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A Unified Treatment of Certain Classes of Combinatorial Identities

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Abstract

We propose and prove some general combinatorics formulas. Applying these formulas, we obtain some new binomial harmonic and harmonic Fibonacci and Lucas number identities. We also recover some known identities included in the works of Frontczak and Boyadzhiev.

1 Introduction

For $s \in \mathbb{C}$ and $n \in \mathbb{N}$, a generalized harmonic number $H_n^{(s)}$ of order s is defined by

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}, \quad \text{and} \quad H_n^{(1)} = H_n, \quad H_0^{(s)} = 0,$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ are the usual harmonic numbers; see [6, 12, 21]. Alternating or skew-harmonic numbers H_n^- are defined by

$$H_n^- = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k},$$

which is the partial sum of $\log 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$; see [8, 17, 20].

In 2009, Boyadzhiev [7] studied some binomial sums involving the harmonic numbers using Euler's transform and obtained the following identity:

$$\sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k} H_k = (a+b)^n H_n - \sum_{k=1}^{n} \frac{b^k}{k}(a+b)^{n-k},$$

which can be equivalently written, by setting $a/b = x$, as follows:

$$\sum_{k=1}^{n} \binom{n}{k} H_k x^k = (x+1)^n \left(H_n - \sum_{k=1}^{n} \frac{1}{k(x+1)^k} \right). \tag{1}$$

Batir, in a recent paper [3], offers many interesting finite sum identities with harmonic numbers, including (1). Frontczak [14, 15] obtained a complement of the identity given in (1) and derived

$$\sum_{k=1}^{n} \binom{n}{k} H_k x^k = (x+1)^n H_n - \sum_{k=0}^{n-1} H_{n-k}(x+1)^k, \tag{2}$$

which can be regarded as a complement of (1). Frontczak [15] also derived an analogue formula for the skew-harmonic numbers. Throughout this paper, we take

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

The Fibonacci sequence $(F_n)_{n\geq 0}$ is given by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$ it satisfies the recursion relation $F_n = F_{n-1} + F_{n-2}$. The Lucas sequence $(L_n)_{n\geq 0}$ satisfies the same recursion relation, that is, $L_n = L_{n-1} + L_{n-2}$ with the initial values $L_0 = 2$ and $L_1 = 1$, where $n \in \mathbb{N}$. The Binet formulas for the Fibonacci numbers F_n and Lucas numbers L_n state that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n.$$

In this article we develop a general formula which contains (1) and many others as special cases. Our results allow to establish many interesting new identities involving the harmonic numbers, Fibonacci numbers, and Lucas numbers. More precisely, for any sequence $(a_n)_{n\geq 1}$ in $\mathbb{C}$ and $x \in \mathbb{C}$ we consider the following general binomial sums:

$$\sum_{k=1}^{n} \binom{n}{k} \sum_{j=1}^{k} a_j x^k.$$

To demonstrate the usefulness of our formulas we give many applications. The following simple, but useful, identities will be required in our ongoing analysis:

$$\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}, \tag{3}$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \tag{4}$$

We continue with the following two lemmas.

Lemma 1. *Let $(a_n)_{n\geq 1}$ be any real or complex sequence. Then we have*

$$\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \binom{k}{j} a_j = \sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k}.$$

This lemma is not new, and a generalization of it can be found in Boyadzhiev [9, Eq. 3]. As stated there, Boyadzhiev attributed this lemma to 't Woord [24]. Also see equation (5.6) in [8]. It is worthy to note that Batır [2] rediscovered this lemma in 2017, and established many interesting combinatorial identities and series involving the harmonic numbers.

Lemma 2. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be any two sequences of real or complex numbers. Then we have*

$$\sum_{k=1}^{n} a_k \sum_{j=1}^{k} b_j = \sum_{p=0}^{n-1} \sum_{k=1}^{n-p} b_k a_{p+k}.$$

This lemma was proved in [4] and helps us to reduce some double sums to a single sum.

2 Main Results

In this section we collect our main results.

