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# Finite Binomial Sum Identities with Harmonic Numbers

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**Abstract**

We prove some general combinatorial identities involving a variable, a parameter, and partial sums of arbitrary sequences. Applying these formulas, we deduce many finite binomial and central binomial sum identities involving the harmonic numbers. Most of our results are new and some known formulas are particular cases of those obtained here.

## 1 Introduction

Let $s$ be a complex number and $n \in \mathbb{N}$. We recall that the generalized harmonic numbers $H_n^{(s)}$ of order $s$ are defined by ,

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}.$$

$H_0^{(s)} = 0$ and $H_n^{(1)}$ is the familiar harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. See [18, Section 6.3]. Alternating or skew-harmonic numbers $H_n^-$ are defined by

$$H_n^- = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k},$$

which is the $n$ th partial sum of $\log 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$, see [4, 18]. The harmonic numbers have been studied since antiquity, and they have involved in a wide range of diverse areas of science and mathematics such as number theory, combinatorics, analysis, computer science, and differential equations. It is a well-known fact that the Riemann zeta function $\zeta$ has many series representations involving harmonic numbers; see, for example, the excellent book by Vălean [31], and [2]. Lagarias [21] proved that the Riemann hypothesis is equivalent to the statement $\sigma(n) \leq H_n + (\log H_n)e^{H_n}$, where $\sigma(n)$ is the sum of positive divisors of $n$. In a very recent paper, Elliot [12] discovered many expressions involving the harmonic numbers, which are equivalent to the Riemann hypothesis. Throughout this paper, we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, -3, \cdots\}$, and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$. We let $C_n$ denote the central binomial coefficients, that is,

$$C_n = \binom{2n}{n}.$$

Let $\Gamma$ be the classical gamma function of Euler, and $\psi(x) = \Gamma'(x)/\Gamma(x)$ $(x > 0)$ be the digamma function. We may recall some basic properties of these important functions, which will be used extensively in this paper. The gamma function satisfies the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad (s \in \mathbb{C}\backslash\mathbb{Z}) \tag{1}$$

(see [23, p. 344] and [11, p. 253]) and the duplication formula

$$\Gamma\left(s + \frac{1}{2}\right) = \frac{\Gamma(2s)\Gamma(1/2)}{2^{2s-1}\Gamma(s)} \tag{2}$$

(see [23, p. 349] and [11, p. 252]). The binomial coefficients $\binom{s}{t}$ $(s, t \in \mathbb{C}\backslash\mathbb{Z}^-)$ are defined by

$$\binom{s}{t} = \frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)}. \tag{3}$$

Making use of formulas (1) and (2) with $s = 1/2 - k$, we obtain

$$\Gamma\left(\frac{1}{2} - k\right) = \frac{(-1)^k \Gamma(1/2) k! 2^{2k}}{(2k)!}. \tag{4}$$

We shall extensively use the following form of the binomial coefficients, which can be shown using the expressions (1), (2) and (3).

$$\binom{-\frac{1}{2}}{k} = \frac{(-1)^k C_k}{4^k} \quad \text{and} \quad \binom{-\frac{3}{2}}{k} = \frac{(-1)^k (2k+1) C_k}{4^k}. \tag{5}$$

The digamma function $\psi$ and harmonic numbers $H_n$ are connected with

$$\psi(n+1) = -\gamma + H_n \quad (n \in \mathbb{N}_0); \tag{6}$$

see [28, p. 31], where $\gamma = 0.57721\cdots$ is the Euler-Mascheroni constant.

The digamma function $\psi$ possesses the following properties:

$$\psi(s) - \psi(1-s) = -\pi \cot(\pi s) \quad (s \in \mathbb{C} \backslash \mathbb{Z}), \tag{7}$$

and

$$\psi\left(s + \frac{1}{2}\right) = 2\psi(2s) - \psi(s) - 2\log 2 \quad (s \in \mathbb{C} \backslash \mathbb{Z}^-); \tag{8}$$

see [28, p. 25]. Using (6), (7) and (8) we get

$$\psi\left(\frac{1}{2} - k\right) = \psi\left(\frac{1}{2} + k\right) = 2H_{2k} - H_k + \psi(1/2). \tag{9}$$

The binomial coefficients satisfy the following useful identities for $n, k \in \mathbb{N}$ with $k \leq n$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{and} \quad \frac{n+1}{k+1}\binom{n+1}{k+1} = \binom{n}{k}. \tag{10}$$

In the literature, there exist many interesting identities for finite sums involving the harmonic numbers and the binomial coefficients. As examples,

