(m, R*) \biguplus \left(\mathcal{C}^j_{n-j}(m, *) \setminus \mathcal{C}^j_{n-j}(m, *, |m')\right) \biguplus \mathcal{C}^j_{n-j}(m, *, |m'), \tag{11}$$

so that

$$\sum_{j=0}^{n}(-1)^j C^j_{n-j}(m) = \sum_{j=0}^{n}(-1)^j |\mathcal{C}^j_{n-j}(m, R*)| + \sum_{j=0}^{n}(-1)^j \left|\left(\mathcal{C}^j_{n-j}(m, *) \setminus \mathcal{C}^j_{n-j}(m, *, |m')\right)\right| + \sum_{j=0}^{n}(-1)^j C^j_{n-j}(m, *, |m').$$

The bijection ϕ readily shows that the first and second sum cancel each other out, upon checking the boundary terms.

The first boundary term is $j = n$ in the first sum. The set $\mathcal{C}^n_0(m, R*)$ is empty (as there are no red element besides $*$, the extra red block cannot be nonempty), so the corresponding term is 0. The second boundary term is $j = 0$ in the second sum. As noted after Definition 19, $\mathcal{C}^0_n(m, *)$ is empty, $\mathcal{C}^0_n(m, *) \setminus \mathcal{C}^0_n(m, *, |m')$ is empty as well, and the corresponding term is 0. □

Lemma 21 implies Proposition 16 when we can show the following identity.

$$\sum_{j=0}^{n}(-1)^j C^j_{n-j}(m, *, |m') = -\sum_{j=0}^{n-2}(-1)^j C^j_{n-2-j}(m+1). \tag{12}$$

We prove Identity (12) in the next section.

5 The bijection ψ

It is convenient to use a slight modified set instead of $\mathcal{C}^k_n(m, *, |m')$ in the combinatorial proof of Identity (12).

Definition 22. We define $\mathcal{C}^k_n(m, *, |m+1)$ like $\mathcal{C}^k_n(m, *, |m')$ in Definition 19, but with $m+1$ instead of $m' = m + k$. As before, denote $C^k_n(m, *, |m+1) = |\mathcal{C}^k_n(m, *, |m+1)|$.

Explicitely, $\mathcal{C}^k_n(m, *, |m')$ is the subset of $\mathcal{C}^k_n(m, *)$ containing elements α such that the minimal blue element, $m + 1$, is in an ordinary Callan pair as a singleton, and there is at least one bar preceding this pair.

There is a bijection from $\mathcal{C}^k_n(m, *, |m')$ to $\mathcal{C}^k_n(m, *, |m+1)$ via a relabelling of blue elements that exchanges $m + k$ and $m + 1$. Using this, (12) is clearly equivalent to:

$$\sum_{j=0}^{n}(-1)^j C^j_{n-j}(m, *, |m+1) = -\sum_{j=0}^{n-2}(-1)^j C^j_{n-2-j}(m+1). \tag{13}$$

In order to show (13), we give a bijection ψ between the sets $\mathcal{C}_n^k(m, *, |m+1)$ and $\mathcal{C}_{n-1}^{k-1}(m+1)$ for $n, k \geq 1$. Note that $\mathcal{C}_0^k(m, *, |m+1) = \mathcal{C}_n^0(m, *, |m+1) = \varnothing$, following the remarks after Definition 19.

We do this in two steps, i.e., as the composition of two operations: $\psi = \psi_r \circ \psi_b$. The first map ψ_b takes the blue element $m+1$ out of a Callan pair and changes it into a labeled blue bar $|_{m+1}$. The second map ψ_r does the same with $m+1$ and $|_{m+1}$.

Definition 23. For $\alpha \in \mathcal{C}_n^k(m, *, |m+1)$, note that α contains an ordinary pair $(m+1, R)$. Let w_1 (respectively, w_2) denote the maximal subsequence of consecutive bars that are directly to the left (respectively, to the right) of $(m+1, R)$. Note that w_1 is nonempty by definition, but w_2 is possibly empty. We define $\psi_b(\alpha)$ as follows.

1. Replace $w_1\ (m+1, R)\ w_2$ with $w_2\ |_{m+1}\ w_1$ in the sequence.
2. Replace the extra red block $\{*\}$ with $R \cup \{*\}$.

Example 24.

$$(4,9)(58,25)|_1(2,347)|_1|_0(37,68)(6*,*) \to_{\psi_b} (4,9)(58,35)|_1|_0|_2|_1(37,68)(6*,347*).$$

Note that an element in the image of ψ_b has a nonempty extra red block.

Definition 25. Let $\alpha \in \mathrm{im}(\psi_b)$. We define $\psi_r(\alpha)$ by the following procedure distinguishing two cases.

- If $m+1 \in R^*$, replace the extra pair (B^*, R^*) with two consecutive elements $|_{m+1}$ and $(B^*, R^* \setminus \{m+1\})$.
- Otherwise, $m+1$ is in an ordinary pair (B_i, R_i). Replace the pair (B_i, R_i) with two consecutive elements $|_{m+1}$ and $(B_i, R^* \setminus \{*\})$, then replace the extra pair (B^*, R^*) with $(B^*, R' \cup \{*\})$ where $R' = R_i \setminus \{m+1\}$.

