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From Fibonacci to Robbins: Series Reversion and Hankel Transforms

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Abstract

The Robbins numbers A_n have an important place in the study of plane partitions and in the study of alternating sign matrices. A simple closed formula exists for these numbers, but its derivation entails quite sophisticated machinery. Building on this basis, we study the Robbins numbers from a more elementary standpoint, based on series reversion and Hankel transforms. We show how a transformation pipeline can lead from the Fibonacci numbers to the Robbins numbers. We employ the language of Riordan arrays to carry out many of the transformations of the generating functions that we use. We establish links between the revert transforms under discussion and certain scaled moment sequences of a family of continuous Hahn polynomials. Finally we show that a family of quasi-Fibonacci polynomials of 7th order play a fundamental role in this theory.

1 Preliminaries

This note concerns the Robbins numbers A_n, which begin

$$1, 1, 2, 7, 429, 7436, \ldots.$$

They can be defined by the summation [1, 5]

$$A_n = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

These numbers count $n \times n$ alternating matrices, as well as the number of descending plane partitions whose parts do not exceed n.

We shall approach the study of these numbers using methods that include the reversion of power series, the sequence Hankel transform [11], and techniques from the area of Riordan arrays (principally the fundamental theorem of Riordan arrays).

Most of the generating functions $g(x) = \sum_{n=0}^{\infty} a_n x^n$ that we shall use will have integer coefficients $a_n \in \mathbb{C}$, though some can be allowed to have coefficients in $\mathbb{C}$.

For a generating function $g(x) = \sum_{n=0} a_n x^n$ with $a_0 \neq 0$, we define its *revert transform* to be the power series

$$r(g)(x) = \frac{1}{x}(xg(x))^{\langle -1 \rangle},$$

where $(xg(x))^{\langle -1 \rangle}$ is the compositional inverse of $xg(x)$. By Lagrange inversion, we have that $r(g)(x) = \sum_{n=0}^{\infty} \alpha_n x^n$ where

$$\alpha_n = \frac{1}{n+1}[x^n]\frac{1}{g(x)^{n+1}}.$$

In these circumstances, we also say that the sequence α_n is the revert transform of the sequence a_n.

The (sequence) *Hankel transform* of a sequence a_n is the sequence h_n with elements $h_n = |a_{i+j}|_{0 \leq i,j \leq n}$. If the generating function of a_n is expressible as a Jacobi continued fraction of the form

$$\cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}},$$

then the Hankel transform of a_n is given by [9, 10] the Heilermann formula

$$h_n = \mu_0^{n+1} \beta_1^n \beta_2^{n-1} \cdots \beta_{n-1}^2 \beta_n.$$

Such a generating function will be denoted by $\mathcal{J}(\alpha_0, \alpha_1, \ldots; \beta_1, \beta_2, \ldots)$. By the above result, the Hankel transform is independent of the sequence $\alpha_0, \alpha_1, \ldots$. In the sequel, we use the flexibility that this confers on us to transform non-integral sequences with desired Hankel transforms to ones with integer elements with the same Hankel transforms.

Given a sequence a_n with generating function $g(x)$, then the following sequences will have the same Hankel transform:

1. $(-1)^n a_n$, with generating function $g(-x)$,
2. The r-th binomial transform $\sum_{=0}^{n} \binom{n}{k} r^{n-k} a_k$ of a_n, with generating function $\frac{1}{1-rx} g\left(\frac{x}{1-rx}\right)$,
3. The r-INVERT transform or INVERT(r) transform of a_n, with generating function $\frac{g(x)}{1-rxg(x)}$.

Thus any combination of these, applied to a sequence, will leave the Hankel transform unchanged. A consequence of this is the following.

Proposition 1. *Given a sequence a_n with $a_0 \neq 0$ with generating function $g(x)$, the Hankel transform of the revert transform of a_n is equal to that of the revert transform of sequences with generating functions of the form*

$$\frac{\frac{1}{1-rx} g\left(\frac{x}{1-rx}\right)}{1 - sx\frac{1}{1-rx} g\left(\frac{x}{1-rx}\right)},$$

and similarly for $g(-x)$.

Proof. This follows because the revert transform of a binomial transform is given by an INVERT transform, and the revert transform of an INVERT transform is given by a binomial transform. □

Note that

$$\frac{\frac{1}{1-rx} g\left(\frac{x}{1-rx}\right)}{1 - sx\frac{1}{1-rx} g\left(\frac{x}{1-rx}\right)} = \frac{g\left(\frac{x}{1-rx}\right)}{1 - rx - sxg\left(\frac{x}{1-rx}\right)}.$$

A *Riordan array* is defined by a pair of power series $(g(x), f(x))$ [4, 13] where

$$g(x) = g_0 + g_1 x + g_2 x^2 + \cdots, \quad g_1 \neq 0,$$

and

$$f(x) = f_1 x + f_2 x^2 + f_3 x^3 + \cdots \quad f_0 = 0, f_1 \neq 0.$$

We represent this pair by the matrix $(a_{n,k})_{0 \le n,k \le \infty}$ where

$$a_{n,k} = [x^n] g(x) f(x)^k.$$

Here, the operator $[x^n]$ denotes the functional that extracts the coefficient of x^n from a power series.