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n+k}{k} (H_k - H_{n-k}) = C_n^2 (H_{2n} - H_n);$$

see [9],

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} H_{n+k}^2 = \frac{1}{nC_n} \left(H_n - H_{2n} - \frac{2}{n}\right);$$

see [32],

$$\sum_{k=1}^{n} \frac{H_k C_k}{4^k} = 2 + \frac{(H_n - 2)(2n+1)C_n}{4^n},$$

see [7], and finally

$$\sum_{k=0}^{n} \frac{(-1)^k H_k}{\binom{n}{k}} = (n+1)\left(\frac{(-1)^n H_{n+1}}{n+2} - \frac{1+(-1)^n}{(n+2)^2}\right),$$

which is a slightly simplified form of [3, Identity 15)]. Recently, many other remarkable finite and infinite sum identities involving the harmonic numbers have been developed by many authors in different forms using a variety of methods. In [9] and [29] the authors used some identities for classical hypergeometric functions. By making use of the residue theorem in complex analysis, Flajolet and Sedgewick [14] established many elegant harmonic sum identities. Wang [32] obtained many nice harmonic number identities using the method of Riordan arrays. In [27] the author used generating function method to evaluate many finite and infinite harmonic sums in closed form. In [7] and [20] the authors evaluated many finite harmonic sums in closed form by using Abel-Gosper algorithm. By using finite difference method, Spivey [26] presented many summation formulas involving the binomial coefficients and harmonic numbers. We refer the interested readers to the papers given in [2, 5, 8, 10, 15, 16, 17, 19, 22, 24, 25, 29, 30] and the references therein for more identities. As we have seen in the brief review, harmonic numbers appear in a variety of useful identities. In a very recent paper, Batır and Sofo [1, Theorem 2.2] proved the following identity: Let $x \in \mathbb{C}$ and $(a_n)_{n\geq 1}$ be any sequence of complex numbers, and $A_n = a_1 + a_2 + \cdots + a_n$. Then

$$\sum_{k=1}^{n} \binom{n}{k} A_k x^k = (1+x)^{n-1} \sum_{k=0}^{n-1} \left( \sum_{j=0}^{k} \binom{k}{j} a_{j+1} x^{j+1} \right) \frac{1}{(1+x)^k}. \tag{11}$$

Using this identity, they deduced many interesting binomial sum identities involving the harmonic numbers, and Fibonacci and Lucas sequences. This article can be regarded as a continuation of the papers [3] and [1]. Our aim in this paper is to generalize formula (11), and to prove the formula (12) noted below. This formula allows us to establish further interesting finite binomial and central binomial sum identities involving harmonic numbers. To demonstrate the usefulness of our formula we present many finite binomial and central binomial sum identities involving harmonic numbers by choosing particular values for $s, x$ and the sequence $(a_n)$. Most of our results are new and many known formulas are special cases of those obtained here.

Now we are ready to present our main results.

## 2 Main Results

**Theorem 1.** *Let $(a_n)$ be a sequence in $\mathbb{C}$, $x \in \mathbb{C}$, and $s \in \mathbb{C}\backslash\mathbb{Z}^-$. Let $A_n = a_1 + a_2 + \cdots + a_n$. Then*

$$\sum_{k=0}^{n} \binom{n+s}{k} A_k x^k = (1+x)^n \Bigg( \sum_{k=0}^{n-1} \binom{s+k}{1+k} A_{k+1} \left(\frac{x}{x+1}\right)^{k+1} + \sum_{k=0}^{n-1} \Bigg( \sum_{j=0}^{k} \binom{s+k}{j} a_{j+1} x^{j+1} \Bigg) \frac{1}{(x+1)^{k+1}} \Bigg). \tag{12}$$

*Proof.* The proof is by mathematical induction on $n$. The theorem is obviously true for $n = 1$. Assume that it is also true for $n$. Let us define

$$f_n(x) = \sum_{k=1}^{n} \binom{n+s}{k} A_k x^k.$$

Then we have by the first identity in (10)

$$\begin{aligned} f_{n+1}(x) &= \sum_{k=1}^{n+1} \binom{n+s+1}{k} A_k x^k \\ &= \binom{n+s+1}{n+1} A_{n+1} x^{n+1} + \sum_{k=1}^{n} \left( \binom{n+s}{k} + \binom{n+s}{k-1} \right) A_k x^k \\ &= \binom{n+s+1}{n+1} A_{n+1} x^{n+1} + f_n(x) + \sum_{k=0}^{n-1} \binom{n+s}{k} A_{k+1} x^{k+1}. \end{aligned}$$

Using $A_{k+1} = A_k + a_{k+1}$, we get after a simple computation

$$f_{n+1}(x) = f_n(x) + \binom{n+s+1}{n+1} A_{n+1} x^{n+1} + \sum_{k=0}^{n-1} \binom{n+s}{k} A_k x^{k+1}$$
$$+ \sum_{k=0}^{n-1} \binom{n+s}{k} a_{k+1} x^{k+1}.$$