Theorem 3. *Let $(a_n)_{n\geq 1}$ be any sequence in $\mathbb{C}$. Then we have*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=1}^{k} a_j = \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n-1}{k} a_{k+1}.$$

Proof. By (4), we obtain

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\sum_{j=1}^{k}a_j = \sum_{k=1}^{n}(-1)^k\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)\sum_{j=1}^{k}a_j$$
$$= \sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k}\sum_{j=1}^{k}a_j + \sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\sum_{j=1}^{k}a_j, \tag{5}$$

where we have used $\binom{n}{n+1}=0$ and $\binom{n-1}{-1}=0$. On the other hand, we have

$$\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\sum_{j=1}^{k}a_j = \sum_{k=0}^{n-1}(-1)^{k-1}\binom{n-1}{k}\sum_{j=1}^{k+1}a_j$$
$$= \sum_{k=0}^{n-1}(-1)^{k-1}\binom{n-1}{k}\left(a_{k+1}+\sum_{j=1}^{k}a_j\right)$$
$$= \sum_{k=0}^{n-1}(-1)^{k-1}\binom{n-1}{k}a_{k+1} - \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\sum_{j=1}^{k}a_j. \tag{6}$$

Replacing (6) in (5), we complete the proof. □

Theorem 4. *Let $(a_n)_{n\geq 1}$ be any sequence in $\mathbb{C}$, $x\in\mathbb{C}$, and A_n be the nth partial sum of $(a_n)_{n\geq 1}$, that is, $A_n = a_1+a_2+\cdots+a_n$. Then*

$$\sum_{k=1}^{n}\binom{n}{k}A_k x^k = (1+x)^{n-1}\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}\binom{k}{j}a_{j+1}x^{j+1}\right)\frac{1}{(1+x)^k}. \tag{7}$$

Here the sum of 0 *terms is taken* 0*, that is,* $A_0 = 0$.

Proof. Let us define

$$g_n(x) = \sum_{k=1}^{n}(-1)^k\binom{n}{k}A_k x^k.$$

Clearly, we have by Theorem 3 that

$$g_n(x) = \sum_{k=1}^{n}(-1)^k\binom{n}{k}A_j x^j = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=1}^{k}\left(A_j x^j - A_{j-1}x^{j-1}\right)$$
$$= \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}\left(A_{k+1}x^{k+1} - A_k x^k\right).$$

Using $A_{k+1} = A_k + a_{k+1}$, we obtain

$$\begin{aligned}
g_n(x) &= \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}\left((A_k + a_{k+1})x^{k+1} - A_k x^k\right) \\
&= \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}A_k x^{k+1} + \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}a_{k+1}x^{k+1} \\
&\quad - \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}A_k x^k \\
&= (1-x)g_{n-1}(x) + \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}a_{k+1}x^{k+1}. \qquad (8)
\end{aligned}$$

Now we shall show the following expression by mathematical induction.

$$g_n(x) = (1-x)^{n-1}\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j}a_{j+1}x^{j+1}\right)\frac{1}{(1-x)^k}. \qquad (9)$$

It is easy to see that (9) is true for $n = 1$. We assume that (9) is true for n. Then by (8) and the induction hypothesis, we have

$$\begin{aligned}
g_{n+1}(x) &= (1-x)^n\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j}a_{j+1}x^{j+1}\right)\frac{1}{(1-x)^k} \\
&\quad + \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}a_{k+1}x^{k+1} \\
&= (1-x)^n\sum_{k=0}^{n}\left(\sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j}a_{j+1}x^{j+1}\right)\frac{1}{(1-x)^k},
\end{aligned}$$

which shows that (9) is valid for $n+1$, and the proof of (9) is complete. Replacing x by $-x$ in (9) we prove (7). Setting $x = 1$ in (7), we obtain the following conclusion: □

Corollary 5. *Let $(a_n)_{n\geq 1}$ be any sequence of complex numbers. Then we have*

$$\sum_{k=1}^{n}\binom{n}{k}A_k = 2^{n-1}\sum_{k=0}^{n-1}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}a_{j+1}. \qquad (10)$$

3 Applications

In this section we present many applications of our main theorems. Our first identity recovers (1).

Identity 6. *Let $n \in \mathbb{N}$ and $x \in \mathbb{C}$. Then*

$$\sum_{k=1}^{n} \binom{n}{k} H_k x^k = (1+x)^n \left(H_n - \sum_{k=1}^{n} \frac{1}{k(1+x)^k} \right). \tag{11}$$

Proof. Putting $a_k = \frac{1}{k}$ in Theorem 4, we get

$$\sum_{k=1}^{n} \binom{n}{k} H_k x^k = (1+x)^{n-1} \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} \binom{k}{j} \frac{x^{j+1}}{j+1} \right) \frac{1}{(1+x)^k}.$$

From (3), it is very easy to see that $\sum_{j=0}^{k} \binom{k}{j} \frac{x^{j+1}}{j+1} = \frac{(1+x)^{k+1}-1}{k+1}$, so that

$$\sum_{k=1}^{n} \binom{n}{k} H_k x^k = (1+x)^n \sum_{k=1}^{n} \frac{(1+x)^k - 1}{k(1+x)^k},$$

which is equivalent to (11). □

In the particular cases of $x = -1$ and $x = 1$ in (11), we get

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} H_k = -\frac{1}{n} \quad \text{and} \quad \sum_{k=1}^{n} \binom{n}{k} H_k = 2^n \sum_{k=1}^{n} \frac{2^k - 1}{k 2^k}. \tag{12}$$

Remark 7. The first identity here is very old and due to Euler [13]. The second identity is also well known and recently, it has been rediscovered by many authors; (see [3, 7, 11, 16, 22]).