Note that we find it convenient to index all matrices in this note so that the top left position of the matrix is the $(0, 0)$-element. This facilitates the use of bivariate generating functions for certain matrices.

The set of pairs $(g(x), f(x))$ defined above is in fact a group, with product

$$(g(x), f(x)) \cdot (u(x), v(x)) = (g(x)u(f(x)), v(f(x)),$$

and inverse

$$(g(x), f(x))^{-1} = \left(\frac{1}{g(\bar{f})}, \bar{f}\right),$$

where $\bar{f} = f^{\langle -1 \rangle}$ is the compositional inverse of f.

The bivariate generating function of the Riordan array $(g(x), f(x))$ is given by $\frac{g(x)}{1-yf(x)}$.

There is an action of the group of Riordan arrays on power series given by

$$(g, f) \cdot h = g.h(f)$$

or using a "dummy variable",

$$(g(x), f(x)) \cdot h(x) = g(x)h(f(x)).$$

This action is called the fundamental theorem of Riordan arrays.

Three mappings on the space of Riordan arrays to the space of semi-infinite matrices will be of significance for us.

The first is that of *rectification.* Given a Riordan array $R = (g(x), f(x))$, the rectification of R is the matrix with general (n, k)-th term $[x^{n+k}]g(x)f(x)^k = [x^n]g(x)(f(x)/x)^k$.

Example 2. The binomial matrix $\mathbf{B}$ (Pascal's triangle) is the matrix $\left(\binom{n}{k}\right)$, corresponding to the Riordan array $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$. This has generating function

$$B(x, y) = \frac{\frac{1}{1-x}}{1 - y\frac{x}{1-x}} = \frac{1}{1 - x - xy}.$$

Then the rectification of $\mathbf{B}$ is the matrix defined by $\left(\frac{1}{1-x}, \frac{1}{1-x}\right)$. This has its generating function given by $\frac{1}{1-x-y}$. This is the matrix $\left(\binom{n+k}{k}\right)$ which begins

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 6 & 10 & 15 & 21 & 28 \\ 1 & 4 & 10 & 20 & 35 & 56 & 84 \\ 1 & 5 & 15 & 35 & 70 & 126 & 210 \\ 1 & 6 & 21 & 56 & 126 & 252 & 462 \\ 1 & 7 & 28 & 84 & 210 & 462 & 924 \end{pmatrix}. \tag{1}$$

The second is a transformation that we call the *square symmetrization* of a Riordan array. If $B(x, y)$ is the bivariate generating function of the Riordan array $(g(x), f(x))$, we define its square symmetrization to be the matrix with generating function

$$B\left(xy, \frac{1}{x}\right) + B\left(xy, \frac{1}{y}\right) - g(xy).$$

Example 3. The square symmetrization of the binomial matrix $\mathbf{B}$ has generating function

$$\frac{1 - 3xy + x^2y^2}{(1 - xy)(1 - x - xy)(1 - y - xy)}.$$

This new matrix begins

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 3 & 6 & 10 & 15 \\ 1 & 3 & 3 & 1 & 4 & 10 & 20 \\ 1 & 4 & 6 & 4 & 1 & 5 & 15 \\ 1 & 5 & 10 & 10 & 5 & 1 & 6 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix}. \tag{2}$$

The third operation is also a symmetrization process, defined as follows. Given a Riordan array $(g(x), f(x))$, we define its *skew symmetrization* to be the matrix with generating function

$$\frac{g(xy)}{1-xf(xy)} + \frac{g(xy)}{1-yf(xy)} - g(xy).$$

Example 4. The generating function of the skew symmetrization of the binomial matrix is given by

$$\frac{1-2xy+x^2y^2-x^3y^3}{(1-xy)(1-xy-xy^2)(1-xy-x^2y)}.$$

This matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 3 & 1 & 4 & 6 \\ 0 & 0 & 0 & 3 & 4 & 1 & 5 \\ 0 & 0 & 0 & 1 & 6 & 5 & 1 \end{pmatrix}. \tag{3}$$

For a semi-infinite matrix $(a_{n,k})_{0\le n,k\le\infty}$ we define its *principal minor sequence* to be the sequence m_n where $m_n = |a_{i,j}|_{0\le i,j\le n}$. The principal minor sequences of the matrices (1), (2) and (3) above are, respectively,

$$1, 1, 1, \ldots,$$

$$1, 0, -1, 8, -71, 656, -4816, 1920, 168784, 43920880, -3315147449, \ldots,$$

and

$$1, 1, 0, -4, 7, 32, -572, 4084, -9084, -33337, -988692, \ldots.$$

We shall refer to sequences by their *On-Line Encyclopedia of Integer Sequences* [14, 15] A-number designation, where known. Thus the binomial matrix is A07318, the sequence $1, 0, -1, 8, \ldots$ above is A292865 and the Robbins numbers A_n are A005130.