Making some algebraic manipulations, this becomes

$$f_{n+1}(x) = (1+x) f_n(x) + \left( \binom{n+s+1}{n+1} - \binom{n+s}{n} \right) A_n x^{n+1}$$
$$+ \left( \binom{n+s+1}{n+1} - \binom{n+s}{n} \right) a_{n+1} x^{n+1} + \sum_{k=0}^{n} \binom{n+s}{k} a_{k+1} x^{k+1}.$$

By the first identity in (10), we have

$$f_{n+1}(x) = (1+x) f_n(x) + \binom{n+s}{n+1} A_{n+1} x^{n+1} + \sum_{k=0}^{n} \binom{n+s}{k} a_{k+1} x^{k+1}. \quad (13)$$

Now using the inductive hypothesis and (13), we obtain

$$f_{n+1}(x) = (1+x)^{n+1} \sum_{k=0}^{n-1} \binom{s+k}{1+k} A_{k+1} \left( \frac{x}{x+1} \right)^{k+1}$$
$$+ (1+x)^{n+1} \sum_{k=0}^{n-1} \left( \sum_{j=0}^{k} \binom{s+k}{j} a_{j+1} x^{j+1} \right) \frac{1}{(x+1)^{k+1}}$$
$$+ \binom{n+s}{n+1} A_{n+1} x^{n+1} + \sum_{k=0}^{n} \binom{n+s}{k} a_{k+1} x^{k+1}.$$

This can be simplified to

$$f_{n+1}(x) = (1+x)^{n+1} \left( \sum_{k=0}^{n} \binom{s+k}{1+k} A_{k+1} \left( \frac{x}{x+1} \right)^{k+1} \right.$$
$$\left. + \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \binom{s+k}{j} a_{j+1} x^{j+1} \right) \frac{1}{(x+1)^{k+1}} \right).$$

So, identity (12) is also valid for $n+1$ and the proof is completed. $\square$

**Corollary 2.** *Letting $x=-1$ in (12) yields for $n\in\mathbb{N}$, and $s\in\mathbb{C}\backslash\mathbb{Z}^-$*

$$\sum_{k=1}^{n}(-1)^k\binom{n+s}{k}A_k$$
$$=(-1)^n\binom{s+n-1}{n}A_n+\sum_{k=0}^{n-1}(-1)^{k+1}\binom{s+n-1}{k}a_{k+1}. \tag{14}$$

**Corollary 3.** *For $s\in\mathbb{C}\backslash\mathbb{Z}^-$, and $n\in\mathbb{N}$ the following identity holds.*

$$\sum_{k=1}^{n}(-1)^k\binom{n+s}{k}H_k$$
$$=(-1)^n\binom{s+n-1}{n}H_n+\frac{(-1)^n}{s+n}\binom{s+n-1}{n}-\frac{1}{s+n}.$$

*Proof.* If we put $a_k=\frac{1}{k}$ in (14) we get

$$\sum_{k=0}^{n}(-1)^k\binom{n+s}{k}H_k$$
$$=(-1)^n\binom{s+n-1}{n}H_n+\sum_{k=0}^{n-1}(-1)^{k+1}\binom{s+n-1}{k}\frac{1}{k+1}.$$

Letting $k+1=k'$ and then deleting the prime, we get by the second identity in (10)

$$\sum_{k=0}^{n}(-1)^k\binom{n+s}{k}H_k$$
$$=(-1)^n\binom{s+n-1}{n}H_n+\frac{1}{s+n}\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}. \tag{15}$$

In [3, Eq. 33] it was proved that

$$\sum_{k=0}^{n}(-1)^k\binom{s+n}{k}=(-1)^n\binom{s+n-1}{n}. \tag{16}$$

Upon using this identity in (15), one easily obtains the desired conclusion. □

Letting $x = 1$ in (12) we obtain

**Corollary 4.** *Let $n \in \mathbb{N}$ and $s \in \mathbb{C}\backslash\mathbb{Z}^-$. Then we have*

$$\sum_{k=1}^{n} \binom{n+s}{k} A_k$$
$$= 2^n \sum_{k=1}^{n} \binom{s+k-1}{k} \frac{A_k}{2^k} + 2^n \sum_{k=1}^{n} \frac{1}{2^k} \sum_{j=0}^{k-1} \binom{s+k-1}{j} a_{j+1}. \tag{17}$$

Differentiating both sides of the equation given in Corollary 3 with respect to $s$, and then setting $a_k = \frac{1}{k}$ we get, by (10), the following corollary.