Identity 8. *Let $n \in \mathbb{N}$ and $x \in \mathbb{C}$. Then we have*

$$\sum_{k=1}^{n} \binom{n}{k} H_k^2 x^k = (1+x)^n \left(H_n^2 - \sum_{k=1}^{n} \frac{H_n - 2H_k + H_{n-k}}{k(1+x)^k} - 2 \sum_{k=1}^{n} \frac{1}{k^2(1+x)^k} \right). \tag{13}$$

Proof. Letting $a_k = (H_k)^2 - (H_{k-1})^2$ in (7) and using the relation $H_{k+1} = H_k + \frac{1}{k+1}$, we get

$$\begin{aligned} \sum_{k=1}^{n} \binom{n}{k} H_k^2 x^k &= \sum_{k=1}^{n} \binom{n}{k} \sum_{j=1}^{k} (H_j^2 - H_{j-1}^2) x^k \\ &= (1+x)^{n-1} \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} \binom{k}{j} \frac{H_{j+1} + H_j}{j+1} x^{j+1} \right) (1+x)^{-k}. \end{aligned} \tag{14}$$

On the other hand, by (3) we have

$$\sum_{j=0}^{k}\binom{k}{j}\frac{H_{j+1}+H_j}{j+1}x^{j+1}=\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j+1}\left(2H_{j+1}-\frac{1}{j+1}\right)x^{j+1}$$
$$=\frac{1}{k+1}\sum_{j=1}^{k+1}\binom{k+1}{j}\left(2H_j-\frac{1}{j}\right)x^j$$
$$=\frac{2}{k+1}\sum_{j=1}^{k+1}\binom{k+1}{j}H_jx^j-\frac{1}{k+1}\sum_{j=1}^{k+1}\binom{k+1}{j}\frac{x^j}{j}. \tag{15}$$

By (11), it follows that

$$\sum_{j=1}^{k+1}\binom{k+1}{j}H_jx^j=(1+x)^{k+1}\sum_{j=1}^{k+1}\frac{(1+x)^j-1}{j(1+x)^j}.$$

Using this identity, we conclude from (14) and (15), and Lemma 1

$$\sum_{k=1}^{n}\binom{n}{k}H_k^2x^k=2(1+x)^n\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{(1+x)^j-1}{j(1+x)^j}$$
$$-(1+x)^n\sum_{k=1}^{n}\frac{1}{k(1+x)^k}\sum_{j=1}^{k}\binom{k}{j}\frac{x^j}{j}. \tag{16}$$

Applying Lemma 1, one easily gets

$$\sum_{j=1}^{k}\binom{k}{j}\frac{x^j}{j}=\sum_{j=1}^{k}\frac{(1+x)^j-1}{j}. \tag{17}$$

Substituting (17) in (16), we get, after some simplifications, that

$$\sum_{k=1}^{n}\binom{n}{k}H_k^2x^k=2(1+x)^n\sum_{k=1}^{n}\frac{H_k}{k}-2(1+x)^n\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j(1+x)^j}$$
$$-(1+x)^n\sum_{k=1}^{n}\frac{1}{k(1+x)^k}\sum_{j=1}^{k}\frac{(1+x)^j}{j}+(1+x)^n\sum_{k=1}^{n}\frac{H_k}{k(1+x)^k}. \tag{18}$$

Setting

$$a_k=\frac{(1+x)^{-k}}{k}\quad\text{and}\quad b_k=\frac{(1+x)^k}{k}$$

in Lemma 2, we obtain

$$\sum_{k=1}^{n}\frac{1}{k(1+x)^k}\sum_{j=1}^{k}\frac{(1+x)^j}{j}=H_n^{(2)}+\sum_{k=1}^{n}\frac{H_k+H_{n-k}-H_n}{k(1+x)^k}. \tag{19}$$

Hassani and Rahimpour [18] proved that for any double sequence $(a_{ij})_{i,j\geq 1}$ the following summation formula is valid.