2 Principal results: Riordan arrays

A primary result in the theory of the Robbins numbers A_n [5] is that the sequence A_{n+1} coincides with the principal minor sequence of the matrix

$$\left(\binom{n+k}{k} - \delta_{n,k-1}\right).$$

This matrix begins

$$\left(\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 6 & 10 & 15 & 21 & 28 \\ 1 & 4 & 10 & 20 & 35 & 56 & 84 \\ 1 & 5 & 15 & 35 & 70 & 126 & 210 \\ 1 & 6 & 21 & 56 & 126 & 252 & 462 \\ 1 & 7 & 28 & 84 & 210 & 462 & 924 \end{array}\right) - \left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

or

$$\left(\begin{array}{ccccccc} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 4 & 5 & 6 & 7 \\ 1 & 3 & 6 & 9 & 15 & 21 & 28 \\ 1 & 4 & 10 & 20 & 34 & 56 & 84 \\ 1 & 5 & 15 & 35 & 70 & 125 & 210 \\ 1 & 6 & 21 & 56 & 126 & 252 & 461 \\ 1 & 7 & 28 & 84 & 210 & 462 & 924 \end{array}\right).$$

As we have seen, the generating function of this matrix is given by

$$f(x,y) = \frac{1}{1-x-y} - \frac{1}{1-xy}.$$

We now have the following result.

Proposition 5. *The Robbins numbers A_{n+1} are given by the principal minors of the matrix obtained by the square symmetrization of the Riordan array*

$$\left(\frac{1}{(1-x)\sqrt{1-4x}}, xc(x)\right),$$

where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ A000108.

Proof. The bivariate generating function of the Riordan array $R = \left(\frac{1}{(1-x)\sqrt{1-4x}}, xc(x)\right)$ is given by

$$R(x,y) = \frac{2}{(1-x)(2-y+y\sqrt{1-4x})\sqrt{1-4x}}.$$

The square symmetrization of this array will then have its generating function given by

$$R\left(xy, \frac{1}{x}\right) + R\left(xy, \frac{1}{y}\right) - \frac{1}{(1-xy)\sqrt{1-4xy}} = \frac{1}{(1-xy)(1-x-y)}.$$

Now by the fundamental theorem of Riordan arrays, we have that

$$\left(\frac{1}{(1-y)\left(\frac{1}{1-y}-y\right)}, y\right) \cdot f(x,y) = \frac{1}{(1-y)\left(\frac{1}{1-y}-y\right)} f(x,y) = \frac{1}{(1-xy)(1-x-y)}.$$

In terms of matrices, this amounts to multiplying the matrix $\left(\binom{n+k}{k} - \delta_{n,k}\right)$ by a Riordan array, which in this case is unipotent. Thus both matrices have the same principal minor sequence. □

Corollary 6. *We have*

$$A_{n+1} = \left| \sum_{j=0}^{n} \binom{2j-n+k}{j-n+k} \right|_{0 \le i,j \le n}.$$

Proof. This follows since the general (n,k) element of the Riordan array $\left(\frac{1}{(1-x)\sqrt{1-4x}}, xc(x)\right)$ is given by

$$\sum_{j=0}^{n} \binom{2j-k}{j-k}.$$

□

In fact, there is a more elementary Riordan array whose square symmetrization yields the Robbins numbers. Thus we have the following result.

Proposition 7. *The principal minor sequence of the square symmetrization of the Riordan array* $\left(\frac{1}{1+x+x^2}, \frac{x}{1+x}\right)$ *is given by the signed Robbins numbers* $(-1)^{\binom{n+1}{2}} A_{n+1}$.

Proof. We know that the matrix $\left(\binom{n+k}{k} - \delta_{n,k-1}\right)$ has a principal minor sequence equal to A_{n+1}. This matrix has generating function $\frac{1}{1-x-y} - \frac{y}{1-xy}$. Thus the matrix $(-1)^k \left(\binom{n+k}{k} - \delta_{n,k-1}\right)$ with generating function $\frac{1}{1-x+y} + \frac{y}{1+xy}$ has a principal minor sequence given by $(-1)^{\binom{n+1}{2}} A_{n+1}$ by the alternating nature property of determinants. The matrix with generating function

$$\left(\frac{1}{1-y}\frac{1}{1+\frac{y^2}{1-y}}, y\right) \cdot \frac{1}{1-x-y} - \frac{y}{1-xy}$$

will have the same principal minor sequence. This gives us the matrix generating function

$$\frac{1+y}{(1-y)(1+y+y^2)}\left(\frac{1}{1-x+y} + \frac{y}{1+xy}\right) = \frac{(1+y)}{(1-y)(1-x-y)(1+xy)}.$$

Now the determinant of a matrix is equal to that of its transpose, so in the last generating function we can exchange x and y without affecting the principal minor sequence. We now operate on the result of this exchange with the unipotent Riordan array $\left(\frac{1-2x}{(1-x)^2}, \frac{x}{1-x}\right)$ to get

$$\left(\frac{1-2x}{(1-x)^2}, \frac{x}{1-x}\right) \cdot \frac{(1+x)}{(1-x)(1-y-x)(1+xy)} = \frac{1}{(1-x-xy)(1-y-xy)}.$$

Now turning to the Riordan array $\left(\frac{1}{1+x+x^2}, \frac{x}{1+x}\right)$, we have that its generating function is given by

$$g(x,y) = \frac{1+x}{(1+x+x^2)(1+x-xy)}.$$

Then we have that

$$g\left(xy, \frac{1}{x}\right) + g\left(xy, \frac{1}{y}\right) + \frac{1}{1+xy+x^2y^2} = \frac{1}{(1-x-xy)(1-y-xy)}.$$