Example 26.

$$(583,46)(29,35)|_1|_0(47,2)(3*,17*) \to_{\psi_r} (583,46)(29,35)|_1|_0(47,2)|_1(3*,7*),$$
$$(583,1)(29,35)|_1|_0(47,46)(3*,27*) \to_{\psi_r} |_1(583,27)(29,35)|_1|_0(47,46)(3*,*).$$

Lemma 27. *For $n, k \geq 1$ and $m \geq 0$, the map $\psi = \psi_r \circ \psi_b$ is a bijection between the sets $\mathcal{C}_n^k(m, *, |m+1)$ and $\mathcal{C}_{n-1}^{k-1}(m+1)$.*

Proof. First, we can check that $\psi(\alpha)$ with labelled bars removed is a $(k-1)\times(n-1)$ Callan sequence where labels are shifted up by $m+1$. As noted above, $\psi_b(\alpha)$ has a nonempty extra red block. In the second case of the definition of ψ_r, we thus see that $R^* \setminus \{*\} \neq \varnothing$, so that $(B_i, R^* \setminus \{*\})$ is a valid Callan pair.

Secondly, we check that the labelled bars in $\psi(\alpha)$ sastisfy the conditions so that $\psi(\alpha) \in \mathcal{C}_{n-1}^{k-1}(m+1)$. We begin with labelled bars in $\psi_b(\alpha)$. We have to check two locations.

- The element after $|_{m+1}$ should be a labelled bar with label strictly smaller than $m+1$. This element is the first one of w_1, as w_1 is nonempty. By definition, it satisfies the required properties.

- If w_2 is empty, either $|_{m+1}$ begins the sequence or it is preceded by a Callan pair. Otherwise, the last element of w_2 is a bar $|_i$ with $i \leq m$, as it is followed by a Callan pair in α. In both cases, this is a valid configuration.

Then, $\psi(\alpha)$ is obtained by adding $|_{m+1}$ to the left of a Callan pair. Thus, either it is a new group of bars in itself, or it is added at the end of an existing group of bars. In the second case, as the last element of this group is a red bar with label i at most m, the configuration $|_i|_{m+1}$ is valid.

We thus have $\psi(\alpha) \in \mathcal{C}_{n-1}^{k-1}(m+1)$. It remains to describe the inverse bijection ψ^{-1} and to check that it is indeed the left and right inverse of ψ. In the definition of ψ_r, the cases can be distinguished: the first one (respectively, second one) results in $|_{m+1}$ being to the left of the extra pair (respectively, of an ordinary pair). Using that, the inverse bijection is straightforward to describe explicitly. □

Remark 28. Our bijections provided a combinatorial proof for the fact that the alternating diagonal sum of C-poly-Bernoulli numbers is given by the Genocchi number (Theorem 9). We used a special technique. Instead of defining a direct involution on the set that is enumerated by the C-poly-Bernoulli numbers, we introduced interpolating sets, the m-barred Callan sequences. Starting with the set of Callan sequences, enumerated by the C-poly-Bernoulli number, we applied the involution ϕ that reduced the number of elements to obtain a new set, on which we again applied the involution, until we reached the set that is known to be counted by the Genocchi number. This tricky solution shows the difficulty of the problem somewhat. It would be also interesting to find a direct bijection on any of the corresponding combinatorial objects that proves Theorem 9 without any iteration steps needed. Moreover, it is still an open problem to find a combinatorial proof for the more general identity related with symmetrized poly-Bernoulli numbers and Gandhi polynomials (see [11]).

6 Acknowledgments

The authors would like to thank the anonymous referees for helpful comments and suggestions.

References

[1] T. Arakawa and M. Kaneko, Multiple Zeta values, poly-Bernoulli numbers, and related zeta functions, *Nagoya Math. J.* **153** (1999), 189–209.

[2] T. Arakawa and M. Kaneko, On poly-Bernoulli numbers, *Comment. Math. Univ. St. Paul.* **48** (1999), 159–167.

[3] B. Bényi and P. Hajnal, Combinatorics of poly-Bernoulli numbers, *Studia Sci. Math. Hungarica* **52** (2015), 537–558.

[4] B. Bényi and P. Hajnal, Combinatorial properties of poly-Bernoulli relatives, *Integers* **17** (2017), Paper A31.

[5] B. Bényi and T. Matsusaka, On the combinatorics of symmetrized poly-Bernoulli numbers, *Electron. J. Combin.* **28** (2021), P1.47.

[6] B. Bényi and G. Nagy, Bijective enumeration of Γ-free 0-1 matrices, *Adv. in Appl. Math.* **96** (2018), 195–215.

[7] C. R. Brewbaker, Lonesum $(0, 1)$ matrices and the poly-Bernoulli numbers of negative index, Master's thesis, Iowa State Univ., 2005. Available at `https://lib.dr.iastate.edu/rtd/18914/`.

[8] D. Dumont, Interprétations combinatoires des nombres de Genocchi, *Duke Math. J.* **41** (1974), 305–318.

[9] M. Kaneko, Poly-Bernoulli numbers, *J. Théor. Nombres Bordeaux* **9** (1997), 221–228.

[10] M. Kaneko, F. Sakurai, and H. Tsumura, On a duality formula for certain sums of values of poly-Bernoulli polynomials and its application, *J. Théorie Nombres Bordeaux* **30** (2018), 203–218.

[11] T. Matsusaka, Symmetrized poly-Bernoulli numbers and combinatorics, *J. Integer Sequences* **23** (2020), Article 20.9.2.

[12] N. J. A. Sloane et al., The on-line encyclopedia of integer sequences, 2021. Available at `https://oeis.org`.

[13] R. P. Stanley, *Enumerative Combinatorics*, Volume 2, Cambridge University Press, 1999.

2010 *Mathematics Subject Classification*: Primary 05A19; Secondary: 11B68.
Keywords: Genocchi number, poly-Bernoulli number, combinatorial identity, bijection, Callan sequence.

(Concerned with sequences A036968, A099594, A136126, and A272644.)

Received January 5 2021; revised versions received May 26 2021; July 17 2021. Published in *Journal of Integer Sequences*, July 18 2021.

Return to Journal of Integer Sequences home page.