$$\sum_{j,k=1}^{n} a_{jk} = \sum_{k=1}^{n}\sum_{j=1}^{k}(a_{jk}+a_{kj}) - \sum_{k=1}^{n} a_{kk}.$$

Setting $a_{ij} = \frac{(1+x)^{-j}}{ij}$ here, it follows that

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j(1+x)^j} = \sum_{k=1}^{n}\frac{H_n - H_k}{k(1+x)^k} + \sum_{k=1}^{n}\frac{1}{k^2(1+x)^k}. \tag{20}$$

Substituting (19) and (20) in (18) the proof is completed. □

Identity 9. *Putting $x=-1$ and $x=1$ in (13), we obtain*

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k^2 = \frac{H_n}{n} - \frac{2}{n^2}, \tag{21}$$

and

$$\sum_{k=1}^{n}\binom{n}{k}H_k^2 = 2^n\left(H_n^2 - \sum_{k=1}^{n}\frac{H_n - 2H_k + H_{n-k}}{k2^k} - 2\sum_{k=1}^{n}\frac{1}{k^2 2^k}\right). \tag{22}$$

Remark 10. The first identity here has previously been obtained by Wang [23] by the method of Riordan arrays, and rediscovered by Boyadzhiev [9].

Identity 11. *For $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k^3 = \frac{3H_n}{n^2} - \frac{H_n^2}{2n} + \frac{5H_n^{(2)}}{2n} - \frac{6}{n^3}.$$

Proof. Using $H_k = H_{k+1} - \frac{1}{k+1}$, we see that

$$H_{k+1}^3 - H_k^3 = \frac{3H_{k+1}^2}{k+1} - \frac{3H_{k+1}}{(k+1)^2} + \frac{1}{(k+1)^3}. \tag{23}$$

Applying Theorem 3 with $a_k = H_k^3 - H_{k-1}^3$ and using (23) we obtain

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k^3 &= \sum_{k=1}^{n}(-1)^k\binom{n}{k}\sum_{j=1}^{k}\left(H_j^3 - H_{j-1}^3\right)\\ &= \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}\left(H_{k+1}^3 - H_k^3\right)\\ &= \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}\left(\frac{3H_{k+1}^2}{k+1} - \frac{3H_{k+1}}{(k+1)^2} + \frac{1}{(k+1)^3}\right).\end{aligned}$$

Letting $k+1=k'$ and then dropping the prime, we get by (3)

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k^3 = \frac{3}{n}\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k^2 - \frac{3}{n}\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{H_k}{k} + \frac{1}{n}\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{1}{k^2}. \tag{24}$$

By Lemma 1 and the first identity in (12) we have

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{H_k}{k} = \sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}(-1)^j\binom{k}{j}H_j = -H_n^{(2)}. \tag{25}$$

We also have from [9, p. 5] or [8, Eq. (9.6)] that

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{1}{k^2} = -\frac{H_n^2+H_n^{(2)}}{2}. \tag{26}$$

Replacing (25) and (26) in (24) and using (21), we obtain the desired result. □

Remark 12. Using Theorem 3 it is possible to evaluate all the sums

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}(H_k)^m,$$

for $m \in \mathbb{N}$, but it requires lengthy calculations when m is large.

Identity 6 and Identity 8 enable us to establish new identities involving the product of the harmonic numbers and Fibonacci numbers, and harmonic numbers and Lucas numbers.

Identity 13. *Let $n \in \mathbb{N}$. Then*

$$\sum_{k=1}^{n}\binom{n}{k}H_kF_k = H_nF_{2n} - \sum_{k=1}^{n}\frac{F_{2n-2k}}{k},$$

$$\text{and} \quad \sum_{k=1}^{n}\binom{n}{k}H_kL_k = H_nL_{2n} - \sum_{k=1}^{n}\frac{L_{2n-2k}}{k}.$$

Proof. Let $P_n(x)$ denote the left-hand side of (11). Evaluate $P_n(x)$ at $x=\alpha$ and $x=\beta$, where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$. This gives

$$P_n(\alpha) = \alpha^{2n}H_n - \sum_{k=1}^{n}\frac{\alpha^{2n-2k}}{k} \quad \text{and} \quad P_n(\beta) = \beta^{2n}H_n - \sum_{k=1}^{n}\frac{\beta^{2n-2k}}{k},$$

where we have used $\alpha^2=\alpha+1$ and $\beta^2=\beta+1$. Now calculate $P_n(\alpha)\mp P_n(\beta)$, and then use the Binet formulas for F_n and L_n. □