This shows that the principal minor sequence of the square symmetrization of $\left(\frac{1}{1+x+x^2}, \frac{x}{1+x}\right)$ is given by $(-1)^{\binom{n+1}{2}} A_{n+1}$. □

Corollary 8. *Let $(a_{n,k})$ be the matrix of the Riordan array $\left(\frac{1}{1+x+x^2}, \frac{x}{1+x}\right)$. Then the principal minor sequence of the matrix $((-1)^k a_{n,k})$ gives the Robbins numbers A_{n+1}.*

3 Principal results: series reversion and Hankel transforms

In this section, we continue to use a generating function approach. The Hankel matrix $(a_{i+j})_{0 \le i,j \le \infty}$ corresponding to a sequence a_n with generating function $A(x)$ has its generating function given by

$$\frac{xA(x) - yA(y)}{x-y}.$$

The Hankel transform is then just the principal minor sequence of this matrix. An application of the fundamental theorem of Riordan arrays now allows us to conclude that this principal minor sequence is again that of the matrix with generating function

$$\frac{x-y}{xr(A)(x) - yr(A)(y)}.$$

This now allows us to state the following proposition.

Proposition 9. *The Hankel transform of the revert transform of the sequence with generating function*

$$\frac{1-x}{1-2x-x^2+x^3}$$

is given by the signed Robbin numbers $(-1)^{\binom{n+1}{2}} A_{n+1}$.

Proof. We let $g(x) = \frac{1-x}{1-2x-x^2+x^3}$ and we form the bivariate generating function

$$\frac{x-y}{xg(x)-yg(y)} = \frac{(1-2x-x^2+x^3)(1-2y-y^2+y^3)}{1-y-(1-3y+y^2)x+y(y-1)x^2}.$$

Without affecting principal minor sequences, we can use the fundamental theorem of Riordan arrays to arrive at the matrix with generating function

$$\frac{1}{1-y-(1-3y+y^2)x+y(y-1)x^2}.$$

But this is the generating function of the square symmetrization of $\left(\frac{1}{1+x+x^2}, \frac{x}{1+x}\right)$. The result is thus proven. □

Corollary 10. *Let* $g(x) = \frac{1-x}{1-2x-x^2+x^3}$. *Then the Robbins numbers* A_{n+1} *are given by the Hankel transform of the revert transform of*

$$g(ix) = \frac{1+3x^2+x^4}{1+6x^2+5x^4+x^6} + \frac{ix}{1+6x^2+5x^4+x^6}, \quad i = \sqrt{-1}.$$

Corollary 11. *Let* $g(x) = \frac{1-x}{1-x-2x^2+x^3}$. *Then the Robbins numbers* A_{n+1} *are given by the Hankel transform of the revert transform of*

$$g(ix) = \frac{1+3x^2+x^4}{1+5x^2+6x^4+x^6} + \frac{ix}{1+5x^2+6x^4+x^6}, \quad i = \sqrt{-1}.$$

Note that in this case, we have

$$g(ix) = \frac{M_{P_4}(x^2)}{M_{P_6}(x^2)} + \frac{ix}{M_{P_6}(x^2)},$$

where $M_{P_r}(x)$ is the matching polynomial of the path graph P_r.

Because of the properties of the binomial and the INVERT transforms, we can now generate an infinity of generating functions with the same Hankel attributes as $g(x) = \frac{1-x}{1-2x-x^2+x^3}$. Such generating functions include

$$\frac{1+x}{1+3x-x^3}, \frac{1-x}{1-3x+x^3}, \frac{1-x}{1-x-2x^2+x^3}, \frac{1+x}{1+2x-x^2-x^3}.$$

Similarly, any element of the family

$$g(x;r) = \frac{1-x}{1-rx-(3-r)x^2+x^3}$$

will lead to the Robbins numbers. In terms of Riordan arrays, we have the following proposition.

Proposition 12. *Let $a_n(r)$ and $b_n(r)$ be, respectively, the initial column elements and the row sum elements of the inverse Riordan array $(g(x;r), xg(x;r))^{-1}$. Then the Hankel transform of both these sequences is given by the signed Robbins numbers $(-1)^{\binom{n+1}{2}} A_{n+1}$.*

Proof. The matrix $(g(x;r), xg(x;r))$ is a Bell matrix. A property of such matrices $(g(x), xg(x))$ is that the initial column of the inverse is precisely the revert transform of the expansion of $g(x)$, while the row sums are given by the INVERT transform of this revert transform. Thus both sequences have the required Hankel transform. □

We can in fact go further: any of the row polynomials of the inverse matrix will have the same Hankel transform, namely the signed Robbins numbers.

4 From Fibonacci to Robbins

In this section we describe a transformation pipeline that begins with the Fibonacci numbers F_{n+1} A000045 with generating function $\mathcal{G}(F_{n+1})(x) = \frac{1}{1-x-x^2}$ and that concludes with the Robbins numbers. The steps of this pipeline are as follows.