**Corollary 5.** *For $s \in \mathbb{C}\backslash\mathbb{Z}^-$ and $n \in \mathbb{N}$ the following identity holds.*

$$\sum_{k=1}^{n} (-1)^k \binom{n+s}{k} (\psi(s+n+1) - \psi(s+n-k+1)) H_k$$
$$= (-1)^n H_n \binom{s+n-1}{n} (\psi(s+n) - \psi(s)) - \frac{(-1)^n}{(s+n)^2} \binom{s+n-1}{n}$$
$$+ \frac{(-1)^n}{s+n} \binom{s+n-1}{n} (\psi(s+n) - \psi(s)) + \frac{1}{(s+n)^2}. \tag{18}$$

**Proposition 6.** *For $s \in \mathbb{C}\backslash\mathbb{Z}^-$ and $n \in \mathbb{N}$ the following identity is valid.*

$$\sum_{k=1}^{n} (-1)^k \binom{s+n}{k} H_k^{(2)}$$
$$= (-1)^n \binom{s+n-1}{n} H_n^{(2)} - \frac{1}{s+n} \left( H_n - s \sum_{k=1}^{n} \frac{(-1)^k \binom{s+k-1}{k-1}}{k^2 \binom{n}{k}} \right). \tag{19}$$

*Proof.* Putting $a_k = \frac{1}{k^2}$ in (14), we get

$$\sum_{k=1}^{n} (-1)^k \binom{s+n}{k} H_k^{(2)}$$
$$= (-1)^n \binom{s+n-1}{n} H_n^{(2)} + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(k+1)^2} \binom{s+n-1}{k}. \tag{20}$$

Using (10), we arrive at

$$\sum_{k=0}^{n-1}\frac{(-1)^{k+1}}{(k+1)^2}\binom{s+n-1}{k}=\frac{1}{s+n}\sum_{k=0}^{n-1}\frac{(-1)^{k+1}}{k+1}\binom{s+n}{k+1}.$$

If we let $k+1=k'$ (and then $k'=k$) we get

$$\sum_{k=0}^{n-1}\frac{(-1)^{k+1}}{(k+1)^2}\binom{s+n-1}{k}=\frac{1}{s+n}\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\frac{1}{k}. \tag{21}$$

The following identity comes from [3, Theorem 4].

$$\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\frac{1}{k}=s\sum_{k=1}^{n}\frac{(-1)^k\binom{s+k-1}{k-1}}{k^2\binom{n}{k}}-H_n. \tag{22}$$

Combining (20), (21) and (22), we see that (19) is valid. □

**Theorem 7.** *For all $s\in\mathbb{C}$, which is not a negative integer, and $n\in\mathbb{N}$ we have*

$$\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}H_k^2=(-1)^n\binom{s+n-1}{n}\left(H_n^2+\frac{2H_n}{s+n}+\frac{2}{(s+n)^2}\right)$$
$$+\frac{H_n}{s+n}-\frac{2}{(s+n)^2}-\frac{s}{s+n}\sum_{k=1}^{n}\frac{(-1)^k\binom{s+k-1}{k-1}}{k^2\binom{n}{k}}. \tag{23}$$

*Proof.* Letting $a_k=H_k^2-H_{k-1}^2$ in Corollary 2, we get

$$\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}H_k^2$$
$$=(-1)^n\binom{s+n-1}{n}H_n^2+\sum_{k=0}^{n-1}(-1)^{k+1}\binom{s+n-1}{k}\left(H_{k+1}^2-H_k^2\right). \tag{24}$$

Using (10) and $H_{k+1}^2-H_k^2=\frac{2H_{k+1}}{k+1}-\frac{1}{(k+1)^2}$, (24) can be easily simplified to

$$\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}H_k^2=(-1)^n\binom{s+n-1}{n}H_n^2$$
$$+\frac{2}{s+n}\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}H_k-\frac{1}{s+n}\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\frac{1}{k}.$$

If we apply Corollary 3 and (22) to this identity, we complete the proof. $\square$

Differentiating both sides of (23) with respect to $s$, we arrive at the following corollary.

**Corollary 8.** *Let $s \in \mathbb{C}\backslash\mathbb{Z}^-$ and $n \in \mathbb{N}$. Then we have*

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\left(\psi(s+n+1)-\psi(s+n-k+1)\right)H_k^2 \\
&= (-1)^n\binom{s+n-1}{n}\left(H_n^2+\frac{2H_n}{s+n}+\frac{2}{(s+n)^2}\right)\left(\psi(s+n)-\psi(s)\right) \\
&- (-1)^n\binom{s+n-1}{n}\left(\frac{2H_n}{(s+n)^2}+\frac{4}{(s+n)^3}\right) \\
&- \frac{H_n}{(s+n)^2}+\frac{4}{(s+n)^3}-\frac{n}{(s+n)^2}\sum_{k=1}^{n}\frac{(-1)^k\binom{s+k-1}{k-1}}{k^2\binom{n}{k}} \\
&+ \frac{s}{s+n}\sum_{k=1}^{n}\frac{(-1)^k\binom{s+k-1}{k-1}}{k^2\binom{n}{k}}\left(\psi(s+k)-\psi(s+1)\right).
\end{aligned} \tag{25}$$

