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Generalized Catalan Recurrences, Riordan Arrays, Elliptic Curves, and Orthogonal Polynomials

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Abstract

We show that the Catalan-Schroeder convolution recurrences and their higher order generalizations can be solved using Riordan arrays and the Catalan numbers. We investigate the Hankel transforms of many of the recurrence solutions, and indicate that Somos-4 sequences often arise. We exhibit relations between recurrences, Riordan arrays, elliptic curves and Somos-4 sequences. We furthermore indicate how one can associate a family of orthogonal polynomials to a point on an elliptic curve, whose moments are related to recurrence solutions.

This paper, which concerns generalized Catalan recurrences, their solutions using Riordan arrays, and applications to elliptic curve sequences, is arranged in the following sections.

1. A motivating example
2. Preliminaries
3. Generalized Catalan recurrences
4. A third-order recurrence
5. A further recurrence
6. Conversion of parameters

1 A motivating example

The elliptic curve

$$E : y^2 - xy + y = x^3 - 2x + x$$

passes through the point $P(0,0)$. The division polynomials of this curve, ψ_n, evaluated at the multiples nP of the point $P(0,0)$, are given by

$$\psi_n(0,0) = (-1)^{\binom{n}{2}} F_n,$$

where F_n is the n-th Fibonacci number A000045. A consequence of this is that the coordinates of nP are

$$x_n = x(nP) = \frac{F_n F_{n+2}}{F_{n+1}^2},$$

and

$$y_n = y(nP) = \frac{(-1)^n F_n}{F_{n+1}^3}.$$

The ratio y_n/x_n is then given by

$$\frac{y_n}{x_n} = \frac{(-1)^n}{F_{n+1}F_{n+2}}.$$

We now form a generating function expressed as a continued fraction

$$\tilde{a}(x) = g_{E,P}(t) = \cfrac{1}{1 + t + \cfrac{x_1 t^2}{1 + \frac{y_1}{x_1} t + \cfrac{x_2 t^2}{1 + \frac{y_2}{x_2} t + \cfrac{x_3 t^2}{1 + \frac{y_3}{x_3} t + \cdots}}}}.$$

In this case, the generating function $g_{E,P}(t)$ expands to give the sequence $\tilde{a}_n$ that begins

$$1, -1, -1, 2, 2, -5, -5, 14, 14, -42, -42, \ldots.$$

The general term of this sequence is

$$\tilde{a}_n = (-1)^{\binom{n+1}{2}} C_{\lfloor \frac{n+1}{2} \rfloor}.$$

Here, we have $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

The Hankel transform $h_n = |a_{i+j}|_{0 \le i,j \le n}$ of $\tilde{a}_n$ begins

$$1, -2, -3, 5, 8, -13, -21, 34, 55, -89, -144, \ldots,$$

with general term

$$h_n = (-1)^{\binom{n+1}{2}} F_{n+2}.$$

Thus we have

$$h_n = (-1)^{n+1} \psi_{n+2}(0,0).$$

On the other hand, we can solve the equation $y^2 - xy + y = x^3 - 2x^2 + x$ for y. We find that

$$y = \frac{(1-x)(\sqrt{1+4x}-1}{2} = (1-x)xC(-x), \quad \text{or} \quad y = -\frac{(1-x)(1+\sqrt{1+4x})}{2}.$$

Here,

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

is the generating function of the Catalan numbers. The first solution expands to give the sequence that begins

$$0, 1, -2, 3, -7, 19, -56, 174, -561, 1859, -6292,$$

while the second solution begins

$$-1, 0, 2, -3, 7, -19, 56, -174, 561, -1859, 6292, \ldots.$$

We are interested in the common part of these sequences, namely the sequence that begins

$$2, -3, 7, -19, 56, -174, 561, -1859, 6292, \ldots.$$

This has its generating function given by

$$f(x) = \left(-\frac{(1-x)(1+\sqrt{1+4x})}{2} + 1 \right) / x^2 = \frac{1+x-(1-x)\sqrt{1+4x}}{2x^2}.$$

We wish to work with a sequence with initial term 1, which has essentially the same Hankel transform as this sequence. Thus we take the generating function

$$f_1(x) = \frac{1}{1-x-x^2 f(x)} = \frac{2}{1-3x+(1-x)\sqrt{(1+4x)}} = \frac{-(x-1)\sqrt{1+4x}-3x+1}{2x(x^2-4x+2)}.$$

Using the fundamental theorem of Riordan arrays [2, 17], we can express this as

$$f_1(x) = \frac{1}{1-3x} C\left(\frac{-x(x^2-4x+2)}{(1-3x^2)^2}\right) = \left(\frac{1}{1-3x}, \frac{-x(x^2-4x+2)}{(1-3x^2)^2}\right) \cdot C(x).$$

We now revert $xf_1(x)$, and divide the result by x, to obtain the generating function $a(x)$ given by

$$a(x) = \frac{1+3x+4x^2-(1+x)\sqrt{1+4x+8x^2}}{2x^3}.$$

This expands to give a sequence a_n that begins

$$1, -1, -1, 8, -22, 33, 7, -212, 702, -1202, -58, \ldots.$$

This sequence satisfies the convolution recurrence (generalized Catalan recurrence)

$$a_n = -3a_{n-1} - 4a_{n-2} + 2a_{n-3} + \sum_{k=1}^{n-4} a_k a_{n-k-3},$$

with $a_0 = 1, a_1 = -1, a_2 = -1, a_3 = 8$. The Hankel transform of this sequence begins

$$1, -2, -3, 5, 8, -13, -21, 34, 55, -89, -144, \ldots.$$

Thus we now have a sequence a_n, constructed from the elliptic curve equation, that has the same Hankel transform as $\tilde{a}_n$. Moreover, we know the generating function of a_n. We can express $a(x)$ as

$$a(x) = \left(\frac{1+2x}{1+3x+4x^2}, \frac{x^3(1+2x)}{(1+3x+4x^2)^2}\right) \cdot C(x).$$

It remains to relate $\tilde{a}(x) = g_{E,P}(x)$ to $a(x)$.

In this special case, the solution is quite easy to state. We have

$$\tilde{a}_n = \sum_{k=0}^{n} \binom{n}{k} (-2)^{n-k} (-1)^k a_k.$$

In other words, the generating function $g_E(x)$ of $\tilde{a}_n = (-1)^{\binom{n+1}{2}} C_{\lfloor \frac{n+1}{2} \rfloor}$ is given by taking the second inverse binomial transform of the generating function of the sequence $(-1)^n a_n$. We can verify this algebraically as follows. We take the generating function

$$a(x) = \frac{1+3x+4x^2-(1+x)\sqrt{1+4x+8x^2}}{2x^3}.$$

We now form the generating function of $(-1)^n a_n$ which is

$$a(-x) = -\frac{1-3x+4x^2-(1-x)\sqrt{1-4x+8x^2}}{2x^3}.$$

Taking the second inverse binomial transform now gives us

$$\frac{1}{1+2x}a\left(\frac{-x}{1+2x}\right) = \frac{(1+x)\sqrt{1+4x^2}-1-x-2x^2}{2x^3}.$$

We can verify independently that this last generating function is indeed the generating function of $\tilde{a}_n = (-1)^{\binom{n+1}{2}} C_{\lfloor \frac{n+1}{2} \rfloor}$.

We have

$$\frac{(1+x)\sqrt{1+4x^2}-1-x-2x^2}{2x^3} = \cfrac{1}{1+x+\cfrac{\frac{F_1F_3}{F_2^2}x^2}{1-\cfrac{1}{F_2F_3}x+\cfrac{\frac{F_2F_4}{F_3^2}x^2}{1+\cfrac{1}{F_3F_4}x+\cfrac{\frac{F_3F_5}{F_4^2}x^2}{1-\cdots}}}}.$$

The sequence $\tilde{a}_n$ is the moment sequence (see the Appendix) of the family of orthogonal polynomials $P_n(t)$ defined by the three-term recurrence

$$\begin{aligned} P_n(t) &= (t+\frac{x_{n-1}}{y_{n-1}})P_{n-1}(t) + x_{n-1}P_{n-2}(t) \\ &= \left(t - \frac{(-1)^n}{F_nF_{n+1}}\right)P_{n-1}(t) + \frac{F_{n-1}F_{n+1}}{F_n^2}P_{n-2}(t), \end{aligned}$$

with $P_0(t) = 1, P_1(t) = t+1$.

The coefficient array of the family $P_n(t)$ then begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3/2 & 1/2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7/3 & 2/3 & 1 & 0 & 0 & 0 \\ 8/5 & 7/5 & 17/5 & 3/5 & 1 & 0 & 0 \\ 1 & 31/8 & 17/8 & 35/8 & 5/8 & 1 & 0 \\ 21/13 & 31/13 & 95/13 & 35/13 & 70/13 & 8/13 & 1 \end{pmatrix}.$$

The inverse of this matrix, which is the moment matrix, begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1/2 & 1 & 0 & 0 & 0 & 0 \\ 2 & -2 & -2/3 & 1 & 0 & 0 & 0 \\ 2 & 3/2 & -3 & -3/5 & 1 & 0 & 0 \\ -5 & 5 & 8/3 & -4 & -5/8 & 1 & 0 \\ -5 & -9/2 & 9 & 3 & -5 & -8/13 & 1 \end{pmatrix}.$$

This exhibits the sequence $\tilde{a}_n$ as a moment sequence.

We note that the denominators in the coefficient matrix of the orthogonal polynomials are the Fibonacci numbers. Scaling up by these, we get the integer matrix that begins as follows.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 7 & 2 & 3 & 0 & 0 & 0 \\ 8 & 7 & 17 & 3 & 5 & 0 & 0 \\ 8 & 31 & 17 & 35 & 5 & 8 & 0 \\ 21 & 31 & 95 & 35 & 70 & 8 & 13 \end{pmatrix}.$$

The row sums of this matrix are given by $F_{n+1}F_{n+2}$.

The generating function of $\tilde{a}_n$ can be described using Riordan arrays as follows.

$$\tilde{a}(x) = \left(\frac{1}{1+x+2x^2}, \frac{-x^3}{(1+x+2x^2)^2}\right) \cdot C(x).$$

A consequence of this is that the sequence $\tilde{a}_n$ satisfies the convolution recurrence

$$\tilde{a}_n = -\tilde{a}_{n-1} - 2\tilde{a}_{n-2} - 2\tilde{a}_{n-3} - \sum_{k=1}^{n-4} \tilde{a}_k \tilde{a}_{n-k-3},$$

with $a_0 = 1, a_1 = -1, a_2 = -1, a_3 = 2$.

2 Preliminaries

The product of two power series

$$a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

and

$$b(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots = \sum_{n=0}^{\infty} b_n x^n$$

is given by

$$a(x)b(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_1 + a_1 b_1 + a_1 b_0)x^2 + \cdots .$$

That is,

$$a(x)b(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n.$$

The term $\sum_{k=0}^{n} a_k b_{n-k}$ is called the convolution of $a_0, a_1, \cdots, a_n$ with $b_0, b_1, \ldots, b_n$. Note that if we multiply the shifted sequences $a_1 + a_2 x + a_3 x^2 + \cdots$ and $b_1 + b_2 x + b_3 x^2 + \cdots$, then the product will begin

$$a_1 b_1 + (a_1 b_2 + a_2 b_1)x + (a_1 b_3 + a_2 b_2 + a_3 b_1)x^2 + \cdots,$$

or

$$\sum_{n=0}^{\infty} \left(\sum_{k=1}^{n+1} a_k b_{n+2-k} \right) x^n.$$

This can be also be written as

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_{k+1} b_{n+1-k} \right) x^n.$$

Expressions such as $\sum_{k=0}^{n} a_k b_{n-k}$ are known as convolutions, and equations involving terms of a sequence and such convolutions are known as convolution recurrences. For instance,

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k},$$

where we stipulate $C_0 = 1$, is a well known convolution recurrence. Note that we have $C_1 = C_0 C_0$, $C_2 = C_0 C_1 + C_1 C_0$, $C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0$, and so on.