Remark 14. Identity 13 can be easily derived from [10, Eq. 10] by letting $c_k = (-1)^{k-1}F_k$; see also [8, Thm. 6.1]. Furthermore, it can be compared with equations (9) and (10) in [14]. If in the first part of Identity 13 we put

$$\sum_{k=1}^{n} \frac{F_{2n-2k}}{k} = \sum_{k=1}^{n} F_{2k-1}H_{n-k},$$

then we obtain the result in [14]. Similarly, if in the second part of Identity 13 we put

$$\sum_{k=1}^{n} \frac{L_{2n-2k}}{k} = 2H_n + \sum_{k=1}^{n} L_{2k-1}H_{n-k},$$

then we obtain the result in [14].

Identity 15. *For $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_kF_{3k} = (-1)^n2^n\left(H_nF_n - \sum_{k=1}^{n}\frac{(-1)^kF_{n-k}}{k2^k}\right),$$

and

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_kL_{3k} = (-1)^n2^n\left(H_nL_n - \sum_{k=1}^{n}\frac{(-1)^kL_{n-k}}{k2^k}\right).$$

Proof. Letting $x = -\alpha^3$ and $x = -\beta^3$ in (11), and using $1 - \alpha^3 = -2\alpha$ and $1 - \beta^3 = -\beta$, respectively, we get

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k\alpha^3 = (-1)^n2^n\alpha^nH_n - 2^n\sum_{k=1}^{n}\frac{(-1)^{n-k}\alpha^{n-k}}{k2^k},$$

and

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k\beta^3 = (-1)^n2^n\beta^nH_n - 2^n\sum_{k=1}^{n}\frac{(-1)^{n-k}\beta^{n-k}}{k2^k}.$$

The combination of these two identities according to the Binet formula yields the desired results. □

Identity 16. *For $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{n}\binom{n}{k}H_kF_{3k} = 2^n\left(H_nF_{2n} - \sum_{k=1}^{n}\frac{F_{2n-2k}}{k2^k}\right),$$

$$\textit{and} \quad \sum_{k=1}^{n}\binom{n}{k}H_kL_{3k} = 2^n\left(H_nL_{2n} - \sum_{k=1}^{n}\frac{L_{2n-2k}}{k2^k}\right).$$

Proof. First we note that $1+\alpha^3 = 2\alpha^2$ and $1+\beta^3 = 2\beta^2$. Computing (11) at $x = \alpha^3$ and $x = \beta^3$ yields

$$\sum_{k=1}^{n} \binom{n}{k} H_k \alpha^{3k} = 2^n \alpha^{2n} H_n - \sum_{k=1}^{n} \frac{2^{n-k}\alpha^{2n-2k}}{k}$$

$$\text{and} \quad \sum_{k=1}^{n} \binom{n}{k} H_k \beta^{3k} = 2^n \beta^{2n} H_n - \sum_{k=1}^{n} \frac{2^{n-k}\beta^{2n-2k}}{k}.$$

Now the proof follows from combining these two sums according to the Binet formula for F_n and L_n. □

By the same method used in the proof of Identity 15 and Identity 16 we can prove the next two results by the help of Identity 8.

Identity 17. *For $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{n} \binom{n}{k} H_k^2 F_k = F_{2n} H_n^2 - \sum_{k=1}^{n} \frac{(H_n - 2H_k + H_{n-k}) F_{2n-2k}}{k} - 2\sum_{k=1}^{n} \frac{F_{2n-2k}}{k^2}$$

and

$$\sum_{k=1}^{n} \binom{n}{k} H_k^2 L_k = L_{2n} H_n^2 - \sum_{k=1}^{n} \frac{(H_n - 2H_k + H_{n-k}) L_{2n-2k}}{k} - 2\sum_{k=1}^{n} \frac{L_{2n-2k}}{k^2},$$

where we have used here $\sum_{k=1}^{n} \frac{H_k}{k} = \frac{H_n^2 + H_n^{(2)}}{2}$; see [1] and [2].