- Apply the binomial transform to $\mathcal{G}(F_{n+1})(x)$ to obtain the generating function $g_0(x) = \frac{1-x}{1-3x+x^2}$.
- Apply the transform $\mathcal{T} : \gamma(x) \to \frac{\gamma(x)}{x+(1-x)\gamma(x)}$ to obtain $\frac{1-x}{1-x-2x^2+x^3}$.
- Apply the revert transform, to obtain the sequence

$$\frac{1}{n+1}[x^n]\left(\frac{1-x-2x^2+x^3}{1-x}\right)^{n+1}.$$

- Apply the Hankel transform, to obtain the signed Robbins numbers $(-1)^{\binom{n+1}{2}} A_{n+1}$.

We can describe the transformation $\mathcal{T}$ as follows.

Proposition 13. *Let $g(x) = g_0 + g_1 x + g_2 x^2 + \dots$ where $g_0 \neq 0$. Then $\mathcal{T}(g)(x)$ is the generating function of the diagonal sums of the Riordan array*

$$\left(1, \frac{g(x)-1}{g(x)}\right) = \left(1, 1 - \frac{1}{g(x)}\right).$$

Proof. We have

$$[x^n]\frac{g(x)}{x+(1-x)g(x)} = [x^n]\frac{1}{1-x\left(\frac{g(x)-1}{g(x)}\right)}$$
$$= [x^n]\sum_{j=0}^{\infty} x^j \left(\frac{g(x)-1}{g(x)}\right)^j$$
$$= \sum_{j=0}^{n}[x^{n-j}]\left(\frac{g(x)-1}{g(x)}\right)^j.$$

□

An application of this to the expansion t_n of $\frac{1-x}{1-x-2x^2+x^3}$ gives us

$$t_n = \sum_{k=0}^{\frac{n}{2}}\sum_{j=0}^{k}\binom{k}{j}\binom{n-k-j-1}{n-k-j}(-1)^j 2^{k-j}.$$

This sequence begins

$$1, 0, 2, 1, 5, 5, 14, 19, 42, 66, 131, \ldots.$$

This is A052547.

Proposition 14. *The Hankel transforms of the revert transform of the $\mathcal{T}$ transform of F_{n+2} and F_{2n+1} are given by the signed Robbins numbers $(-1)^{\binom{n+1}{2}}A_{n+1}$.*

Proof. The $\mathcal{T}$ transform of the generating function $\frac{1+x}{1-x-x^2}$ of F_{n+2} is given by $\frac{1+x}{1+x-2x^2-x^3}$, while the $\mathcal{T}$ transform of the generating function $\frac{1-x}{1-3x+x^2}$ of F_{2n+1} is given by $\frac{1-x}{1-x-2x^2+x^3}$. □

Due to Heilermann's formula for the Hankel transform of sequences whose generating function can be expressed as a Jacobi continued fraction, we know that the Hankel transform depends only on the coefficients of x^2 in the continued fraction expression. This property subsists through the reversion process.

Example 15. We have

$$\frac{1-x}{1-x-2x^2+x^3} = \cfrac{1}{1-\cfrac{2x^2}{1-\frac{1}{2}x-\cfrac{\frac{1}{4}x^2}{1-\cfrac{1}{2}x}}},$$

and

$$\frac{1-x}{1-2x-x^2+x^3} = \cfrac{1}{1-x-\cfrac{2x^2}{1-\frac{1}{2}x-\cfrac{\frac{1}{4}x^2}{1-\cfrac{1}{2}x}}}.$$

In general, any sequence whose generating function has a Jacobi continued fraction expansion with the sequence $-2, -\frac{1}{4}, 0, 0, 0, \ldots$ of coefficients of x^2 will have a revert transform whose Hankel transform will be the signed Robbins numbers. To see the Heilermann formula in action, we look at the example of $\frac{1-x}{1-2x-x^2+x^3}$. The revert transform of the expansion of this will have a generating function with continued fraction expression that begins

$$\cfrac{1}{1+x+\cfrac{2x^2}{1+\frac{1}{2}x+\cfrac{\frac{7}{2}x^2}{1+\frac{1}{2}x+\cfrac{\frac{12}{7}x^2}{1+\frac{1}{2}x+\cfrac{\frac{143}{84}x^2}{1+\cdots}}}}}.$$

Thus we get

$$1, 1^2 \cdot (-2) = -2, 1^3 \cdot (-2)^2 \cdot \left(-\frac{7}{4}\right)^2 = -7, 1^4 \cdot (-2)^3 \cdot \left(-\frac{7}{4}\right)^2 \cdot \left(-\frac{12}{7}\right) = 42, \ldots.$$

5 The sequences U_n and V_n

There are a number of sequences closely associated to the Robbins numbers A_n. Two such sequences are [7]

$$U_n : 1, 2, 11, 170, 7429, 920460, \ldots \quad \text{A051255},$$

and

$$V_n : 1, 3, 26, 646, 45885, 9304650, \ldots \quad \text{A005156}.$$

The sequence U_n counts the number of cyclically symmetric transpose complement plane partitions whose Ferrers diagrams fit in an $n \times n \times n$ box. The sequence V_n counts the number of $(2n+1) \times (2n+1)$ alternating sign matrices that are invariant under vertical reflection. The sequence V_n is the Hankel transform of the revert transform of a sequence with a rational generating function, namely $\frac{1}{(1+x)^3}$. Thus this sequence is susceptible to the same type of analysis as A_n above.