Starting from $C_0 = 1$, we obtain $C_0 = 1, 1, 2, 5, 14, 42, \ldots$. These are the Catalan numbers A000108. If we let $C(x) = \sum_{n=0}^{\infty} C_n x^n$, then we see that

$$C_1 + C_2 x + C_3 x^2 + \cdots = C_0 C_0 + (C_0 C_1 + C_1 C_0)x + \ldots = C(x) \times C(x) = C(x)^2.$$

Thus,

$$C(x) = C_0 + C_1 x + C_2 x^2 + \cdots = C_0 + x(C_1 + C_2 x + C_3 x^2 + \ldots) = 1 + xC(x)^2.$$

This means that $C(x)$, the *generating function* of the Catalan numbers, is a solution of the quadratic equation

$$u = 1 + xu^2, \quad \text{or} \quad xu^2 - u + 1 = 0.$$

We obtain two solutions,

$$u(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \quad \text{and} \quad \frac{1 + \sqrt{1 - 4x}}{2x}.$$

When $x = 0$, the second solution leads to a division by zero, so we choose the first solution. Thus

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is the generating function of the Catalan numbers. In this note, shall seek to solve more generalized convolution recurrences. The Catalan numbers C_n and their generating function $C(x)$ will play an important role in this discussion. This will arise because we will often be solving quadratic equations, as above. Thus if we consider a quadratic equation such as

$$au^2 + bu + c = 0$$

then the solution

$$u = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

can be expressed as follows. We have

$$\begin{aligned}
\frac{-c}{b} C\left(\frac{ac}{b^2}\right) &= \frac{c}{b}\left(\frac{1 - \sqrt{1 - 4\frac{ac}{b^2}}}{2\frac{ac}{b^2}}\right) \\
&= \frac{-b}{2a}\left(1 - \sqrt{1 - 4\frac{ac}{b^2}}\right) \\
&= \frac{1}{2a}\left(-b + b\sqrt{1 - 4\frac{ac}{b^2}}\right) \\
&= \frac{1}{2a}\left(-b + \sqrt{b^2 - 4ac}\right).
\end{aligned}$$

The squared terms that give rise to the quadratic expressions to be solved will in turn be explained in part by the presence of the convolution elements. For instance, if we multiply a shifted power sequence $a_1 + a_2x + a_3x^2 + \cdots$ by itself, thus getting a power series that begins

$$a_1a_1 + (a_1a_2 + a_2a_1)x + (a_1a_3 + a_2a_2 + a_3a_1)x^2 + \cdots,$$

then the generating function of this expression will be

$$\frac{a(x) - 1}{x} \cdot \frac{a(x) - 1}{x} = \frac{(a(x) - 1)^2}{x^2} = \frac{a(x)^2 - 2a(x) + 1}{x^2},$$

where

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

is the generating function of the sequence $a_0, a_1, a_2, \ldots$. (We have taken $a_0 = 1$ for simplicity here).

The expression $\frac{-c}{b} C\left(\frac{ac}{b^2}\right)$ is reminiscent of the fundamental theorem of Riordan arrays [2, 17], and in the cases that we shall consider, it will indeed be an instance of this result. We shall therefore use Riordan arrays [2, 16, 17] extensively. A Riordan array can be defined by a pair (g, f) of power series

$$g(x) = g_0 + g_1x + g_2x^2 + \cdots, \quad g_0 \neq 0,$$

and

$$f(x) = f_1 x + f_2 x^2 + f_3 x^3 + \cdots,$$

with indeterminate x and coefficients drawn from a suitable ring. To this pair we can associate the lower-triangular matrix with general (n,k)-th term $T_{n,k}$ given by

$$T_{n,k} = [x^n] g(x) f(x)^k.$$

Here, $[x^n]$ is the functional on power series that extracts the coefficient of x^n [14]. Thus this matrix is the "Riordan array" associated with the pair (g,f). In practice, we refer to either as a Riordan array. The fundamental theorem of Riordan arrays asserts that the action of the pair (g,f) acting on the generating function $h(x)$ as defined by

$$(g(x), f(x)) \cdot h(x) = g(x) f(h(x))$$

is mirrored by multiplying the (infinite) vector whose elements are given by the expansion of $h(x)$ by the matrix $(T_{n,k})$.

When $f_1 \neq 0$ the matrices have a non-zero diagonal and hence they are invertible. If $f_1 = 0$ we have arrays that are often described as "stretched", which are not invertible in the usual sense. The fundamental theorem still holds.

Somos-4 sequences are defined using the Weierstrass σ function for appropriate elliptic curves. The link between this σ function and the Hankel transform [11, 12, 13] explains the importance of the Hankel transform in this note. The motivating example shows how the $x-$ and $y-$ coordinates of the multiples of special points on an elliptic curve can give rise to integer sequences whose Hankel transform is then the Somos-4 sequence for that point. The link to the title of this note is that these sequences satisfy a recurrence of the desired type.

Where we encounter sequences that appear in the *On-Line Encyclopedia of Integer Sequences* [18, 19], we refer to them by their *Annnnnn* number.

The r-th binomial transform b_n of the sequence a_n has general term $b_n = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} a_k$. If the sequence a_n has a generating sequence $g(x)$, then the sequence b_n has generating function $\frac{1}{1-rx} g\left(\frac{x}{1-rx}\right)$.

The $\textsc{Invert}(a)$ transform of the sequence a_n whose generating function is $g(x)$ will have a generating function given by $\frac{g(x)}{1-axg(x)} = (g(x), xg(x)) \cdot \frac{1}{1-ax}$.

If the generating function $g(x)$ of a sequence a_n can be expressed as a Stieltjes continued fraction

$$g(x) = \cfrac{1}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cdots}}},$$

then the r-th binomial transform of a_n will have a generating function given by

$$\cfrac{1}{1 - (r+\alpha_0) x - \cfrac{\beta_1 x^2}{1 - (r+\alpha_1) x - \cfrac{\beta_2 x^2}{1 - (r+\alpha_2) x - \cdots}}}.$$

The INVERT(a) transform of a_n will have a generating function given by

$$\cfrac{1}{1-(a+\alpha_0)x-\cfrac{\beta_1 x^2}{1-\alpha_1 x-\cfrac{\beta_2 x^2}{1-\alpha_2 x-\cdots}}}.$$

The generating function $g(x)$ above will be designated by $\mathcal{J}(\alpha_0,\alpha_1,\ldots;\beta_1,\beta_2,\ldots)$.

If a sequence $a_0, a_1, \ldots$ with generating function $g(x)$ has a Hankel transform that begins $h_0, h_1, \ldots$ then the sequence with generating function $\frac{1}{1-x-x^2 g(x)}$ will have a Hankel transform that begins $1, h_0, h_1, \ldots$. In addition, this new sequence will have its initial term equal to 1.

More information on orthogonal polynomials, Riordan arrays and Hankel transforms may be found in the Appendix.

3 Generalized Catalan recurrences

We let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the n-th Catalan numbers A000108. The sequence of Catalan numbers has generating function $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. The Catalan numbers begin

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots.$$

The large Schroeder numbers S_n A006318 can then be defined by

$$S_n = \sum_{k=0}^{n} \binom{n+k}{2k} C_k.$$

These numbers begin

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, \ldots.$$

Using the theory of Riordan arrays, we see that the generating function $S(x)$ of the large Schroeder numbers is given by an application of the fundamental theorem of Riordan arrays [2, 17], which in this case says that

$$S(x) = \left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right) \cdot C(x) = \frac{1}{1-x} C\left(\frac{x}{(1-x)^2}\right).$$

This follows since the matrix representation of the Riordan array $\left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right)$ has general term

$$[x^n] \frac{1}{1-x} \frac{x^k}{(1-x)^{2k}} = \binom{n+k}{2k}.$$

Simplifying, we find that

$$S(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}.$$

We now consider the generating function $\frac{C(x)+C(-x)}{2} = \frac{\sqrt{1+4x}+\sqrt{1-4x}}{2x}$, which expands to give the sequence

$$1, 0, 2, 0, 14, 0, 132, 0, 1430, 0, 16796, \ldots,$$

which is the aerated sequence of Catalan numbers of even index A048990. We once again apply the Riordan array $\left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right)$ to this sequence, resulting in the sequence that begins

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, \ldots.$$

The generating function of this sequence, which is called the sequence of little Schroeder numbers s_n A001003, is then given by

$$\left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right) \cdot \frac{\sqrt{1+4x} + \sqrt{1-4x}}{2x} = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4x}.$$

We deduce that

$$s_n = \sum_{k=0}^{n} \binom{n+k}{2k} C_k \frac{1 + (-1)^k}{2} = \sum_{k=0}^{n} \binom{n+2k}{4k} C_{2k}.$$

We can further use Riordan arrays to characterize these two sequences. The theory of Riordan arrays and orthogonal polynomials [2, 3, 4] allows us to derive the following result.

Proposition 1. *The large Schroeder numbers S_n are the moments of the family of orthogonal polynomials whose coefficient array is given by the Riordan array*

$$\left(\frac{1}{1+2x}, \frac{x}{1+3x+2x^2}\right)$$

and the little Schroeder numbers s_n are the moments of the family of orthogonal polynomials whose coefficient array is given by the Riordan array

$$\left(\frac{1}{1+x}, \frac{x}{1+3x+2x^2}\right).$$

One consequence of this result is the following.

Corollary 2. *The Hankel transform $H_n = |S_{i+j}|_{0 \le i,j \le n}$ of the large Schroeder numbers, and the Hankel transform $h_n = |s_{i+j}|_{0 \le i,j \le n}$ of the little Schroeder numbers satisfy*

$$H_n = h_n = 2^{\binom{n+1}{2}}.$$

Another corollary, which we can prove independently, is that the generating function $S(x)$ of the large Schroeder numbers has the following Jacobi continued fraction [5, 23] form:

$$S(x) = \cfrac{1}{1 - 2x - \cfrac{2x^2}{1 - 3x - \cfrac{2x^2}{1 - 3x - \cfrac{2x^2}{1 - 3x - \cdots}}}},$$

which we express as

$$S(x) = \mathcal{J}(2, 3, 3, 3, \ldots; 2, 2, 2, \ldots).$$

We solve for v in the equation

$$v = \frac{1}{1 - 3x - 2x^2 v},$$

then we verify that

$$S(x) = \frac{1}{1 - 2x - 2x^2 v}.$$

We similarly have

$$s(x) = \mathcal{J}(1, 3, 3, 3, \ldots; 2, 2, 2, \ldots).$$

The Schroeder number sequences S_n and s_n each satisfy a simple convolution recurrence. For S_n, we have [15]

$$S_n = 3S_{n-1} + \sum_{k=0}^{n-3} S_{k+1} S_{n-k-2},$$

with initial conditions $S_0 = 1$, $S_1 = 2$. Note that we can write this as

$$S_n = 3S_{n-1} + \sum_{k=1}^{n-2} S_k S_{n-k-1}.$$

The summation in the recurrence is then equal to

$$S_1 S_{n-2} + S_2 S_{n-3} + \cdots + S_{n-2} S_1.$$

For s_n, we have

$$s_n = 3s_{n-1} + 2\sum_{k=0}^{n-3} s_{k+1} s_{n-k-2},$$

with initial conditions $s_0 = 1$, $s_1 = 1$. These recurrences, and their generalizations, will be the subject of this note.