Identity 18. *Putting $x = \alpha^3$ in (13), and using $1+\alpha^3 = 2\alpha^2$, $1-\alpha^3 = -2\alpha$ and $1+\beta^3 = 2\beta^2$, $1-\beta^3 = -2\beta$, the evaluation of (13) yields for $n \in \mathbb{N}$*

$$\begin{aligned}&\sum_{k=1}^{n} \binom{n}{k} H_k^2 F_{3k} \\ &= 2^n F_{2n} H_n^2 - 2^n \sum_{k=1}^{n} \frac{(H_n - 2H_k + H_{n-k}) F_{2n-2k}}{k2^k} - 2^{n+1} \sum_{k=1}^{n} \frac{F_{2n-2k}}{k^2 2^k}\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=1}^{n} \binom{n}{k} H_k^2 L_{3k} \\ &= 2^n L_{2n} H_n^2 - 2^n \sum_{k=1}^{n} \frac{(H_n - 2H_k + H_{n-k}) L_{2n-2k}}{k2^k} - 2^{n+1} \sum_{k=1}^{n} \frac{L_{2n-2k}}{k^2 2^k}.\end{aligned}$$

The proof of the following identity can be done similar to the proof of Identity 15 by letting $x = -\alpha^3$ and $x = -\beta^3$ in Identity 8, respectively.

Identity 19. *For $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k} H_k^2 F_{3k} = (-1)^n 2^n \Bigg(F_n H_n^2 - 2\sum_{k=1}^{n} \frac{(-1)^k F_{n-k}}{k^2 2^k} - \sum_{k=1}^{n} \frac{(-1)^k (H_n - 2H_k + H_{n-k}) F_{n-k}}{k 2^k} \Bigg),$$

and

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k} H_k^2 L_{3k} = (-1)^n 2^n \Bigg(L_n H_n^2 - 2\sum_{k=1}^{n} \frac{(-1)^k L_{n-k}}{k^2 2^k} - \sum_{k=1}^{n} \frac{(-1)^k (H_n - 2H_k + H_{n-k}) L_{n-k}}{k 2^k} \Bigg).$$

Identity 20. *For $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{n} \binom{n}{k} H_k^- x^k = (1+x)^n \sum_{k=1}^{n} \frac{1-(1-x)^k}{k(1+x)^k}. \tag{27}$$

Proof. The proof follows from Theorem 4 by setting $a_k = \frac{(-1)^{k+1}}{k}$. □

Remark 21. Identity 20 can be easily obtained from [10, Cor. 1] by setting $a_k = (-1)^k H_k^-$; see also [8, Thm. 1.6]. Letting $x = -1$ and $x = 1$ in (27), respectively, we get

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k} H_k^- = \frac{1-2^n}{n} \quad \text{and} \quad \sum_{k=1}^{n} \binom{n}{k} H_k^- = 2^n \sum_{k=1}^{n} \frac{1}{k 2^k}.$$

The first identity here can be found in [8, Eq. (9.20)], and [9, Eq. (14)]. The second identity recovers (2).

The next identity is not new and can be found in entry (9.43) in [8].

Identity 22. *For integers $n \geq 1$ we have*

$$\sum_{k=1}^{n} \binom{n}{k} H_k^- F_{3k} = 2^n \sum_{k=1}^{n} \frac{F_{2n-2k}}{2^k k} - 2^n \sum_{k=1}^{n} \frac{(-1)^k F_{2n-k}}{k}$$

and $$\sum_{k=1}^{n} \binom{n}{k} H_k^- L_{3k} = 2^n \sum_{k=1}^{n} \frac{L_{2n-2k}}{2^k k} - 2^n \sum_{k=1}^{n} \frac{(-1)^k L_{2n-k}}{k}.$$

Proof. Clearly, we have $1+\alpha^3 = 2\alpha^2$, $1-\alpha^3 = -2\alpha$ and $1+\beta^3 = 2\beta^2$, $1-\beta^3 = -2\beta$. Thus, evaluating (27) at $x = \alpha^3$ yields

$$Q_n(\alpha) := \sum_{k=1}^{n} \binom{n}{k} H_k^- \alpha^{3k} = 2^n \sum_{k=1}^{n} \frac{\alpha^{2n-2k}}{k2^k} - \sum_{k=1}^{n} \frac{(-1)^k \alpha^{2n-k}}{k},$$

and similarly

$$Q_n(\beta) := \sum_{k=1}^{n} \binom{n}{k} H_k^- \beta^{3k} = 2^n \sum_{k=1}^{n} \frac{\beta^{2n-2k}}{k2^k} - 2^n \sum_{k=1}^{n} \frac{(-1)^k \beta^{2n-k}}{k}.$$

The evaluation of $Q_n(\alpha) \mp Q_n(\beta)$ leads to the desired results. □

The proof of the following result is similar to that of Identity 15.