The sequence U_n is the Hankel transform of the revert transform of the sequence with generating function $\frac{1}{c(x)}$, which is not rational. Thus we need another approach to its study. We have the following result.

Proposition 16. *The sequence U_n is the principal minor sequence of the skew symmetrization of the Riordan array*

$$\left(\frac{1}{\sqrt{1-4x}}, xc(x)^3\right).$$

Proof. The generating function of the skew symmetrization of $\left(\frac{1}{\sqrt{1-4x}}, xc(x)^3\right)$ is given by

$$\frac{g(xy)}{1-xf(xy)} + \frac{g(xy)}{1-yf(xy)} - g(xy),$$

where

$$g(x) = \frac{1}{\sqrt{1-4x}}, \quad f(x) = xc(x)^3.$$

This simplifies to the expression

$$\frac{1-xy}{1-(y+3)xy-x^2y}.$$

This is the generating function of the matrix $\left(\binom{i+j}{2i-j}\right)_{0\le i,j\le\infty}$, whose principal minor sequence is known to equal U_n. □

6 Orthogonal polynomials

A main result of this note is to exhibit the signed Robbins numbers $(-1)^{\binom{n+1}{2}}A_{n+1}$ as the Hankel transform of a sequence with integer coefficients. A consequence of this is that we can also exhibit the Robbins numbers A_{n+1} as the Hankel transform of a sequence with Gaussian integer coefficients. We can find in the literature expressions for A_n as a Hankel transform of sequences with rational coefficients. For instance, the Robbins numbers A_n are related to the Hankel transform of the moments of the continuous Hahn polynomials [6] $p_n\left(\frac{x}{6}; \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, from which we deduce the formula

$$A_n = \prod_{k=0}^{n-1} \frac{k!(3k+1)!}{(2k)!(2k+1)!}.$$

The polynomials $p_n\left(\frac{x}{6}; \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ begin

$$1, \frac{x}{3}, \frac{3x^2-8}{36}, \frac{5x(x^2-12)}{324}, \ldots$$

with coefficient array that begins

$$\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
-\frac{4}{9} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & -\frac{10}{9} & 0 & \frac{5}{54} & 0 & 0 & 0 \\
\frac{80}{27} & 0 & -\frac{95}{54} & 0 & \frac{35}{648} & 0 & 0 \\
0 & \frac{1288}{81} & 0 & -\frac{1085}{486} & 0 & \frac{7}{216} & 0 \\
-\frac{44800}{729} & 0 & \frac{11228}{243} & 0 & -\frac{805}{324} & 0 & \frac{77}{3888}
\end{array}\right).$$

The inverse of this coefficient matrix begins

$$\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 \\
\frac{8}{3} & 0 & 6 & 0 & 0 & 0 & 0 \\
0 & 36 & 0 & \frac{54}{5} & 0 & 0 & 0 \\
32 & 0 & \frac{1368}{7} & 0 & \frac{648}{35} & 0 & 0 \\
0 & 1008 & 0 & 744 & 0 & \frac{216}{7} & 0 \\
896 & 0 & \frac{73632}{7} & 0 & \frac{178848}{77} & 0 & \frac{3888}{77}
\end{array}\right).$$

The (scaled) moments are thus given by the sequence e_n that begins

$$1, 0, \frac{8}{3}, 0, 32, 0, 896, 0, \frac{414208}{9}, 0, 3782656, 0, \ldots.$$

We find that the Hankel transform h_n of this sequence begins

$$1, \frac{8}{3}, \frac{1792}{27}, \frac{917504}{27}, \frac{153545080832}{243}, \frac{1308154954262773760 0}{19683}, \ldots.$$

The theory [6] now tells us that this is the sequence

$$h_n = A_{n+1} \left(\frac{4}{3}\right)^{\binom{n+1}{2}} \prod_{i=0}^{n} i!^2.$$

Thus we have

$$A_{n+1} = \frac{h_n}{\left(\frac{4}{3}\right)^{\binom{n+1}{2}} \prod_{i=0}^{n} i!^2}.$$

In fact, we have that

$$e_n = \frac{9 \cdot 6^n}{2\sqrt{3}\pi} \mu_n \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right),$$

where $\mu_n(a, b, c, d)$ represents the canonical moment sequence of the continuous Hahn polynomials $p_n(x, a, b, c, d)$ [12]. Multiplying by $\left(\frac{3}{4}\right)^{\frac{n}{2}}$ gives us a sequence $\tilde{e}_n$ of integers

$$1, 0, 2, 0, 18, 0, 378, 0, 14562, 0, \ldots$$

whose Hankel transform $\tilde{h}_n$ satisfies

$$\tilde{h}_n = A_{n+1} \prod_{i=0}^{n} i!^2.$$

The sequence $\tilde{e}_n$ has its generating function given by

$$\mathcal{J}\left(0, 0, 0, \ldots; 2, 7, \frac{108}{7}, \frac{572}{21}, \ldots\right).$$