Proposition 3. *We consider the generalized Catalan-Schroeder recurrence*

$$a_n = sa_{n-1} + t\sum_{k=0}^{n-3} a_{k+1}a_{n-k-2},$$

with initial conditions $a_0 = 1$, $a_1 = p$. Then the generating function of the sequence a_n is given by

$$\left(\frac{1+(p-s+t)x}{1-(s-2t)x}, \frac{tx(1+(p-s+t)}{(1-(s-2t)x)^2}\right)\cdot C(x).$$

Note that when $p = 2$, $s = 3$ and $t = 1$, we obtain

$$\left(\frac{1+(2-3+1)x}{1-(3-2)x}, \frac{x(1+(2-3+1)x)}{(1-(3-2)x)^2}\right)\cdot C(x) = \left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right)\cdot C(x) = S(x),$$

while when $p = 1$, $s = 3$ and $t = 2$, we obtain

$$\left(\frac{1+(1-3+2)x}{1+(2\cdot 2-3)x}, \frac{2x(1+(1-3+2)x)}{(1+(2\cdot 2-3)x)^2}\right)\cdot C(x) = \left(\frac{1}{1+x}, \frac{2x}{(1+x)^2}\right)\cdot C(x) = s(x).$$

Note that this last result gives us that

$$s_n = \sum_{k=0}^{n} \binom{n+k}{2k}(-1)^{n-k}2^k C_k.$$

Proof. The terms $a_1a_{n-2} + \cdots + a_{n-2}a_1$, for $n \geq 3$, successively contribute the terms

$$\begin{aligned} &a_1a_1, \\ &a_1a_2 + a_2a_1, \\ &a_1a_3 + a_2a_2 + a_3a_1, \ldots\ . \end{aligned}$$

These correspond to the coefficients in the product

$$(a_1 + a_2x + a_3x^2 + \cdots)(a_1 + a_2x + a_3x^2 + \cdots).$$

This product has generating function

$$\left(\frac{f(x)-1}{x}\right)^2 = \frac{f(x)^2 - 2f(x) + 1}{x^2}.$$

Thus we can convert the recurrence

$$a_n = sa_{n-1} + t\sum_{k=0}^{n-3} a_{k+1}a_{n-k-2}, \quad n \geq 2$$

into the following statement about the generating function $u(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + px + \sum_{n=2}^{\infty} a_n x^n$:

$$u = 1 + px + sx(u-1) + tx(u^2 - 2u + 1).$$

Solving this, we get

$$u(x) = \frac{1 - (s-2t)x - \sqrt{1 - 2sx + (s^2 - 4pt)x^2}}{2tx}.$$

This is equal to

$$\left(\frac{1 + (p-s+t)x}{1-(s-2t)x}, \frac{tx(1+(p-s+t)x)}{1-(s-2t)x}\right) \cdot C(x).$$

□

Corollary 4. *The generating function*

$$\left(\frac{1+ax}{1+bx}, \frac{mx(1+ax)}{(1+bx)^2}\right) \cdot C(x)$$

expands to give the sequence solution of the convolution recurrence

$$a_n = (-b+2m)a_{n-1} + m\sum_{k=0}^{n-3} a_{k+1}a_{n-k-3},$$

with $a_0 = 1, a_1 = a - b + m$.

Example 5. We calculate the general term $T_{n,k}$ of the Riordan array $\left(\frac{1+ax}{1+bx}, \frac{mx(1+ax)}{(1+bx)^2}\right)$. We have

$$\begin{aligned}
T_{n,k} &= [x^n]\frac{1+ax}{1+bx}\left(\frac{mx(1+ax)}{1+bx}\right)^k \\
&= m^k[x^{n-k}]\frac{(1+ax)^{k+1}}{(1+bx)^{2k+1}} \\
&= m^k[x^{n-k}]\sum_{j=0}^{k+1}\binom{k+1}{j}a^j x^j \sum_{i=0}^{\infty}\binom{-(2k+1)}{i}b^i x^i \\
&= m^k[x^{n-k}]\sum_{j=0}^{k+1}\binom{k+1}{j}a^j x^j \sum_{i=0}^{\infty}\binom{2k+1+i-1}{i}(-1)^i b^i x^i \\
&= m^k\sum_{j=0}^{k+1}\binom{k+1}{j}a^j\binom{2k+n-k-j}{n-k-j}(-b)^{n-k-j} \\
&= m^k\sum_{j=0}^{k+1}\binom{k+1}{j}\binom{n+k-j}{2k}a^j(-b)^{n-k-j}.
\end{aligned}$$

Thus the expansion of $\frac{1+ax}{1+bx}C\left(\frac{mx(1+ax)}{1+bx}\right)$ has general term given by

$$a_n = \sum_{k=0}^{n}(\sum_{j=0}^{k+1}\binom{k+1}{j}\binom{n+k-j}{2k}a^j(-b)^{n-k-j})m^k C_k.$$

Note that in practice care has to be taken with the terms $T_{0,k}$ using the above formula so we have the following correction (for $a \neq 0$).

$$a_n = \sum_{k=0}^{n}\left(\sum_{j=0}^{k+1}\binom{k+1}{j}\binom{n+k-j}{2k}a^j(-b)^{n-k-j}\right)m^k C_k + \frac{b}{a}0^k.$$

We denote the general solution of the recurrence

$$a_n = sa_{n-1} + t\sum_{k=0}^{n-3} a_{k+1}a_{n-k-2},$$

with initial conditions $a_0 = 1$, $a_1 = p$ by $a_n(p, s, t)$.

Example 6. When $(p, s, t) = (1, 1, 1)$, we get the sequence $a_n(1, 1, 1)$ that begins

$$1, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, \ldots.$$

These are the Motzkin numbers, with a 1 prepended. The Hankel transform of this sequence has generating function $\frac{1-x}{1-x+x^2}$, beginning

$$1, 0, -1, -1, 0, 1, 1, 0, -1, \ldots.$$

The Hankel transform of the shifted sequence (the Motzkin numbers) is known to be the given by the all 1's sequence $1, 1, 1, 1, \ldots$. The generating function of the Motzkin numbers is $\mathcal{J}(1, 1, 1, \ldots; 1, 1, 1, \ldots)$.

Example 7. When $(p, s, t) = (1, 1, 2)$, we get the sequence $a_n(1, 1, 2)$ that begins

$$1, 1, 1, 3, 7, 21, 61, 191, 603, 1961, 6457, 21595, \ldots.$$

This is the sequence A025235 with a prepended 1. The sequence A025235 counts Motzkin paths with the up step in two colors. The Hankel transform of $a_n(1, 1, 2)$ begins

$$1, 0, -4, -16, 128, 6144, 65536, -20971520, -3758096384, \ldots$$

while that of the shifted sequence $u_{n+1}(1, 1, 2)$ or A025235 is $2^{\binom{n+1}{2}}$. This follows from the fact that the generating function of this latter sequence is $\mathcal{J}(1, 1, 1, \ldots; 2, 2, 2, \ldots)$.

Example 8. When $(p,s,t) = (1,2,1)$, the sequence $a_n(1,2,1)$ that we get is the sequence of Catalan numbers C_n. This corresponds to a bijection between 2-colored Motzkin paths (where the level step can have one of two colors) of length $n-1$ and Dyck paths. It is a classical result that the Hankel transforms of this sequence and its shift C_{n+1} are the all 1's sequence.

With regard to the shifted sequence $a_{n+1}(p,s,t)$, we have the following result.

Proposition 9. *The sequence $\frac{1}{p}a_{n+1}(p,s,t)$ has generating function $\mathcal{J}(s,s,s,\ldots;pt,pt,pt,\ldots)$, and Hankel transform $(pt)^{\binom{n+1}{2}}$ (for $p \neq 0$).*

Proof. The sequence with generating function $\mathcal{J}(s,s,s,\ldots;pt,pt,pt,\ldots)$ has a generating function $g(x)$ that satisfies

$$g(x) = \frac{1}{1 - sx - ptx^2 g(x)}.$$

Solving this equation gives us

$$g(x) = \frac{1 - sx - \sqrt{1 - 2sx + (s^2 - 4pt)x^2}}{2ptx^2}.$$

We see that this is the generating function of $\frac{1}{p}a_{n+1}(p,s,t)$. □

Corollary 10. *The sequence $\frac{1}{p}a_{n+1}(p,s,t)$ is the moment sequence for the family of orthogonal polynomials whose coefficient array is given by the Riordan array*

$$\left(\frac{1}{1+sx+ptx^2}, \frac{x}{1+sx+ptx^2}\right).$$

Corollary 11. *We have the moment representation*

$$\frac{1}{p}a_{n+1}(p,s,t) = \frac{1}{2\pi}\int_{s-2\sqrt{pt}}^{s+2\sqrt{pt}} x^n \frac{\sqrt{-x^2+2sx+4pt-s^2}}{pt}\,dt.$$

Proof. We apply the Stieltjes-Perron transform [9] to the generating function of a_{n+1}. □

Corollary 12. *If $s = 0$ and $t \neq 0$, we have*

$$a_n(p,0,t) = \frac{1}{2\pi}\int_{-2\sqrt{pt}}^{2\sqrt{pt}} x^n \frac{\sqrt{4pt-x^2}}{tx}\,dx + 0^n.$$

Corollary 13. *If $s \neq 0$ and $t \neq 0$, we have*

$$a_n(p,s,t) = \frac{1}{2\pi}\int_{s-2\sqrt{pt}}^{s+2\sqrt{pt}} x^n \frac{\sqrt{-x^2+2sx+4pt-s^2}}{tx}\,dx + 0^n\left(1 - \frac{s}{2t} - \operatorname{sgn}(4pt-s^2)\frac{\sqrt{s^2-4pt}}{2t}\right).$$

We can formulate the following conjecture concerning the Hankel transform of the sequence $a_n(p,s,t)$.

Conjecture 14. The Hankel transform $h_n(p,s,t)$ of the sequence $a_n(p,s,t)$ is given by

$$\begin{aligned} h_n(p,s,t) &= t^{\binom{n}{2}} p^{\binom{n+1}{2}} [x^n] \frac{1-px}{1-sx+ptx^2} \\ &= t^{\binom{n}{2}} p^{\binom{n+1}{2}} \left(\sum_{i=0}^{n} \binom{i}{n-i} (-pt)^{n-i} s^{2i-n} - p \sum_{i=0}^{n-1} \binom{i}{n-i-1} (-pt)^{n-i-1} s^{2i-n+1} \right). \end{aligned}$$

We note that we have

$$\left(\frac{1-x}{1+ptx^2}, \frac{x}{1+ptx^2} \right) \cdot \frac{1}{1-sx} = \frac{1-px}{1-sx+ptx^2}.$$

Thus the Hankel transform $h_n(p,s,t)$ is given by the evaluation of a sequence of scaled orthogonal polynomials, since the Riordan array $\left(\frac{1-x}{1+ptx^2}, \frac{x}{1+ptx^2} \right)$ is the coefficient matrix of a family of orthogonal polynomials [4].

We have the following conjecture concerning the Hankel transforms $h_n(p,s,t)$ and Somos-4 sequences. Recall that a (α,β) Somos-4 sequence is a sequence e_n that satisfies

$$e_n = \frac{\alpha e_{n-1} e_{n-3} + \beta e_{n-2}^2}{e_{n-4}}$$

for given e_0, e_1, e_2, e_3. Somos-4 sequences are closely related to elliptic curves [10, 25].