**Theorem 9.** *Let $s \in \mathbb{C}\backslash\mathbb{Z}^-$ and $n \in \mathbb{N}$. Then we have*

$$\begin{aligned}
\sum_{k=1}^{n}\binom{n+s}{k}H_k^- &= 2^n\sum_{k=1}^{n}\binom{s+k-1}{k}\frac{H_k^-}{2^k}+2^n\sum_{k=1}^{n}\frac{2^{-k}}{s+k} \\
&\quad + 2^n\sum_{k=1}^{n}\frac{(-1)^{k+1}}{s+k}\binom{s+k-1}{k}\frac{1}{2^k}.
\end{aligned} \tag{26}$$

*Proof.* Setting $a_k = \frac{(-1)^{k+1}}{k}$ in (14), we obtain

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n+s}{k}H_k^- &= 2^n\sum_{k=0}^{n-1}\binom{s+k}{1+k}\frac{H_{k+1}^-}{2^{k+1}} \\
&\quad + 2^n\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}\binom{s+k}{j}\frac{(-1)^j}{j+1}\right)\frac{1}{2^{k+1}}.
\end{aligned} \tag{27}$$

According to (10) and using (16), we deduce

$$\begin{aligned}\sum_{j=0}^{k}\binom{s+k}{j}\frac{(-1)^j}{j+1} &= \frac{1}{s+k+1}\sum_{j=1}^{k+1}(-1)^{j+1}\binom{s+k+1}{j}\\ &= \frac{(-1)^k}{s+k+1}\binom{s+k}{k+1}+\frac{1}{s+k+1}.\end{aligned}$$

Substituting this in (27), and letting $k+1=k'$ (and then $k'=k$), we complete the proof. □

**Theorem 10.** *Let $n\in\mathbb{N}$ and $s\in\mathbb{C}\backslash\mathbb{Z}^-$. Then we have*

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\left(H_k^-\right)^2 = (-1)^n\binom{s+n-1}{n}\left(H_n^-\right)^2\\ &-\frac{2^{n+1}}{s+n}\left(\sum_{k=1}^{n}\binom{s+k-1}{k}\frac{H_k^-}{2^k}-\sum_{k=1}^{n}\frac{(-1)^k}{2^k(s+k)}\binom{s+k-1}{k}\right.\\ &\left.+\sum_{k=1}^{n}\frac{1}{2^k(s+k)}\right)-\frac{1}{s+n}\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\frac{1}{k}.\end{aligned}$$

*Proof.* Putting $a_k=\left(H_k^-\right)^2-\left(H_{k-1}^-\right)^2$ in (14), we obtain

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\left(H_k^-\right)^2 &= \sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\sum_{j=1}^{k}\left(\left(H_j^-\right)^2-\left(H_{j-1}^-\right)^2\right)\\ &= (-1)^n\binom{s+n-1}{n}\left(H_n^-\right)^2\\ &-\sum_{k=0}^{n-1}(-1)^k\binom{s+n-1}{k}\left(\left(H_{k+1}^-\right)^2-\left(H_k^-\right)^2\right).\end{aligned}\tag{28}$$

But since

$$\left(H_{k+1}^-\right)^2-\left(H_k^-\right)^2=\frac{(-1)^k}{k+1}\left(2H_{k+1}^- - \frac{(-1)^k}{k+1}\right)$$

we deduce from (28)

$$\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\left(H_k^-\right)^2=(-1)^n\binom{s+n-1}{n}\left(H_n^-\right)^2$$
$$-\sum_{k=0}^{n-1}(-1)^k\binom{s+n-1}{k}\frac{(-1)^k}{k+1}\left(2H_{k+1}^- -\frac{(-1)^k}{k+1}\right).$$

Making use of the second identity in (10), and simplifying the result, It follows from this identity that

$$\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\left(H_k^-\right)^2=(-1)^n\binom{s+n-1}{n}\left(H_n^-\right)^2$$
$$-\frac{2}{s+n}\sum_{k=0}^{n-1}\binom{s+n}{k+1}H_{k+1}^- -\frac{1}{s+n}\sum_{k=0}^{n-1}(-1)^{k+1}\binom{s+n}{k+1}\frac{1}{k+1}.$$

Letting $k+1=k'$ (and then $k'=k$), this can be simplified to

$$\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\left(H_k^-\right)^2=(-1)^n\binom{s+n-1}{n}\left(H_n^-\right)^2$$
$$-\frac{2}{s+n}\sum_{k=1}^{n}\binom{s+n}{k}H_k^- -\frac{1}{s+n}\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\frac{1}{k}. \tag{29}$$

We therefore conclude from Theorem 9

$$\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\left(H_k^-\right)^2=(-1)^n\binom{s+n-1}{n}\left(H_n^-\right)^2$$
$$-\frac{2^{n+1}}{s+n}\left(\sum_{k=1}^{n}\binom{s+k-1}{k}\frac{H_k^-}{2^k}-\sum_{k=1}^{n}\frac{(-1)^k}{2^k(s+k)}\binom{s+k-1}{k}\right.$$
$$\left.+\sum_{k=1}^{n}\frac{1}{2^k(s+k)}\right)-\frac{1}{s+n}\sum_{k=1}^{n}(-1)^k\binom{s+n}{k}\frac{1}{k},$$

which is the desired conclusion. $\square$

## 3 Applications

In this section we present some applications of our results. Our first identity is well-known and due to Euler [13].