Identity 23. *For $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} H_k^- F_{3k} = (-1)^n 2^n \left(\sum_{k=1}^{n} \frac{(-1)^k F_{n-k}}{2^k k} - \sum_{k=1}^{n} \frac{(-1)^k F_{n+k}}{k} \right),$$

and

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} H_k^- L_{3k} = (-1)^n 2^n \left(\sum_{k=1}^{n} \frac{(-1)^k L_{n-k}}{2^k k} - \sum_{k=1}^{n} \frac{(-1)^k L_{n+k}}{k} \right).$$

The proof of the next two identities can be done by proceeding as in the proof of Identity 8 by setting $a_k = \frac{1}{k^2}$ and $a_k = \left(H_k^-\right)^2 - \left(H_{k-1}^-\right)^2$ in Theorem 4. We omit the proofs.

Identity 24. *Letting $a_k = \frac{1}{k^2}$ in (7), we get after some simple computations*

$$\sum_{k=1}^{n} \binom{n}{k} H_k^{(2)} x^k = (1+x)^n \left(H_n^{(2)} - \sum_{k=1}^{n} \frac{H_n - H_{n-k}}{k(1+x)^k} \right). \tag{28}$$

Considering the two values of $x = 1$ and $x = -1$, we have

$$\sum_{k=1}^{n} \binom{n}{k} H_k^{(2)} = 2^n \left(H_n^{(2)} - \sum_{k=1}^{n} \frac{H_n - H_{n-k}}{k2^k} \right)$$

$$\text{and} \quad \sum_{k=1}^{n} (-1)^k \binom{n}{k} H_k^{(2)} = -\frac{H_n}{n}.$$

Identity 25. *Letting* $a_k = \frac{(-1)^{k+1}}{k}$ *in Theorem 4, we get for* $n \in \mathbb{N}$ *and* $x \in \mathbb{C}$

$$\sum_{k=1}^{n} \binom{n}{k} \left(H_k^-\right)^2 x^k = (1+x)^n \Bigg(2\sum_{k=1}^{n} \frac{H_k + H_{n-k} - H_n}{k} \left(\frac{1-x}{1+x}\right)^k$$
$$+ H_n^{(2)} + \sum_{k=1}^{n} \frac{H_n - H_{n-k}}{k(1+x)^k} - 2\sum_{k=1}^{n} \frac{1}{k} \left(\frac{1-x}{1+x}\right)^k \sum_{j=1}^{k} \frac{(1-x)^{-j}}{j} \Bigg). \tag{29}$$

Considering the two values of $x = 1$ *and* $x = -1$, *we get*

$$\sum_{k=1}^{n} \binom{n}{k} \left(H_k^-\right)^2 = 2^n \left(H_n^{(2)} + \sum_{k=1}^{n} \frac{H_n - H_{n-k}}{2^k k} - 2\sum_{k=1}^{n} \frac{1}{2^k k^2} \right),$$

$$\textit{and} \quad \sum_{k=1}^{n} (-1)^k \binom{n}{k} \left(H_k^-\right)^2 = \frac{H_n}{n} - \frac{2^{n+1}}{n} \sum_{k=1}^{n} \frac{1}{k 2^k}.$$

Identity 26. *Letting* $x = \alpha$ *and* $x = \beta$ *in (28) and combining the resulting expressions according to the Binet formula one gets*

$$\sum_{k=1}^{n} \binom{n}{k} H_k^{(2)} F_k = F_{2n} H_n^{(2)} - \sum_{k=1}^{n} \frac{(H_n - H_{n-k}) F_{2n-2k}}{k},$$

$$\textit{and} \quad \sum_{k=1}^{n} \binom{n}{k} H_k^{(2)} L_k = L_{2n} H_n^{(2)} - \sum_{k=1}^{n} \frac{(H_n - H_{n-k}) L_{2n-2k}}{k}.$$

Identity 27. *Letting* $x = \alpha^3$ *and* $x = \beta^3$ *in (28) and combining the results according to the Binet formula one gets*

$$\sum_{k=1}^{n} \binom{n}{k} H_k^{(2)} F_{3k} = 2^n H_n^{(2)} F_{2n} - 2^n \sum_{k=1}^{n} \frac{(H_n - H_{n-k}) F_{2n-2k}}{k 2^k}$$

$$\textit{and} \quad \sum_{k=1}^{n} \binom{n}{k} H_k^{(2)} L_{3k} = 2^n H_n^{(2)} L_{2n} - 2^n \sum_{k=1}^{n} \frac{(H_n - H_{n-k}) L_{2n-2k}}{k 2^k}.$$

Identity 28. *Setting* $x = -\alpha^3$ *and* $x = -\beta^3$ *in (28), respectively, and then combining the resulting identities according to the Binet formula, we get*