Conjecture 15. The Hankel transform $h_n(p,s,t)$ of the generalized Schroeder numbers $a_n(p,s,t)$ is a $((pst)^2, (pt)^3(pt-s^2))$ Somos-4 sequence.

Example 16. We consider the sequence $a_n(1,-1,-1)$ which begins

$$1, 1, -1, 0, 2, -3, -1, 11, -15, -13, 77, \ldots.$$

Its Hankel transform is the signed Fibonacci sequence

$$1, -2, -3, 5, 8, -13, \ldots$$

with general term $(-1)^{\binom{n}{2}} F_{n+1}$. This is a (trivial) $(1,2)$ Somos-4 sequence.

Example 17. The sequence $a_n(2,1,1)$ begins

$$1, 2, 2, 6, 14, 42, 122, 382, 1206, 3922, 12914, \ldots.$$

This is A014431. Its Hankel transform begins

$$1, -2, -24, -64, 5120, 229376, \ldots.$$

This is a $(4,8)$ Somos-4 sequence.

4 A third-order recurrence

We begin this section by looking at the generalized Catalan recurrence

$$a_n = ra_{n-1} + sa_{n-2} + t\sum_{k=1}^{n-3} a_k a_{n-k-2},$$

with $a_0 = 1$, $a_1 = p$ and $a_2 = q$. This may equivalently be written as

$$a_n = ra_{n-1} + sa_{n-2} + t\sum_{i=0}^{n-4} a_{i+1} a_{n-i-3},$$

with $a_0 = 1$, $a_1 = p$ and $a_2 = q$. Thus in the summation part, the sum is

$$a_1 a_{n-3} + \cdots + a_{n-3} a_1.$$

We shall denote the solution of this recurrence by $a_n(p, q, r, s, t)$. This sequence begins

$$1, p, q, ps + qr, p^2t + prs + q(r^2 + s), p^2rt + p(2qt + r^2s + s^2) + qr(r^2 + 2s), \ldots.$$

We let $u(x) = u(x; p, q, r, s, t)$ be the generating function of this solution sequence. We have the following proposition.

Proposition 18. *The generating function $u(x)$ is given by*

$$\left(\frac{1 - (r-p)x - (-q + pr + s - t)x^2}{1 - rx - (s - 2t)x^2}, \frac{tx^2(1 - (r-p)x - (-q + pr + s - t)x^2)}{(1 - rx - (s - 2t)x^2)^2}\right) \cdot C(x).$$

Proof. We translate the recurrence (for $n \geq 3$) and the initial conditions into the following equation for the generating function $u(x) = \sum_{n=0}^{\infty} a_n(p, q, r, s, t)x^n$.

$$u(x) = 1 + px + qx^2 + rx(u - 1 - xp) + sx^2(u - 1) + tx^2(u^2 - 2u + 1).$$

Then the solution is given by

$$u(x) = \frac{1 - rx - (s - 2t)x^2 - \sqrt{1 - 2rx + (r^2 - 2s)x^2 + 2(rs - 2pt)x^3 + (s^2 - 4qt + 4prt)x^4}}{2tx^2}.$$

This is equal to

$$\left(\frac{1 - (r-p)x - (-q + pr + s - t)x^2}{1 - rx - (s - 2t)x^2}, \frac{tx^2(1 - (r-p)x - (-q + pr + s - t)x^2}{(1 - rx - (s - 2t)x^2)^2}\right) \cdot C(x).$$

□

Example 19. The sequence $a_n(1,2,2,1,1)$ begins

$$1, 1, 2, 5, 13, 35, 97, 275, 794, 2327, 6905, \ldots.$$

This is A086581, which counts the number of Dyck paths of semi-length n that avoid DDUU. In this case we have

$$a_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+k}{3k} C_k.$$

The generating function is

$$\frac{1}{1-x} C\left(\frac{x^2}{(1-x)^3}\right).$$

Example 20. The sequence $a_n(1,2,1,2,1)$ coincides with the Motzkin numbers M_n. The generating function is

$$\frac{1}{1-x} C\left(\frac{x^2}{(1-x)^2}\right),$$

so that

$$M_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k.$$

Example 21. The sequence $a_n(1,2,2,2,1)$ begins

$$1, 1, 2, 6, 17, 50, 150, 458, 1420, 4460, 14165, \ldots.$$

This is A025272$(n+1)$. It has as generating function the power series given by

$$\frac{1-x-x^2}{1-2x} C\left(\frac{x^2(1-x-x^2)}{(1-2x)^2}\right).$$

The Hankel transform of the shifted sequence a_{n+1} in this case begins

$$1, 2, 3, -5, -28, -67, -411, -506, 10855, 74231, 664776, \ldots.$$

This is the $(1,-2)$ Somos-4 sequence A178376$(n+1)$, which is defined by the elliptic curve $y^2 + y = x^3 + 3x^2 + x$.

Example 22. The sequence $-a_{n+1}(-1,2,-2,-1,-1)$ begins

$$1, -2, 3, -3, -1, 15, -47, 98, -133, 17, 579, \ldots,$$

It has its generating function given by

$$\frac{1+x}{1+2x-x^2} C\left(\frac{-x^2(1+x)}{(1+2x-x^2)^2}\right).$$

Its Hankel transform begins

$$1, -1, 1, 2, -1, -3, \ldots$$

which is a $(1, 1)$ Somos-4 sequence. This is a variant of A006769, which is the elliptic divisibility sequence associated with elliptic curve $E : y^2 + y = x^3 - x$ and multiples of the point $(0, 0)$. In this case, the sequence $a_n(-1, 2, -2, -1, -1)$ begins

$$1, -1, 2, -3, 3, 1, -15, 47, -98, 133, -17, -579, \ldots$$

and its Hankel transform is the $(1, 1)$-Somos sequence beginning

$$1, 1, -2, -1, 3, -5, \ldots,$$

which is $(-1)^{n+1}$A006769$(n + 3)$.

Example 23. The sequence $a_n(-1, -2, 2, -1, -1)$ begins

$$1, -1, -2, -3, -5, -11, -27, -65, -154, -371, -917, -2303, \ldots.$$

Its generating function is given by

$$\frac{1 - 3x}{1 - 2x - x^2} C\left(\frac{x^2(1 - 3x)}{(1 - 2x - x^2)^2}\right).$$

Its Hankel transform begins

$$1, -3, 2, 11, -29, -21, \ldots$$

which is a $(1, 1)$ Somos-4 sequence. It is a variant of A178384, which is associated to the elliptic curve $y^2 + y = x^3 + x$. The shifted sequence $-a_{n+1}$ in this case has a Hankel transform that begins

$$1, -1, -3, -2, 11, 29, \ldots.$$

We have the following two conjectures regarding these sequences, their Hankel transforms, and Somos-4 sequences. The first conjecture claims that in all cases, the sequence $\frac{1}{p}a_{n+1}(p, q, r, s, t)$ has a Somos-4 Hankel transform.

Conjecture 24. The Hankel transform of the sequence $\frac{1}{p}a_{n+1}(p, q, r, s, t)$ is a $((pt)^2, -t^2(p^2 s + pqr - q^2))$ Somos-4 sequence.

Conjecture 25. If $r = 1$, $t = 1$ and $s = q - p + 1$, then the sequence $a_n(p, q, r, s, t) = a_n(p, q, 1, q - p + 1, 1)$ has a Hankel transform that is a $(p^2, p^3 - pq + q^2 - p^2(1 + q))$ Somos-4 sequence.

We note that in this last case, the generating function of the sequence a_n will be given by

$$\left(\frac{1 - (1 - p)x}{1 - x + (p + q - 1)x^2}, \frac{x^2(1 - (1 - p)x)}{(1 - x + (p + q - 1)x^2)^2}\right) \cdot C(x).$$

In another direction, it is interesting to consider sequences with generating functions of the form

$$\frac{1-x-\alpha x^2}{1-x-x^2}C\left(\frac{x^2(1-x-\alpha x^2)}{(1-x-x^2)^2}\right).$$

For these, we have the following conjecture.

Conjecture 26. The Hankel transform h_n of the sequence with generating function

$$\frac{1-x-\alpha x^2}{1-x-x^2}C\left(\frac{x^2(1-x-\alpha x^2)}{(1-x-x^2)^2}\right)$$

is given by

$$h_n = (2-\alpha)^{\lfloor\frac{(n+1)^2}{4}\rfloor}[x^n]\frac{(1+x)(1+(\alpha-2)x^2)}{1-3x^2-(\alpha-2)x^4}.$$

Note that

$$\frac{(1+x)(1+(\alpha-2)x^2)}{1-3x^2-(\alpha-2)x^4} = (1+x)f(x^2;\alpha),$$

where

$$f(x;\alpha) = \frac{1+(\alpha-2)x}{1-3x-(\alpha-2)x^2}.$$

For instance, when $\alpha = 1$, we get the generating function $\frac{1-x}{1-3x+x^2}$ of the bisected Fibonacci numbers F_{2n+1} $1, 2, 5, 13, 34, 89, \ldots$ A001519$(n+1)$. In general, if we set

$$\begin{aligned} d_n &= \sum_{k=0}^{n}\binom{k}{n-k}(\alpha-2)^{n-k}3^{2k-n} + (\alpha-2)\sum_{k=0}^{n}\binom{k}{n-k-1}(\alpha-2)^{n-k-1}3^{2k-n+1} \\ &= \sum_{k=0}^{n}\left(\binom{k}{n-k}+3\binom{k}{n-k-1}\right)(\alpha-2)^{n-k}3^{2k-n}, \end{aligned}$$

then we have

$$h_n = (2-\alpha)^{\lfloor\frac{(n+1)^2}{4}\rfloor}d_{\lfloor\frac{n}{2}\rfloor}.$$

Note that here $\binom{k}{n-k}+3\binom{k}{n-k-1}$ is the general term of the Riordan array $(1+3x, x(1+x))$. In fact, we have

$$f(x;\alpha) = \left(1+(\alpha-2)x, x\left(1+\frac{\alpha-2}{3}x\right)\right)\cdot\frac{1}{1-3x}.$$

Example 27. We take $\alpha = 1$, to get the equation

$$\left(1-x, x\left(1-\frac{x}{3}\right)\right)\cdot\frac{1}{1-3x} = \frac{1-x}{1-3x+x^2} = \sum_{n=0}^{\infty}F_{2n+1}x^n,$$

which is the generating function of the Fibonacci bisection F_{2n+1} that begins $1, 2, 5, 13, 34, \ldots$. We have $2 - \alpha = 1$. In this case the sequence we get begins

$$1, 0, 1, 1, 4, 7, 20, 43, 112, 263, 669, 1640, 4166, \ldots$$

with generating function

$$C\left(\frac{x^2}{1-x-x^2}\right).$$

The Hankel transform is then given by $F_{2\lfloor \frac{n}{2} \rfloor + 1}$, or

$$1, 1, 2, 2, 5, 5, 13, 13, 34, 34, \ldots.$$

Example 28. We take $\alpha = -1$. Then $2 - \alpha = 3$. The sequence in question has generating function

$$\left(\frac{1-x+x^2}{1-x-x^2}, \frac{x^2(1-x+x^2}{(1-x-x^2)^2}\right) \cdot C(x).$$

We obtain the sequence that begins

$$1, 0, 3, 3, 12, 21, 66, 147, 426, 1065, 3009, 7986, 22476, \ldots.$$

The Hankel transform begins

$$1, 3, 0, 0, -2187, -59049, -4782969, \ldots.$$

Dividing this by $3^{\lfloor \frac{(n+1)^2}{4} \rfloor}$, we obtain

$$1, 1, 0, 0, -3, -3, -9, \ldots$$

which is the expansion of $(1 + x)(1 - 3x^2)/(1 - 3x^2 + 3x^4)$. This is the doubling of the sequence

$$1, 0, -3, -9, -18, -27, -27, 0, 81, 243, 486, \ldots$$

which has generating function $\frac{1-3x}{1-3x+3x^2}$.