**Identity 11.** Let $n \in \mathbb{N}$. If we set $s=0$ in Corollary 4, we get the well-known identity

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k = -\frac{1}{n}.$$

**Identity 12.** Let $n \in \mathbb{N}$. Then setting $a_k = \frac{1}{k}$, $x=1$ and $s=0$ in Theorem 1, we obtain

$$\sum_{k=1}^{n}\binom{n}{k}H_k = 2^n\left(H_n - \sum_{k=1}^{n}\frac{1}{k2^k}\right).$$

*Remark* 13. This identity is also well-known; see [1, 3, 5, 17, 29].

**Identity 14.** For $n \in \mathbb{N}$ the following identity holds.

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}H_kH_{n-k} = \frac{\left(1+(-1)^n\right)\left(nH_n+1\right)}{n^2}.$$

*Proof.* Setting $s=0$ in Corollary 5, we get

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\{\psi(n+1)-\psi(n-k+1)\}H_k\\
&= (-1)^nH_n\binom{n-1}{n}\psi(n) - (-1)^nH_n\binom{s+n-1}{n}\psi(s)\bigg|_{s=0} + \frac{1}{n^2}\\
&- \frac{(-1)^n}{n^2}\binom{n-1}{n} + \frac{(-1)^n}{n}\binom{n-1}{n}\psi(n)\\
&- \frac{(-1)^n}{n}\binom{s+n-1}{n}\psi(s)\bigg|_{s=0}.
\end{aligned} \tag{30}$$

Since $\binom{n-1}{n}=0$, $\psi(n+1)-\psi(n-k+1) = H_n - H_{n-k}$ by (6), and

$$\begin{aligned}
\binom{s+n-1}{n}\psi(s)\bigg|_{s=0} &= \frac{\Gamma(s+n)}{n!}\frac{\psi(s)}{\Gamma(s)}\bigg|_{s=0}\\
&= \frac{1}{n}\lim_{s\to 0}\frac{s\psi(s+1)-1}{\Gamma(s+1)} = -\frac{1}{n},
\end{aligned} \tag{31}$$

we conclude from (30) and (31)

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\left(H_n - H_{n-k}\right)H_k = \frac{1+(-1)^n(nH_n+1)}{n^2}.$$

Now the proof follows from Identity 11. □

**Identity 15.** Let $n$ be a positive integer. Then we have

$$\sum_{k=1}^{n}\frac{H_kC_k}{4^k} = 2 + \frac{(H_n-2)(2n+1)C_n}{4^n}.$$

*Proof.* Setting $s=-\frac{1}{2}-n$ in Corollary 3 we find

$$\sum_{k=1}^{n}(-1)^k\binom{-\frac{1}{2}}{k}H_k = (-1)^n\binom{-\frac{3}{2}}{n}H_n - 2(-1)^n\binom{-\frac{3}{2}}{n} + 2.$$

The proof immediately follows from (5). □

*Remark* 16. Identity 15 is not new and it has been proved by Chen et al. [7] by a method combining the Abel's summation formula and Gosper's algorithm elegantly.

**Identity 17.** Let $n$ be a positive integer. Then we have

$$\sum_{k=1}^{n}\frac{H_{2k}C_k}{4^k} = 1 + \frac{(2n+1)(H_{2n+1}-2)C_n}{4^n}.$$

*Proof.* In [25, Identity 3] it was proved that

$$\sum_{k=1}^{n}\frac{(2H_{2k}-H_k)C_k}{4^k} = \frac{(2n+1)(2H_{2n+1}-H_n-2)C_n}{4^n}. \tag{32}$$

Now we complete the proof by combining Identity 15 and (32). □

Setting $s=0$ in (23), we get the following known result; see [1, 3, 20, 32].

**Identity 18.** Let $n$ be a positive integer. Then we have

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k^2 = \frac{H_n}{n} - \frac{2}{n^2}.$$

**Identity 19.** Let $n$ be a positive integer. Then we have

$$\sum_{k=1}^{n} \frac{H_k^2 C_k}{4^k} = -2H_n - 8$$
$$+ \frac{(2n+1)C_n}{4^n}\left(H_n^2 - 4H_n + 8 + 2\sum_{k=0}^{n-1} \frac{4^k}{(n-k)(2k+1)C_k}\right).$$