Now we recognize in $\prod_{i=0}^{n} i!^2$ A055209 the Hankel transform of the Euler numbers A000364, since we have

$$\prod_{i=0}^{n} i!^2 = \prod_{k=0}^{n} (k+1)^{2(n-k)},$$

where the Euler numbers have generating function $\mathcal{J}(0, 0, 0, \ldots; 1, 4, 9, 16, \ldots)$. Thus by the Heilermann formula, the sequence with generating function

$$\mathcal{J}\left(0, 0, 0, \ldots; \frac{2}{1}, \frac{7}{4}, \frac{108}{7 \cdot 9}, \frac{572}{21 \cdot 16}, \ldots\right) = \mathcal{J}\left(0, 0, 0, \ldots; \frac{2}{1}, \frac{7}{4}, \frac{12}{7}, \frac{143}{84}, \ldots\right)$$

will have a Hankel transform equal to A_{n+1}. This is the sequence that begins

$$1, 0, 2, 0, \frac{15}{2}, 0, \frac{273}{8}, 0, \frac{5471}{32}, 0, \ldots.$$

Proposition 17. *The Hankel transform of the revert transform of the sequence with generating function*

$$\frac{x^2+4}{9x^2+4}$$

is given by A_{n+1}.

Proof. We have

$$\frac{x^2+4}{9x^2+4} = \frac{1}{1 + \frac{2x^2}{1+\frac{1}{4}x^2}}.$$

Thus

$$\frac{x^2+4}{9x^2+4} = \mathcal{J}\left(0, 0, 0, \ldots; -2, -\frac{1}{4}, 0, 0, \ldots\right).$$

On the other hand, we have

$$\frac{1-x}{1-2x-x^2+x^3} = \mathcal{J}\left(1, \frac{1}{2}, \frac{1}{2}, 0, 0, \ldots; 2, \frac{1}{4}, 0, 0, \ldots\right).$$

We conclude that the Hankel transform of the revert transform of expansion of $\frac{x^2+4}{9x^2+4}$ is A_{n+1}. □

The revert transform of the expansion of $\frac{x^2+4}{9x^2+4}$ has generating function

$$g(x) = 3 + \frac{2}{\sqrt{3}x}\sqrt{27x^2-4}\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{\sqrt{-27}x}{2}\right)\right)$$

and begins

$$1, 0, 2, 0, \frac{15}{2}, 0, \frac{273}{8}, 0, \frac{5471}{32}, 0, \ldots.$$

Denote this sequence by t_n. We consider the sequence $\tilde{t}_n = 2^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^{n-k} t_k$ obtained by taking the $(1/2)$-binomial transform of $g(x)$ and scaling the expansion of this new generating function by 2^n. Thus we have

$$\tilde{t}_n = 2^n [x^n] \frac{1}{1-\frac{x}{2}} g\left(\frac{x}{1-\frac{x}{2}}\right) = [x^n] \frac{1}{1-x} g\left(\frac{2x}{1-x}\right).$$

This sequence begins

$$1, 1, 9, 25, 169, 681, 4105, 19657, 113545, 592777, \ldots.$$

Proposition 18. *The Hankel transform h_n of the sequence $\tilde{t}_n$ is given by*

$$h_n = 4^{\binom{n+1}{2}} A_{n+1}.$$

The sequence $\tilde{t}_n$ has its generating function given by

$$\tilde{g}(x) = \frac{1}{1-x}\left(3 + \frac{2\sqrt{26x^2+2x-1}}{\sqrt{3}x}\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{\sqrt{27}ix}{1-x}\right)\right)\right).$$

The sequence $\tilde{t}_n$ is the revert transform of the expansion of

$$\frac{1+x^2}{1+x+9x^2+x^3}.$$

Proof. The binomial transform does not change the Hankel transform, while the scaling by 2^n introduces the Hankel scaling factor of $4^{\binom{n+1}{2}}$. The generating function is found by an application of the fundamental theorem of Riordan arrays. □

Proposition 19. *The sequence with n-th term $2^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^{n-k} (-1)^{\frac{k}{2}} t_k$, which begins*

$$1, 1, -7, -23, 73, 521, -503, -11255, -9207, 225289, 606217, \ldots,$$

is the revert transform of

$$\frac{1-x^2}{1+x-9x^2-x^3}$$

and its Hankel transform is given by $(-4)^{\binom{n+1}{2}} A_{n+1}$.

Proof. This follows since we have applied a binomial transform which does not affect the Hankel transform, and the scaling factor in the Hankel transform comes from the scaling factor 2^n in the sequence. The sign in the scaling factor is a result of the sign change coming from the term $(-1)^{\frac{k}{2}}$. □

7 Compositions and quasi-Fibonacci polynomials of the 7th order

We have seen that the generating function

$$\frac{1-x}{1-x-2x^2+x^3}$$

and its INVERT transform

$$\frac{1-x}{1-2x-x^2+x^3}$$

lead to the signed Robbins numbers $(-1)^{\binom{n+1}{2}}A_{n+1}$ through reversion and the Hankel transform. The complexification $x \to ix$ of these generating functions lead to the Robbins numbers A_{n+1}. Although they are just two of an infinity of generating functions (all related combinations of sign change, INVERT transform and binomial transform) they have been given a certain prominence in a number of other contexts. One application is to the theory of compositions. We have the following.

Proposition 20. *The generating function of the number of compositions of n with r types of 1 and two types of 2 is given by*

$$\frac{1-x}{1-(r+1)x+(r-2)x^2+x^3}.$$

For any r, the revert transform of this sequence has a Hankel transform given by the signed Robbins numbers $(-1)^{\binom{n+1}{2}}A_{n+1}$.