Example 29. The case of $\alpha = 2$ is of special interest. We have $2 - \alpha = 0$, so that the Hankel transform is just the sequence $1, 0, 0, 0, \ldots$. This is explained by the fact that

$$\left(\frac{1-x-2x^2}{1-x-x^2}, \frac{x^2(1-x-2x^2)}{(1-x-x^2)^2}\right) \cdot C(x) = \frac{1-x-2x^2}{1-x-x^2} C\left(\frac{x^2(1-x-2x^2)}{(1-x-x^2)^2}\right) = 1.$$

A choice of parameters (p, q, r, s, t) that satisfies this context is given by $(p, q, r, s, t) = (0, 2 - \alpha, 1, 3, 1)$. Thus the recurrence

$$a_n = a_{n-1} + 3a_{n-2} + \sum_{k=1}^{n-3} a_k a_{n-k-2},$$

with $a_0 = 1, a_1 = 0, a_2 = \alpha - 2$ will have a solution with generating function given by $\frac{1-x-\alpha x^2}{1-x-x^2} C\left(\frac{x^2(1-x-\alpha x^2)}{(1-x-x^2)^2}\right)$. We can reformulate our conjecture in the following manner.

Conjecture 30. The solution $a_n(\beta)$ of the recurrence

$$a_n = a_{n-1} + 3a_{n-2} + \sum_{k=1}^{n-3} a_k a_{n-k-2},$$

with $a_0 = 1, a_1 = 0, a_2 = \beta$ has a Hankel transform $h_n(\beta)$ given by

$$h_n(\beta) = \beta^{\lfloor \frac{(n+1)^2}{4} \rfloor} [x^n] \frac{(1+x)(1-\beta x^2)}{1-3x^2+\beta x^4}.$$

We can convert from the recurrence to the Riordan array and vice versa. Thus given (p, q, r, s, t), we obtain (a, b, c, d, m) as follows:

$$\begin{aligned} a &= p - r \\ b &= q - pr - s + t \\ c &= -r \\ d &= -s + 2t \\ m &= t \end{aligned}$$

Given (a, b, c, d, m), we obtain (p, q, r, s, t) as follows.

$$\begin{aligned} p &= a - c \\ q &= -ac + b + c^2 - d + m \\ r &= -c \\ s &= -d + 2m \\ t &= m. \end{aligned}$$

A related result is the following.

Proposition 31. *Assume that a_n satisfies the following convolution recursion relation:*

$$a_n = \begin{cases} 1, & \text{if } n = 0; \\ r, & \text{if } n = 1; \\ ra_{n-1} + sa_{n-2} + t\sum_{i=0}^{n-2} a_i a_{n-2-i}, & \text{if } n > 1. \end{cases}$$

Then the sequence a_n has generating function

$$\left(\frac{1}{1-rx-sx^2}, \frac{tx^2}{(1-rx-sx^2)^2} \right) \cdot C(x),$$

and the Hankel transform of a_n is a $((pt)^2, t^2(t+s)^2 - r^2t^3)$ Somos-4 sequence.

Note that $\sum_{i=0}^{n-2} a_i a_{n-2-i}$ expands to give $(a_0 a_{n-2} + \cdots + a_{n-2} a_0)$.

Proof. For a proof of the Hankel transform assertion, see [6]. We translate the recurrence to the equality

$$u(x) = 1 + rxu(x) + sx^2u(x) + tx^2u(x)^2.$$

Solving for $u(x)$ now gives the result. □

Examples of such sequences are given in the following table.

(r, s, t)	Annnnnn	Description
$(1, 1, 1)$	A128720	Skew Dyck paths avoiding UUU
$(2, 1, 1)$	A085139$(n+1)$	G.f. is $\mathcal{J}(2, 1, 2, 2, 1, 2, 2, 1, 2, \ldots; 2, 2, 1, 2, 2, 1, \ldots)$
$(1, 2, 1)$	A174171	Chebyshev transform of Motzkin numbers
$(2, 2, 1)$	A174403	Hankel transform is A174404
$(3, 1, 1)$	A084782$(n+1)$	A084782$(n) = \sum_{j=0}^{n} \sum_{i=0}^{j} a_i a_{j-i} F_{n-j}$
$(1, 2, -1)$	A187256	Peakless Motzkin paths, with level steps in two colors

The sequence a_n begins

$$1, r, r^2 + s + t, r(r^2 + 2s + 3t), r^4 + r^2(3s + 6t) + s^2 + 3st + 2t^2, \ldots.$$

The sequence with prepended 1 that begins $1, 1, r, r^2 + s + t, \ldots$ then has generating function

$$\left(\frac{1 - (r - t - 1)x - sx^2}{1 - (r - 2t)x - sx^2}, \frac{x(1 - (r - t - 1)x - sx^2}{(1 - (r - 2t)x - sx^2)^2}\right) \cdot C(x).$$

When $t = 1$, we get

$$\left(1, \frac{x}{1 - (r - 2)x - sx^2}\right) \cdot C(x).$$

In fact, we can modify the above proposition as follows.

Proposition 32. *Assume that a_n satisfies the following convolution recursion relation:*

$$a_n = \begin{cases} 1, & \text{if } n = 0; \\ p, & \text{if } n = 1; \\ ra_{n-1} + sa_{n-2} + t\sum_{i=0}^{n-2} a_i a_{n-2-i}, & \text{if } n > 1. \end{cases}$$

Then the sequence a_n has generating function

$$\left(\frac{1 + (p - r)x}{1 - rx - sx^2}, \frac{tx^2(1 + (p - r)x}{(1 - rx - sx^2)^2}\right) \cdot C(x),$$

and the Hankel transform of a_n is a $((pt)^2, t^2((t + s)^2 - p^2s + prs + prt - 2p^2t)$ Somos-4 sequence.

5 A further recurrence

We now consider the recurrence

$$a_n = ua_{n-1} + va_{n-2} + wa_{n-3} + t\sum_{k=1}^{n-4} a_k a_{n-k-3},$$

with $a_0 = 1$, $a_1 = p$, $a_2 = q$, and $a_3 = s$. We note that the term $\sum_{k=1}^{n-4} a_k a_{n-k-3}$ here expands to give $(a_1 a_{n-4} + \cdots + a_{n-4}a_1)$. We denote the solution of this recurrence by $a_n(p, q, s, u, v, w, t)$. We then have the following result.

Proposition 33. *The generating function of the sequence $a_n(p, q, s, u, v, w, t)$ is given by*

$$\left(\frac{1 + (p-u)x - (v + pu - q)x^2 - (w - s - t + qu + pv)x^3}{1 - ux - vx^2 - (w - 2t)x^3},\right.$$

$$\left.\frac{tx^3(1 + (p-u)x - (v + pu - q)x^2 - (w - s - t + qu + pv)x^3)}{(1 - ux - vx^2 - (w - 2t)x^3)^2}\right) \cdot C(x).$$

Example 34. The sequence $a_n(1, 1, 2, 1, 1, 1, 1)$ begins

$$1, 1, 1, 2, 4, 8, 16, 33, 69, 146, 312, 673, 1463, \ldots.$$

This is A004149$(n+1)$, which gives the number of Motzkin paths of length n with no peaks or valleys. The generating function of this sequence is given by

$$\frac{1}{1-x} C\left(\frac{x^3}{(1-x^2)(1-x)^2}\right).$$

Example 35. The sequence $a_n(1, 2, 3, 1, 1, 1, 1)$ which begins

$$1, 1, 2, 3, 6, 12, 25, 53, 114, 249, 550, 1227, 276, \ldots$$

is A162985. It counts the number of Dyck paths of semi-length n avoiding UUU, DDD and UUDUDD (Deutsch). Its generating function is given by

$$\frac{1}{1 - x - x^2 + x^3} C\left(\frac{x^3}{(1 - x - x^2 + x^3)^2}\right).$$

Note that this generating function can be expressed as the continued fraction

$$\cfrac{1}{1 - x - x^2 + x^3 - \cfrac{x^3}{1 - x - x^2 + x^3 - \cfrac{x^3}{1 - x - x^2 + x^3 \cdots}}}.$$

Example 36. The sequence $a_n(1,1,2,1,0,2,1)$ which begins

$$1,1,1,2,4,7,13,26,52,104,212,438,910,1903,4009,8494,\ldots$$

is A023431. It counts the number of Motzkin paths of length n with no peaks and no double rises. Its generating function is given by

$$g(x) = \left(\frac{1}{1-x}, \frac{x^3}{(1-x)^2}\right) \cdot C(x).$$

We then have

$$a_n(1,1,2,1,0,2,1) = \sum_{k=0}^{n} \binom{n-k}{2k} C_k.$$

It is interesting to note that the Riordan array $(g(x), xg(x))$ is a pseudo-involution in the Riordan group [1]. More generally, if

$$g(x;\alpha) = \left(\frac{1}{1-\alpha x}, \frac{x^3}{(1-\alpha x)^2}\right) \cdot C(x)$$

then the Riordan array $(g(x;\alpha), xg(x;\alpha))$ is a pseudo-involution. For example, $g(x;2)$ is the generating function of $a_n(2,4,9,2,0,2,1)$. We will then have

$$a_n(2,4,9,2,0,2,1) = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-k}{2k} 2^{n-3k} C_k.$$

This sequence is A091561 [1]. Its Hankel transform is the sequence

$$1,0,-1,-1,0,1,1,0,-1,-1,0,\ldots$$

with generating function $\frac{1-x}{1-x+x^2}$.

Example 37. The sequence $a_n(1,2,4,1,1,2,1)$ is the RNA sequence that begins

$$1,1,2,4,8,17,37,82,185,423,978,2283,5373,\ldots.$$

This is A004148, with generating function

$$\frac{1}{1-x-x^2} C\left(\frac{x^3}{(1-x-x^2)^2}\right).$$

The related sequence with generating function

$$\frac{1-x}{1-x-x^2} C\left(\frac{x^3(1-x)}{(1-x-x^2)^2}\right)$$

begins

$$1, 0, 1, 2, 3, 7, 14, 28, 60, 126, 268, 579, 1253, \ldots.$$

Its Hankel transform begins

$$1, 1, -2, -3, -7, 5, 32, 83, 87, -821, -2366, \ldots.$$

This is a $(1, -1)$ Somos-4 sequence.

Example 38. The sequence $a_n(1, 3, 6, 1, 2, 2, 1)$ begins

$$1, 1, 3, 6, 14, 33, 79, 194, 482, 1214, 3090, 7936, \ldots.$$

This sequence has its generating function given by

$$\frac{1}{1 - x - 2x^2} C\left(\frac{x^3}{(1 - x - 2x^2)^2}\right).$$

Its Hankel transform begins

$$1, 2, 1, -7, -16, -57, -113, 670, 3983, 23647, \ldots.$$

This is a $(1, -2)$ Somos-4 sequence.