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} H_k^{(2)} F_{3k} = (-1)^n 2^n \left(H_n^{(2)} F_n - \sum_{k=1}^{n} \frac{(-1)^k (H_n - H_{n-k}) F_{n-k}}{k} \right),$$

and

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} H_k^{(2)} L_{3k} = (-1)^n 2^n \left(H_n^{(2)} L_n - \sum_{k=1}^{n} \frac{(-1)^k (H_n - H_{n-k}) L_{n-k}}{k} \right).$$

Identity 29. *Setting $x = \alpha^3$ and $x = \beta^3$ in (29), respectively, and then combining the resulting identities according to the Binet formula, we get*

$$\sum_{k=1}^{n}\binom{n}{k}\left(H_k^-\right)^2 F_{3k} = 2^n F_{2n} H_n^{(2)} - 2^{n+1}\sum_{k=1}^{n}\frac{(-1)^k}{k}\sum_{j=1}^{k}\frac{(-1)^j F_{2n+k-j}}{j}$$
$$+ 2^{n+1}\sum_{k=1}^{n}\frac{(-1)^k(H_k + H_{n-k} - H_n)F_{2n-k}}{k2^k} + 2^n\sum_{k=1}^{n}\frac{(H_n - H_k)F_{2n-k}}{k2^k}.$$

and

$$\sum_{k=1}^{n}\binom{n}{k}\left(H_k^-\right)^2 L_{3k} = 2^n L_{2n} H_n^{(2)} - 2^{n+1}\sum_{k=1}^{n}\frac{(-1)^k}{k}\sum_{j=1}^{k}\frac{(-1)^j L_{2n+k-j}}{j}$$
$$+ 2^{n+1}\sum_{k=1}^{n}\frac{(-1)^k(H_k + H_{n-k} - H_n)L_{2n-k}}{k2^k} + 2^n\sum_{k=1}^{n}\frac{(H_n - H_k)L_{2n-k}}{k2^k}.$$

The following identity has been posed as a problem by Ohtsuka in [19].

Identity 30. *For all positive integers n the following identity holds:*

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{1}{ij} = \sum_{1\le i\le j\le n}\frac{2^n - 2^{n-i}}{ij}.$$

Proof. Since

$$\sum_{1\le i\le j\le k}\frac{1}{ij} = 1 + \frac{1}{2}\left(1+\frac{1}{2}\right) + \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \cdots$$
$$+ \frac{1}{k}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right) = \sum_{k=1}^{n}\frac{H_k}{k}$$

and similarly

$$\sum_{1\le i\le j\le n}\frac{2^n - 2^{n-i}}{ij} = \sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{2^n - 2^{k-j}}{j},$$

it suffices to show

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{k=1}^{n}\frac{H_k}{k} = \sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{2^n - 2^{k-j}}{j}.$$

Substituting $a_k = \frac{H_k}{k}$ in (10), we get by the help of (3)

$$\sum_{k=1}^{n} \binom{n}{k} \sum_{j=1}^{k} \frac{H_j}{j} = 2^{n-1} + 2^{n-1} \sum_{k=1}^{n-1} \frac{1}{2^k} \sum_{j=1}^{k} \binom{k}{j} \frac{H_{j+1}}{j+1}$$
$$= 2^{n-1} + 2^n \sum_{k=2}^{n} \frac{1}{k2^k} \sum_{j=1}^{k} \binom{k}{j} H_j.$$

Using the second identity in (12), we find

$$\sum_{k=1}^{n} \binom{n}{k} \sum_{j=1}^{k} \frac{H_j}{j} = 2^{n-1} + 2^n \sum_{k=2}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{2^j - 1}{j2^j}$$
$$= 2^n \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{2^j - 1}{j2^j} = \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{2^n - 2^{n-j}}{j},$$

as claimed. □

As usual let $S(n,k)$ denote the Stirling numbers of the second kind, that is

$$S(n,k) = \frac{(-1)^k}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n.$$

The following identity provides a new proof, based on Theorem 3, of a well-known formula for the sum of powers of consecutive positive integers; see, for example, [5].

Identity 31. *For $m, n \in \mathbb{N}$ the following identity holds:*

$$\sum_{k=1}^{n} k^m = \sum_{k=1}^{n} \binom{n}{k} (k-1)! S(m+1,k).$$

Proof. Putting $a_k = k^m$ in Theorem 3, we get

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} \sum_{k=1}^{n} k^m = \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n-1}{k} (k+1)^m$$
$$= \frac{1}{n} \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^{m+1} = (-1)^n (n-1)! S(m+1,n).$$

Employing the well-known binomial inversion formula the proof is completed. □

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