*Proof.* Setting $s = \frac{1}{2} - n$ in (23) we get

$$\sum_{k=1}^{n}(-1)^k \binom{-\frac{1}{2}}{k} H_k^2 = (-1)^n \binom{-\frac{3}{2}}{n} \left(H_n^2 - 4H_n + 8\right) - 2H_n - 8$$
$$- \frac{2n+1}{2} \sum_{k=1}^{n} \frac{(-1)^k}{k^2\binom{n}{k}} \binom{-\frac{3}{2} - n + k}{k-1}. \tag{33}$$

Using (9) we get, after simplifying

$$\binom{-\frac{3}{2} - n + k}{k-1} = \frac{\Gamma\left(-\frac{1}{2} - n + k\right)}{(k-1)!\Gamma\left(\frac{1}{2} - n\right)} = \frac{-2}{2n-2k+1} \frac{\Gamma\left(\frac{1}{2} - n + k\right)}{(k-1)!\Gamma\left(\frac{1}{2} - n\right)}$$
$$= \frac{2(-1)^{k+1}}{2n-2k+1} \frac{(n-k)!}{(k-1)!(2n-2k)!} \frac{(2n)!}{n!4^k}.$$

Thus,

$$\frac{(-1)^k}{k^2\binom{n}{k}} \binom{-\frac{3}{2} - n + k}{k-1}$$
$$= -\frac{2}{2n-2k+1} \cdot \frac{(n-k)!}{(k-1)!(2n-2k)!} \cdot \frac{(2n)!}{n!4^k} \cdot \frac{k!(n-k)!}{k^2 n!}$$

Upon simplifying this becomes

$$\frac{(-1)^k}{k^2\binom{n}{k}} \binom{-\frac{3}{2} - n + k}{k-1} = -\frac{2}{k(2n-2k+1)} \frac{\binom{2n}{n}}{\binom{2n-2k}{n-k} 4^k}.$$

Substituting this in (33), and changing the index, we arrive at the conclusion by taking into account (10). □

**Identity 20.** For $n \in \mathbb{N}$ the following identity holds.

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_{n-k}H_k^2$$
$$= \frac{\left(1-(-1)^n\right)H_n^2}{n} - \frac{2\left(1+(-1)^n\right)H_n}{n^2} - \frac{4+(-1)^n}{n^3} - \frac{1}{n}\sum_{k=1}^{n}\frac{(-1)^k}{k^2\binom{n}{k}}.$$

*Proof.* The proof is done by the same way with the proof of Identity 18 if we take $s = 0$ in (25). $\square$

**Identity 21.** For $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n}\frac{C_k H_k^{(2)}}{4^k} = 2H_n + \frac{(2n+1)C_n}{4^n}\left(H_n^{(2)} - 2\sum_{k=0}^{n-1}\frac{4^k}{(n-k)(2k+1)C_k}\right).$$

*Proof.* The proof can be done by the same method with the proof of Identity 18 by setting $s = -\frac{1}{2} - n$ in (19). $\square$

**Identity 22.** Let $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{n}\frac{\left(H_k^2 + H_k^{(2)}\right)C_k}{4^k} = \frac{(2n+1)C_n}{4^n}\left(H_n^2 + H_n^{(2)} - 4H_n + 8\right) - 8.$$

*Proof.* If we sum the equations given in Identity 19 and Identity 21 side by side, the proof isimmediately follows. $\square$

Putting $s = -\frac{1}{2} - n$ in (18), we get

**Identity 23.** For $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n}\frac{\left(H_k - 2H_{2k}\right)H_k C_k}{4^k}$$
$$= \frac{(2n+1)C_n}{4^n}\left(H_n - 2H_{2n} + \frac{4n}{2n+1}\right)(H_n - 2) - \frac{(2n+1)C_n}{4^{n-1}} + 4.$$

Combining Identity 19 and Identity 23, we get the following conclusion.

**Identity 24.** For $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \frac{H_k H_{2k} C_k}{4^k} = \frac{(2n+1)C_n}{4^n}\Bigg(H_n H_{2n} - H_n - 2H_{2n} + 6 + \frac{2n(2-H_n)}{2n+1} + \sum_{k=0}^{n-1} \frac{4^k}{(n-k)(2k+1)C_k}\Bigg) - H_n - 6.$$

**Identity 25.** Let $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{n} \frac{(-1)^k H_k^- C_k}{4^k} = \frac{(-1)^n(2n+1)C_n}{4^n} \sum_{k=0}^{n-1} \frac{(-8)^k \binom{n}{k} H_{n-k}^-}{(2k+1)C_k} + \frac{2(2n+1)C_n}{4^n} \sum_{k=0}^{n-1} \frac{\binom{n}{k} 8^k}{(2k+1)^2 C_k} - 2\sum_{k=0}^{n-1} \frac{2^k}{2k+1}.$$

*Proof.* The proof follows from (26) by putting $s = -\frac{1}{2} - n$. We omit the details. □

**Identity 26.** Let $n \in \mathbb{N}$. If we set $s = 0$ in (26), we find

$$\sum_{k=1}^{n} \binom{n}{k} H_k^- = 2^n \sum_{k=1}^{n} \frac{1}{k 2^k}.$$

*Remark* 27. This identity is not new and an inductive proof of it can be found in [6, Eq. (14)] and [4, Eq. (9.20)]. See also [1, Remark 21].