In general, we can conjecture that the sequences with generating function

$$\frac{1}{1 - x - \alpha x^2} C\left(\frac{x^3}{(1 - x - \alpha x^2)^2}\right)$$

have Hankel transforms that are $(1, -\alpha)$ Somos-4 sequences. Such Hankel transforms begin

$$1, \alpha, -1 + \alpha, -1 + \alpha - \alpha^3, -2\alpha + 3\alpha^2 - \alpha^3 - \alpha^4, 1 - 3\alpha + 3\alpha^2 - 2\alpha^3 + \alpha^4 - \alpha^6,$$

$$1 - 3\alpha + 3\alpha^2 + 3\alpha^3 - 9\alpha^4 + 6\alpha^5 + \alpha^6 - 2\alpha^7,$$

$$3\alpha - 12\alpha^2 + 19\alpha^3 - 11\alpha^4 - 3\alpha^5 + 5\alpha^6 + 2\alpha^7 - 3\alpha^8 + \alpha^{10}, \ldots.$$

For instance, the Hankel transform of the sequence with generating function

$$\frac{1}{1 - x + x^2} C\left(\frac{x^3}{(1 - x + x^2)^2}\right)$$

is the $(1, 1)$ Somos-4 sequence A178627$(n + 1)$. The sequence A178627 is defined by the elliptic curve

$$E : y^2 + xy - y = x^3 - x^2 + x.$$

These examples show that such sequences merit more study. Further evidence of this is given by the following conjecture concerning their Hankel transforms.

Conjecture 39. The Hankel transform of the sequence with generating function

$$\frac{1-x-x^2-\alpha x^3}{1-x-x^2-x^3}C\left(\frac{x^3(1-x-x^2-\alpha x^3)}{(1-x-x^2-x^3)^2}\right)$$

is given by

$$h_n = A_n(\alpha)(2-\alpha)^{B_n}$$

where

$$A_n(\alpha) = [x^n]\frac{1+(\alpha-2)x^2-(\alpha-2)x^3+(4\alpha-5)x^5-(\alpha-1)(\alpha-2)x^8}{1+3x^3-(\alpha-2)x^6},$$

and

$$B_n = [x^n]\frac{x(1-x+2x^2-2x^3+3x^4-3x^5+x^6)}{(1-x)^2(1-x^3)}.$$

The sequence B_n begins

$$0, 1, 1, 3, 4, 7, 9, 11, 15, 18, 21, 26, 30, \ldots.$$

The quasi-polynomial sequence A236337 is related to this.

Example 40. For $\alpha = 2$, we find that

$$\frac{1-x-x^2-2x^3}{1-x-x^2-x^3}C\left(\frac{x^3(1-x-x^2-2x^3)}{(1-x-x^2-x^3)^2}\right) = 1.$$

Thus the Hankel transform is the sequence $1, 0, 0, 0, \ldots$.

Example 41. When $\alpha = -1$, we obtain the sequence that begins

$$1, 0, 0, 3, 3, 6, 18, 33, 69, 165, 351, 768, 1758, 3921, 8811, 20130, \ldots.$$

The Hankel transform of this begins

$$1, 0, -9, 0, 0, 0, -59049, 0, 43046721, 3486784401, 0, \ldots.$$

This is given by

$$h_n = 3^{B_n}[x^n]\frac{1-3x^2+3x^3-9x^5-6x^8}{1+3x^3+3x^6}.$$

6 Conversion of parameters

Given a generating function of the form

$$g(x) = \left(\frac{1 + ax + bx^2 + cx^3}{1 + dx + ex^2 + fx^3}, \frac{mx^3(1 + ax + bx^2 + cx^3)}{(1 + dx + ex^2 + fx^3)^2} \right) \cdot C(x),$$

what are the corresponding parameters p, q, s, u, v, w, t? We find the following.

$$\begin{aligned} p &= a - d \\ q &= -ad + b + d^2 - e \\ s &= a(d^2 - e) + c - bd - d^3 + 2de - f + m \\ u &= -d \\ v &= -e \\ w &= -f + 2m \\ t &= m. \end{aligned}$$

Example 42. The doubly aerated large Schroeder numbers, with generating function

$$\left(\frac{1}{1 - x^3}, \frac{x^2}{(1 - x^3)^2} \right) \cdot C(x),$$

correspond to the recurrence with parameters $(0, 0, 2, 0, 0, 3, 1)$. This is the recurrence

$$a_n = 3a_{n-3} + \sum_{k=1}^{n-4} a_k a_{n-k-3},$$

with $a_0 = 1$, $a_1 = 0$, $a_2 = 0$ and $a_3 = 2$. The resulting sequence

$$1, 0, 0, 2, 0, 0, 6, 0, 0, 22, 0, 0, 90, 0, 0, 394, 0, 0, \ldots$$

has a Hankel transform which begins

$$1, 0, -4, -8, 0, 128, 512, 0, -32768, -262144, 0, \ldots.$$

Dividing this by 2^n gives us the sequence that begins

$$1, 0, -1, -1, 0, 4, 8, 0, -128, -512, 0, \ldots.$$

This suggests that the generating function of this Hankel transform may satisfy the functional equation

$$1 - x^2 - x^3 f(x) - f(x/2) = 0.$$

7 From elliptic curve to recurrences and Somos sequences

We consider a particular example which illustrates a process that starts with an elliptic curve. Thus we consider the elliptic curve

$$E : y^2 - 3xy - y = x^3 - x.$$

We start by solving this for y. We get two solutions,

$$y^- = \frac{1 + 3x - \sqrt{1 + 2x + 9x^2 + 4x^3}}{2},$$

and

$$y^+ = \frac{1 + 3x + \sqrt{1 + 2x + 9x^2 + 4x^3}}{2}.$$

Expanding the right hand side of the first solution y^-, we get

$$0, 1, -2, 1, 3, -7, -4, 38, -27, -175, 384, \ldots.$$

Expanding the right hand side of the second solution y^+, we get

$$1, 2, 2, -1, -3, 7, 4, -38, 27, 175, -384, \ldots.$$

The solution y^- has a generating function that can be expressed as

$$\frac{x(1-x^2)}{1+3x} C\left(\frac{x(1-x)^2}{(1+3x)^2}\right).$$

Thus the sequence that begins

$$1, -2, 1, 3, -7, -4, 38, -27, -175, 384, \ldots$$

has n-th term given by

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k+1} \binom{k+1}{j} \binom{n+k-2j}{n-k-2j} (-1)^j (-3)^{n-k-2j} \right) C_k.$$

We are interested in the terms (apart from sign) that are common to these two solutions. We thus truncate the sequence to start with $2, -1, -3, 7, 4, -38, 27, 175, -384, \ldots$. This leads to the new generating function

$$g(x) = \left(\frac{1 + 3x + \sqrt{1 + 2x + 9x^2 + 4x^3}}{2} - 1 - 2x \right) / x^2 = \frac{\sqrt{1 + 2x + 9x^2 + 4x^3} - x - 1}{2x^2}.$$

This sequence has general term

$$\sum_{k=0}^{n}\sum_{j=0}^{k+1}\binom{k+1}{j}\binom{n-j}{n-2k-j}(-1)^{n-k-j}2^{k+1-j}C_k,$$

or equivalently,

$$\sum_{k=0}^{n+1}\left(\sum_{j=0}^{k+1}\binom{k+1}{j}\binom{n+1+k-2j}{n+1-k-2j}(-1)^{j+1}(-3)^{n+1-k-2j}\right)C_k.$$

Now we find that

$$g(x)=\frac{2+x}{1+x}C\left(\frac{-x^2(2+x)}{(1+x)^2}\right)=\frac{\sqrt{1+2x+9x^2+4x^3}-x-1}{2x^2}.$$

Re-writing this as

$$2\frac{1+\frac{x}{2}}{1+x}C\left(\frac{-2x^2(1+\frac{x}{2})}{(1+x)^2}\right),$$

we see that the sequence $2,-1,-3,7,4,-38,27,175,-384,\ldots$ defined by the elliptic curve is $2d_n$, where d_n is the solution of the recurrence

$$d_n=-d_{n-1}-4d_{n-2}-2\sum_{k=1}^{n-3}d_kd_{n-k-2},$$

with

$$d_0=1,d_1=-\frac{1}{2},d_2=-\frac{3}{2}.$$

The Hankel transform of this sequence $2d_n$ begins

$$2,-7,-57,670,23647,-833503,-147165662,\ldots$$

which is a $(1,16)$ Somos-4 sequence.

We now form the generating function

$$\frac{1}{1-x+x^2g(x)}=\frac{2}{1-3x+\sqrt{1+2x+9x^2+4x^3}}.$$

This gives us a sequence whose initial term is 1 and whose Hankel transform prepends a 1 to the previous transform. Multiplying this by x and reverting, and then dividing by x, we obtain the generating function

$$\frac{1+3x-\sqrt{1+6x+9x^2-4x^3-8x^4}}{2x^3}=\frac{1+2x}{1+3x}C\left(\frac{x^2(1+2x)}{(1+3x)^2}\right).$$

This expands to give the sequence a_n that begins

$$1, -1, 3, -8, 22, -59, 155, -396, 978, -2310, 5122, \ldots,$$

which is thus the solution to the recurrence

$$a_n = -3a_{n-1} + 2a_{n-3} + \sum_{k=1}^{n-4} a_k a_{n-k-3},$$

with $a_0 = 1, a_1 = -1, a_2 = 3, a_3 = -8$. The Hankel transform of this sequence begins

$$1, 2, 1, -7, -16, -57, -113, 670, 3983, 23647, 140576, \ldots.$$

This is a $(1, -2)$ Somos-4 sequence. It is A178622$(n+2)$. We note that the previous Hankel transform $2, -7, -57, 670, 23647, -833503, -147165662, \ldots$ is a bisection of this latter sequence.

Taking the second binomial transform $\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} a_k$ of the sequence a_n, we obtain the sequence that begins

$$1, 1, 3, 6, 14, 33, 79, 194, 482, 1214, 3090, 7936, \ldots.$$

We have met this already. It is $a_n(1, 3, 6, 1, 2, 2, 1)$ with generating function

$$\frac{1}{1 - x - 2x^2} C\left(\frac{x^3}{1 - x - 2x^2}\right).$$

It is interesting to note that the sequence b_n that begins

$$0, 2, 1, 1, 3, 6, 14, 33, 79, 194, 482, 1214, 3090, 7936, \ldots$$

satisfies the recurrence

$$b_n = b_{n-1} + \sum_{i=0}^{n-3} b_i b_{n-i-1},$$

with $b_0 = 0, b_1 = 2, b_2 = 1$. See also A025243.

We have the following coordinates for the points $nP(0, 0)$ on the elliptic curve

$$E : y^2 - 3xy - y = x^3 - x.$$

$x(nP)$	0	-2	$-\frac{1}{4}$	14	$\frac{16}{49}$	$\frac{-399}{256}$	$\frac{-1808}{3249}$
$y(nP)$	0	-3	$\frac{5}{8}$	78	$\frac{55}{343}$	$\frac{-11921}{4096}$	$\frac{68464}{185193}$
$\frac{y}{x}$	1	$\frac{3}{2}$	$-\frac{5}{2}$	$\frac{39}{7}$	$\frac{55}{122}$	$\frac{703}{912}$	$-\frac{4279}{6441}$

We form the continued fraction

$$\cfrac{1}{1+x-\cfrac{2x^2}{1+\frac{3x}{2}-\cfrac{\frac{x^2}{4}}{1-\frac{5x}{2}+\cfrac{14x^2}{1+\frac{39x}{7}+\cfrac{\frac{16x^2}{49}}{1+\frac{55x}{112}-\cfrac{\frac{399x^2}{256}}{1-\ldots}}}}}}.$$

This expands to give the sequence $\tilde{a}_n$

$$1, -1, 3, -8, 22, -59, 155, -396, 978, -2310, 5122, \ldots$$

with g.f

$$\left(\frac{1+2x}{1+3x}, \frac{x^3(1+2x)}{(1+3x)^2}\right) \cdot C(x).$$

We have

$$\tilde{a}_n = \sum_{k=0}^{n} \sum_{j=0}^{k+1} \binom{k+1}{j} \binom{n-k-j}{n-3k-j} 2^j (-3)^{n-3k-j} C_k.$$

In this case, we have $\tilde{a}_n = a_n$, where a_n is the sequence obtained starting from solving the elliptic curve equation.