Proof. By [8, Theorem 2.10], the generating function we seek is

$$\frac{1}{1-x\left(r+2x+\frac{x^2}{1-x}\right)} = \frac{1-x}{1-(r+1)x+(r-2)x^2+x^3}.$$

The assertion concerning the Robbins numbers follows from the fact that

$$\frac{1-x}{1-(r+1)x+(r-2)x^2+x^3} = \mathcal{J}\left(r, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \ldots; 2, \frac{1}{4}, 0, 0, 0, \ldots\right).$$

□

We have

$$r\left(\frac{1-x}{1-x-2x^2+x^3}\right) = \mathcal{J}\left(0, \frac{1}{2}, \frac{1}{2}, \ldots; -2, -\frac{7}{4}, -\frac{12}{7}, -\frac{143}{84}, \ldots\right),$$

and

$$r\left(\frac{1-x}{1-2x-x^2+x^3}\right) = \mathcal{J}\left(-1, -\frac{1}{2}, -\frac{1}{2}, \ldots; -2, -\frac{7}{4}, -\frac{12}{7}, -\frac{143}{84}, \ldots\right).$$

We similarly find that

$$r\left(\frac{1-3x+2x^2}{1-4x+3x^2+x^3}\right)=\mathcal{J}\left(-1,\frac{1}{2},\frac{1}{2},\ldots;-2,-\frac{7}{4},-\frac{12}{7},-\frac{143}{84},\ldots\right)$$

and

$$r\left(\frac{1-3x+2x^2}{1-5x+6x^2-x^3}\right)=\mathcal{J}\left(-2,-\frac{1}{2},-\frac{1}{2},\ldots;-2,-\frac{7}{4},-\frac{12}{7},-\frac{143}{84},\ldots\right).$$

We can place these generating functions in a general context by using the concept of quasi-Fibonacci polynomials [16]. Wituła et al introduce three parameterized families of sequences $\mathcal{A}_n(\delta),\mathcal{B}_n(\delta)$ and $\mathcal{C}_n(\delta)$. We concentrate on the family $\mathcal{A}_n(\delta)$. We use the language of Riordan arrays to describe the polynomials $A_n(\delta)$.

Proposition 21. *The polynomials $y^n\mathcal{A}_n\left(\frac{1}{y}\right)$ are the row polynomials of the inversion of the Riordan array*

$$\left(\frac{1-x}{1-x-2x^2+x^3},-\frac{x(1-x)}{1-x-2x^2+x^3}\right).$$

Proof. The inversion of the Riordan array $\left(\frac{1-x}{1-x-2x^2+x^3},-\frac{x(1-x)}{1-x-2x^2+x^3}\right)$ is given by the exponential Riordan array [3] that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 6 & 0 & 1 & 0 & 0 \\ 5 & -4 & 12 & 0 & 1 & 0 \\ -5 & 25 & -10 & 20 & 0 & 1 \end{pmatrix}.$$

The reversal of this matrix is the coefficient matrix of the quasi-Fibonacci polynomials $\mathcal{A}_n(\delta)$ ([16], Table 1). □

The polynomials $y^n\mathcal{A}_n\left(\frac{1}{y}\right)$ have generating function

$$\frac{1-(2y-1)x+y(y-1)x^2}{1-(3y-1)x+(3y^2-2y-2)x^2-(1-2y-y^2+y^3)x^3}.$$

Examples of generating functions of this form are

$$\frac{1+3x+2x^2}{1+4x+3x^2-x^3},\frac{1+x}{1+x-2x^2-x^3},\frac{1-x}{1-2x-x^2+x^3},\frac{1-3x+2x^2}{1-5x+6x^2-x^3}$$

(respectively, A121449, A052547, A077998, and A052975) for values of $y=-1,\ldots,2$. The Hankel transform of the revert transform of the sequence $y^n\mathcal{A}_n\left(\frac{1}{y}\right)$ is given by the signed

Robbins numbers $(-1)^{\binom{n+1}{2}}A_{n+1}$ (independent of y). The polynomials $\mathcal{A}_n(y)$ have generating function

$$\frac{1-(2-y)x-(y-1)x^2}{1-(3-y)x-(-3+2y-2y^2)x^2-(1-y-2y^2+y^3)x^3}.$$

The Hankel transform of the revert transform of the polynomial sequence $\mathcal{A}_n(y)$ is given by $(-y^2)^{\binom{n+1}{2}}A_{n+1}$.

8 Conclusions

In this note, we have indicated that the Robbins numbers can be determined by the Hankel transform of the revert transform of a family of sequences related to the so-called quasi-Fibonacci polynomials of 7th order $A_n(\delta)$ [16]. These reversions seem to be related to the moment sequences of certain continuous Hahn polynomials, at least in one case [6] of the defining parameters (namely $\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}$). Links have also been established with compositions and quasi-Fibonacci polynomials. The language of Riordan arrays and of Jacobi continued fractions has brought interesting perspectives to this work. We have also used the technique of translating the algebra of matrix multiplication into the algebra of bivariate generating functions, previously used in a related context by Gessel and Xin [7].

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