**Identity 28.** For $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \binom{n}{k} H_{n-k} H_k^- = 2^n \sum_{k=1}^{n} \frac{H_n - H_k^-}{2^k k} + 2^n \sum_{k=1}^{n} \frac{1 + (-1)^k}{2^k k^2}.$$

*Proof.* Differentiating both sides of (26) with respect to $s$ and then setting

$s=0$, we get by (6)

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(\psi(n+1)-\psi(n-k+1))H_k^{-}\\
&H_n\sum_{k=0}^{n}\binom{n}{k}H_k^{-}-\sum_{k=0}^{n}\binom{n}{k}H_{n-k}H_k^{-}\\
&=-2^n\sum_{k=1}^{n}\binom{s+k-1}{k}\psi(s)\Big|_{s=0}\frac{H_k^{-}}{2^k}-2^n\sum_{k=1}^{n}\frac{1}{2^k k^2}\\
&+2^n\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{s+k-1}{k}\psi(s)\Big|_{s=0}\frac{1}{2^k},
\end{aligned}
$$

where we have used $\binom{k-1}{k}=0$. Using (31) with $n=k$ it follows that

$$
\begin{aligned}
H_n\sum_{k=1}^{n}\binom{n}{k}H_k^{-}-\sum_{k=0}^{n}\binom{n}{k}H_{n-k}H_k^{-}&=2^n\sum_{k=1}^{n}\frac{H_k^{-}}{k2^k}-2^n\sum_{k=1}^{n}\frac{1}{2^k k^2}\\
&-2^n\sum_{k=1}^{n}\frac{(-1)^k}{k^2 2^k}.
\end{aligned}
$$

Employing Identity 26 the conclusion follows. $\square$

**Identity 29.** Let $n\in\mathbb{N}$. If we set $s=0$ in (28) we deduce by Identity 11

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\left(H_k^{-}\right)^2=\frac{H_n}{n}-\frac{2^{n+1}}{n}\sum_{k=1}^{n}\frac{1}{k2^k}.$$

*Remark* 30. Identity 29 can be found in [1, Identity 25].

Setting $s=\frac{1}{2}$ in Corollary 3 and using the formula (5), after a short computation, we get

**Identity 31.** Let $n\in\mathbb{N}$. Then

$$\sum_{k=0}^{n}\frac{(-1)^k 4^k\binom{n}{k}H_{n-k}}{(2k+1)C_k}=\frac{H_n}{2n+1}+\frac{2}{(2n+1)^2}-\frac{2(-1)^n4^n}{(2n+1)^2C_n}.$$

If we set $s=\frac{1}{2}-n$ in Corollary 3, and using the first identity in (10) we obtain

**Identity 32.** For $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n-1} \frac{C_k H_{k+1}}{(k+1)4^k} = 4 - \frac{2(H_n+2)C_n}{4^n}.$$

**Identity 33.** Setting $s=-\frac{1}{2}$ in Corollary 3, we get for $n \in \mathbb{N}$

$$\sum_{k=0}^{n} \frac{(-1)^{k+1}4^k\binom{n}{k}H_{n-k}}{C_k} = \frac{H_n}{2n-1} + \frac{2}{(2n-1)^2} + \frac{2\cdot 4^n(-1)^n}{(2n-1)C_n}.$$

**Identity 34.** Setting $s=-\frac{1}{2}$ in Proposition 6, we get

$$\sum_{k=0}^{n} \frac{(-1)^k 4^k\binom{n}{k}H_{n-k}^{(2)}}{C_k} = \frac{2(-1)^{n+1}4^n}{(2n-1)C_n}\left(H_n + \sum_{k=1}^{n} \frac{(-1)^k C_k}{k(2k-1)4^k\binom{n}{k}}\right) - \frac{H_n^{(2)}}{2n-1}.$$

**Identity 35.** Let $n$ be a non-negative integer. Then

$$\sum_{k=1}^{n} \binom{n}{k} H_k H_{n-k} = 2^n\left(H_n^2 + H_n^{(2)} - 2H_n\sum_{k=1}^{n}\frac{1}{k2^k} - 2\sum_{k=1}^{n}\frac{1}{k^2 2^k}\right).$$

*Proof.* If we differentiate both sides of (17) with respect to $s$, and put $s=0$ and $a_k=\frac{1}{k}$, we can easily obtain the desired result after some easy calculations. □

*Remark* 36. Applying our results given in this paper, we may obtain many other finite binomial and central binomial sum identities involving harmonic numbers, but for briefness we are satisfied with these examples.

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