We briefly look at the shifted sequence

$$-1, -3, 7, 4, -38, 27, 175, -384, \ldots$$

of the sequence first encountered in this section. This has a generating function given by

$$-\left(\frac{1+4x}{1+x+4x^2}, \frac{x^3(1+4x)}{(1+x+4x^2)^2}\right) \cdot C(x).$$

It has a Hankel transform that begins

$$-1, -16, 113, 3983, -140576, -14871471, \ldots.$$

Again, this is a $(1, 16)$ Somos-4 sequence.

The elliptic curve

$$E : y^2 - 3xy - y = x^3 - x$$

thus gives rise to the following Riordan arrays

- $\left(\frac{1-x^2}{1+3x}, \frac{x(1-x^2)}{(1+3x)^2}\right)$

- $\left(\frac{2+x}{1+x}, -\frac{x^2(2+x)}{(1+x)^2}\right)$
- $\left(\frac{1+4x}{1+x+4x^2}, \frac{x^3(1+4x)}{(1+x+4x^2)^2}\right)$
- $\left(\frac{1}{1-3x}, \frac{x(2+x^2)}{(1-3x)^2}\right)$
- $\left(\frac{1+2x}{1+3x}, \frac{x^3(1+2x)}{(1+3x)^2}\right)$
- $\left(\frac{1}{1-x-2x^2}, \frac{x^3}{(1-x-2x^2)^2}\right)$

with their corresponding recurrences.

8 The family $E_t : y^2 + 4xy + y = x^3 + (t-1)x^2 + tx$

The family of elliptic curves

$$E_t : y^2 + 4xy + y = x^3 + (t-1)x^2 + tx$$

has the property that each of its curves passes through the points $(0,0), (-1,1)$ and $(-1,2)$. Solving the equation

$$y^2 + 4xy + y = x^3 + (t-1)x^2 + tx$$

gives

$$y = -\frac{1 + 4x \pm \sqrt{1 + 4(t+2)x + 4(t+3)x^2 + 4x^3}}{2}.$$

For instance, $-\frac{1+4x-\sqrt{1+4(t+2)x+4(t+3)x^2+4x^3}}{2}$ expands to give

$$0, t, -(t^2 + 3t + 1), 2t^3 + 10t^2 + 14t + 5, \ldots.$$

The generating function of this sequence can be expressed as

$$\frac{x(t + x(t-1) + x^2}{1 + 4x} C\left(\frac{-x(t + x(t-1) + x^2)}{(1+4x)^2}\right).$$

As before, we are interested in the terms (up to sign) that are common to the two solutions. Thus we focus on the sequence b_n that begins

$$(t^2 + 3t + 1), -(2t^3 + 10t^2 + 14t + 5), \ldots.$$

This has generating function

$$g(x) = \frac{1 + 3t + t^2 - x}{1 + 2(t+2)x} C\left(\frac{x^2(1 + 3t + t^2 - x)}{(1 + 2(t+2)x)^2}\right).$$

We have the following conjecture.

Conjecture 43. The Hankel transform of b_n is an

$$((2t^3+10t^2+14t+5)^2, -3t^8-40t^7-222t^6-666t^5-$$

$$1173t^4-1230t^3-740t^2-29(8t+1))$$

Somos-4 sequence.

The generating function $\frac{1}{1-x-x^2g(x)}$ will now give us a sequence whose initial term is 1, which has the same Hankel transform, with a prepended 1. We obtain that

$$\frac{1}{1-x-x^2g(x)} = \frac{1}{1-2(t+3)x}C\left(\frac{-x(x^2-(t+2)(t+3)x+2t+5)}{(1-2(t+3)x)^2}\right).$$

We now revert the generating function $\frac{x}{1-x-x^2g(x)}$, and divide the result by x, to get the generating function

$$a(x) = \frac{1+(2t+5)x}{1+2(t+3)x+(t+2)(t+3)x^2}C\left(\frac{x^3(1+(2t+5)x)}{(1+2(t+3)x+(t+2)(t+3)x^2)^2}\right).$$

This expands to give a sequence a_n that begins

$$1, -1, -t(t+3), 2t^3+13t^2+23t+7, -3t^4-30t^3-103t^2-134t-44, \ldots.$$

The sequence a_n therefore satisfies the following recurrence.

$$a_n = -2(t+3)a_{n-1} - (t+2)(t+3)a_{n-2} + 2a_{n-3} + \sum_{k=1}^{n-k-4} a_k a_{n-k-3},$$

with $a_0 = 1, a_1 = -1, a_2 = -t(t+3), a_3 = 2t^3+13t^2+23t+7$. We have the following conjecture concerning the Hankel transform of this sequence.

Conjecture 44. The Hankel transform of the sequence whose generating function is given by

$$\frac{1+(2t+5)x}{1+2(t+3)x+(t+2)(t+3)x^2}C\left(\frac{x^3(1+(2t+5)x)}{(1+2(t+3)x+(t+2)(t+3)x^2)^2}\right)$$

is a $(1, t^2+3t+1)$ Somos-4 sequence.

Example 45. When $t=0$, we get the sequence with generating function

$$\frac{1+5x}{1+6x+6x^2}C\left(\frac{x^3(1+5x)}{(1+6x+6x^2)^2}\right),$$

which begins

$$1, -1, 0, 7, -44, 223, -1060, 4920, -22626, 103719, -475214, \ldots.$$

This sequence satisfies then the recurrence

$$a_n = -6a_{n-1} - 6a_{n-3} + 2a_{n-3} + \sum_{k=1}^{n-4} a_k a_{n-k-3}$$

with

$$a_0 = 1, a_1 = -1, a_2 = 0, a_3 = 7.$$

The Hankel transform of this sequence is then the $(1,1)$ Somos-4 sequence A157101 which begins

$$1, -1, -5, -4, 29, 129, -65, -3689, -16264, \ldots.$$

The associated elliptic curve is

$$E_0 : y^2 + 4xy + y = x^3 - x^2.$$

Example 46. When $t = -3$, we obtain the sequence a_n with generating function

$$(1-x)C(x^3(1-x)) = \frac{1 - \sqrt{1 - 4x^3 + 4x^4}}{2x^3}$$

which begins

$$1, -1, 0, 1, -2, 1, 2, -6, 6, 3, -20, \ldots.$$

This sequence then satisfies the recurrence

$$a_n = 2a_{n-3} + \sum_{k=1}^{n-4} a_k a_{n-k-3}$$

with

$$a_0 = 1, a_1 = -1, a_2 = 0, a_3 = 1.$$

Its Hankel transform is the $(1,1)$ Somos-4 sequence A006769$(n+2)$ that begins

$$1, -1, 1, 2, -1, -3, -5, 7, -4, -23, 29, \ldots.$$

The associated elliptic curve is

$$E_{(-3)} : y^2 + 4xy + y = x^3 - 4x^2 - 3x.$$

We can relate the sequence to the coordinates of the multiples of the point $(0,0)$ on this curve in the following way. Taking the second binomial transform of $(-1)^n a_n$, given by

$$d_n = \sum_{k=0}^{n} 2^{n-k}(-1)^k a_k,$$

and then taking the INVERT(4) transform of d_n, we arrive at the sequence $\tilde{a}_n$ that begins

$$1, -1, 0, -1, -2, -5, -10, -14, 6, 145, 720, 2618, 7850, 19389, 35016, \ldots.$$

Due to the invariance of the Hankel transform under binomial and INVERT transforms, this sequence has the same Somos-4 Hankel transform as a_n. The sequence $\tilde{a}_n$ has a generating function that can be expressed as

$$\frac{1-x}{1+4x-4x^2} C\left(\frac{x(4-4x-x^2-x^3)}{(1+4x-4x^2)^2}\right).$$

We can express this as the continued fraction

$$\cfrac{1}{1+x+\cfrac{x^2}{1-2x-\cfrac{x^2}{1-x+\cfrac{2x^2}{1-\frac{7x}{2}+\cfrac{\frac{x^2}{4}}{1-\frac{9x}{2}+\cfrac{6x^2}{1-\ldots}}}}}}.$$

This corresponds to the following x and y coordinates of the multiples of $P(0,0)$ on $E_{(-3)}$.

$x(nP)$	0	1	-1	2	$\frac{1}{4}$	6	$-\frac{5}{9}$
$y(nP)$	0	-2	1	-7	$-\frac{9}{8}$	2	$\frac{38}{27}$
$\frac{y}{x}$	1	2	-1	$\frac{-7}{2}$	$\frac{-9}{2}$	$\frac{1}{3}$	$-\frac{38}{15}$

Thus we can write the generating function $\sum_{n=0}^{\infty} \tilde{a}_n t^n$ as

$$\cfrac{1}{1+z(0)t+\cfrac{x(1)t^2}{1+z(1)t+\cfrac{x(2)t^2}{1+z(2)t+\cdots}}},$$

where $z(n) = \frac{y(nP)}{x(nP)}$ and $x(n) = x(nP)$. The sequence $\tilde{a}_n$ is the moment sequence for the family of orthogonal polynomials $Q_n(x)$ with a coefficient matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 & 0 & 0 \\ -2 & 3/2 & 8 & -11/2 & 1 & 0 & 0 \\ 9 & -9 & -35 & 35 & -10 & 1 & 0 \\ -9 & 15 & 82/3 & -57 & 107/3 & -29/3 & 1 \end{pmatrix}.$$

The inverse M of this matrix (the moment matrix) contains $\tilde{a}_n$ as its first column. The matrix M begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 2 & 1 & 0 & 0 & 0 \\ -2 & 7 & 3 & 11/2 & 1 & 0 & 0 \\ -5 & 15 & -1 & 22 & 10 & 1 & 0 \\ -10 & 24 & -30 & 147/2 & 61 & 29/3 & 1 \end{pmatrix}.$$

The production matrix of M then begins

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 7/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/4 & 9/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -6 & -1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5/9 & 38/15 \end{pmatrix}.$$

The tri-diagonal nature of this production matrix shows that we are in the presence of a family of orthogonal polynomials. The diagonal and sub-diagonal contain the x coordinates and the y/x-ratios of the multiples of the point $P(0,0)$ on $E_{(-3)}$.

The family $Q_n(t)$ is defined by the three-term recurrence

$$Q_n(t) = \left(t + \frac{x((n-1)P)}{y((n-1)P)}\right) Q_{n-1}(t) + x((n-1)P)Q_{n-2}(t),$$

where $P = (0,0)$ on $E_{(-3)}$ and we have $Q_0(t) = 1, Q_1(t) = t + 1$.

Using results about the divisibility polynomials ψ_n of an elliptic curve [20], we can express the x and y coordinates of multiples of the point $P(0,0)$ for the curve $E_{(-3)}$ as follows.

$$x(nP) = \frac{-\psi_{n-1}\psi_{n+1}}{\psi_n^2},$$

and

$$y(nP) = \frac{1}{2}\left(\frac{\psi_{2n}}{\psi_n^4} - \left(1 - 4\frac{\psi_{n-1}\psi_{n+1}}{\psi_n^2}\right)\right),$$

where we have used the shorthand $\psi_n = \psi_n(0,0)$ in the above expressions. Alternatively, we have

$$x((n+1)P) = \frac{-\tilde{h}_{n-1}\tilde{h}_{n+1}}{\tilde{h}_n^2},$$

and

$$y((n+1)P) = \frac{1}{2}\left(\frac{\tilde{h}_{2(n+1)}}{\tilde{h}_n^4} + a_1\frac{\tilde{h}_{n-1}\tilde{h}_{n+1}}{\tilde{h}_n^2} - a_3\right),$$

where $a_1 = 4$ and $a_3 = 1$. Here, $\tilde{h}_n$ is the Hankel transform of $\tilde{a}_n$.

We note finally that an application of the Stieltjes-Perron transform suggests that the absolutely continuous part of the associated measure is given by

$$\frac{1}{\pi}\frac{\sqrt{-x^4-4(2x^3+6x^2+7x+3)}}{2(4x^2-1)}.$$

9 The case of $E : y^2 + axy + y = x^3 + bx^2 + cx$

We let $P = P(0,0)$ be the point $(0,0)$ on the elliptic curve

$$E : y^2 + axy + y = x^3 + bx^2 + cx,$$

and let $\tilde{a}_n$ be the sequence with generating function

$$\tilde{a}(t) = \cfrac{1}{1+t+\cfrac{x_1t^2}{1+\frac{y_1}{x_1}t+\cfrac{x_2t^2}{1+\frac{y_2}{x_2}t+\cfrac{x_3t^2}{1+\frac{y_3}{x_3}t+\cdots}}}},$$

where $x_n = x(nP)$ and $y_n = y(nP)$ are the coordinates of the multiples nP of the point P on the curve. We can conjecture that the generating function of $\tilde{a}_n$ is given by

$$(g(x), f(x)) \cdot C(x),$$

where

$$g(x) = \frac{1}{1-(a+2\gamma)x-(b-2\gamma(a+c))x^2},$$

and

$$f(x) = \frac{x(x^3(a+c)(\gamma^2(a+c)-b\gamma+1)-x^2(2c\gamma^2+\gamma(a^2+a(3\Delta c-2)-b)+1)+x\gamma(2a+2c-1)-\gamma)}{(1-(a+2\gamma)x-(b-2\gamma(a+c))x^2)^2}.$$

Here, we have used the notation

$$\gamma = c - 1.$$

We see that there are simplifications when $c = 1$. In this case, we find that the generating function of $\tilde{a}_n$ is given by

$$\frac{1-(a+1)x}{1-ax-bx^2}C\left(\frac{-x^3(1-(a+1)x)}{(1-ax-bx^2)^2}\right).$$

This allows us to give a closed form expression for $\tilde{a}_n$ when $c = 1$.

$$\tilde{a}_n = \sum_{k=0}^{n}(\sum_{j=0}^{k+1}\binom{k+1}{j}(-a-1)^j\sum_{i=0}^{n-3k-j}\binom{2k+i}{i}\binom{i}{n-3k-i-j}b^{n-3k-i-j}a^{2i+3k+j-n})(-1)^kC_k.$$

This sequence begins

$$1, -1, b-a, -a^2+ab-b-1, -a^3+a^2b-a(2b+1)+b^2+2,$$

$$-a^4+a^3b-a^2(3b+1)+a(2b^2+4)-b^2-3b-1, \ldots,$$

with a Hankel transform that begins

$$1, -a+b-1, -a^2+a(b-3)+2b-3,$$

$$a^3+a^2(2-3b)+a(3b^2-5b)-b^3+3b^2-b-2, \ldots,$$

which we conjecture to be a $(1, a-b+1)$ Somos-4 sequence.

The sequence $\tilde{a}_n$ for $c=1$ satisfies the convolution recurrence

$$\tilde{a}_n = a\tilde{a}_{n-1} + b\tilde{a}_{n-2} - 2\tilde{a}_{n-3} - \sum_{k=1}^{n-k-4} \tilde{a}_k \tilde{a}_{n-k-3},$$

with $\tilde{a}_0 = 1, \tilde{a}_1 = -1, \tilde{a}_2 = b-a, \tilde{a}_3 = -a^2+ab-b-1$.

For the related sequence a_n, calculated from the solution of the elliptic curve equation, we can calculate that its generating function is given by

$$(g(x), f(x)) \cdot C(x)$$

where

$$g(x) = \frac{1+(1+a+2c)x}{1+(a+2c+2)x+(a(c+1)+(c+1)^2-b)x^2},$$

$$f(x) = \frac{x^3(1+(1+a+2c)x)}{(1+(a+2c+2)x+(a(c+1)+(c+1)^2-b)x^2)^2}.$$

This means that a_n satisfies the following convolution recurrence.

$$a_n = -(a+2(c+1))a_{n-1} + (b-a(c+1)-(c+1)^2)a_{n-2} + 2a_{n-3} + \sum_{k=1}^{n-4} a_k a_{n-k-3},$$

with $a_0 = 1, a_1 = -1, a_2 = 1+b-ac-c^2, a_3 = (3+a+2c)(-b+c(a+c))$.

The relationship between the generating function $\tilde{a}(x) = \sum_{n=0}^{\infty} \tilde{a}_n x^n$ and the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$ is the following: $\tilde{a}(x)$ is the INVERT$(-(c+1))$ of the $(c+1)$-st binomial transform of $a(-x)$.

10 Conclusions

We have found that the language of Riordan arrays is an appropriate one to investigate structural aspects of the solutions of generalized Catalan or Schroeder recurrences. The appearance of the Catalan numbers is explained by the quadratic nature of the convolution recurrences. This quadratic theme is continued in terms of solving elliptic curve equations in the quadratic term, which leads to generating functions that are Riordan array solutions of convolution equations. Often, the Hankel transform of the solution sequences give rise to Somos-4 sequences. By using the x and y coordinates of multiples of a special point on the elliptic curves, we can obtain Stieltjes continued fraction expressions for the generating functions of related integer sequences. In specific instances, this leads to orthogonal polynomials and their three-term recurrences, opening the door to the investigation of associated measures. Note that we work with points on curves for which the division polynomials never evaluate to 0.

The association of Somos-4 sequences and the Hankel transform is implicit in previous work (see the Appendix) [10, 21, 20]. What is apparently new is the use of the $\frac{y(nP)}{x(nP)}$ ratios to define *integer* sequences whose Hankel transforms furnish Somos sequences.

Proofs of results concerning Hankel transforms and Somos sequences can be elusive (but see [10, 11, 12, 24, 25], hence we couch some proposed results as conjectures. It is hoped that further insight will remedy this in the future.

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12 Appendix: Orthogonal polynomials, Riordan arrays and the Hankel transform

By an *orthogonal polynomial sequence* $(p_n(x))_{n\geq 0}$ we shall understand [7, 8] an infinite sequence of polynomials $p_n(x)$, $n \geq 0$, of degree n, with real coefficients (often integer coefficients) that are mutually orthogonal on an interval $[x_0, x_1]$ (where $x_0 = -\infty$ is allowed, as well as $x_1 = \infty$), with respect to a weight function $w : [x_0, x_1] \to \mathbb{R}$:

$$\int_{x_0}^{x_1} p_n(x)p_m(x)w(x)dx = \delta_{nm}\sqrt{h_n h_m},$$

where

$$\int_{x_0}^{x_1} p_n^2(x)w(x)dx = h_n.$$

We assume that w is strictly positive on the interval (x_0, x_1). Every such sequence obeys a so-called "three-term recurrence":

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x)$$

for coefficients a_n, b_n and c_n that depend on n but not x. We note that if

$$p_j(x) = k_j x^j + k'_j x^{j-1} + \cdots \qquad j = 0, 1, \ldots$$

then

$$a_n = \frac{k_{n+1}}{k_n}, \qquad b_n = a_n \left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n} \right), \qquad c_n = a_n \left(\frac{k_{n-1} h_n}{k_n h_{n-1}} \right),$$

where

$$h_i = \int_{x_0}^{x_1} p_i(x)^2 w(x)\, dx.$$

Since the degree of $p_n(x)$ is n, the coefficient array of the polynomials is a lower triangular (infinite) matrix. In the case of monic orthogonal polynomials the diagonal elements of this array will all be 1. In this case, we can write the three-term recurrence as

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0.$$

The *moments* associated to the orthogonal polynomial sequence are the numbers

$$\mu_n = \int_{x_0}^{x_1} x^n w(x) dx.$$

We can find $p_n(x)$, α_n and β_n from a knowledge of these moments. To do this, we let Δ_n be the Hankel determinant $|\mu_{i+j}|_{i,j\geq 0}^n$ and $\Delta_{n,x}$ be the same determinant, but with the last row equal to $1, x, x^2, \ldots$. Then

$$p_n(x) = \frac{\Delta_{n,x}}{\Delta_{n-1}}.$$

More generally, we let $H \begin{pmatrix} u_1 & \ldots & u_k \\ v_1 & \ldots & v_k \end{pmatrix}$ be the determinant of Hankel type with (i,j)-th term $\mu_{u_i+v_j}$. Let

$$\Delta_n = H \begin{pmatrix} 0 & 1 & \ldots & n \\ 0 & 1 & \ldots & n \end{pmatrix}, \qquad \Delta' = H \begin{pmatrix} 0 & 1 & \ldots & n-1 & n \\ 0 & 1 & \ldots & n-1 & n+1 \end{pmatrix}.$$

Then we have

$$\alpha_n = \frac{\Delta'_n}{\Delta_n} - \frac{\Delta'_{n-1}}{\Delta_{n-1}}, \qquad \beta_n = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}.$$

Of importance to this study are the following results (the first is the well-known "Favard theorem"), which we essentially reproduce from [12]:

Theorem 47. *[12] (Cf. [22], Théorème 9 on p.I-4, or [23], Theorem 50.1). Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree $n = 0, 1, \ldots$ Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$ with $\beta_n \neq 0$ for all $n \geq 1$, such that the three-term recurrence*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n(x), \quad \text{for} \quad n \geq 1,$$

holds, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \alpha_0$.

Theorem 48. *[12] (Cf. [22], Proposition 1, (7), on p. V-5, or [23], Theorem 51.1). Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, which is orthogonal with respect to some functional L. Let*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n(x), \quad \text{for} \quad n \geq 1,$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = L(x^k)$ satisfies

$$g(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$

Given a family of monic orthogonal polynomials

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0,$$

we can write

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k.$$

Then we have

$$\sum_{k=0}^{n+1} a_{n+1,k} x^k = (x - \alpha_n) \sum_{k=0}^{n} a_{n,k} x^k - \beta_n \sum_{k=0}^{n-1} a_{n-1,k} x^k$$

from which we deduce

$$a_{n+1,0} = -\alpha_n a_{n,0} - \beta_n a_{n-1,0} \tag{1}$$

and

$$a_{n+1,k} = a_{n,k-1} - \alpha_n a_{n,k} - \beta_n a_{n-1,k} \tag{2}$$

The question immediately arises as to the conditions under which a Riordan array (g, f) can be the coefficient array of a family of orthogonal polynomials. A partial answer is given by the following proposition.

Proposition 49. *Every Riordan array of the form*

$$\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$$

is the coefficient array of a family of monic orthogonal polynomials.

We note that in this case the three-term recurrence coefficients α_n and β_n are constants. We can strengthen this result as follows.

Proposition 50. *Every Riordan array of the form*

$$\left(\frac{1-\lambda x-\mu x^2}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$$

is the coefficient array of a family of monic orthogonal polynomials.

Proposition 51. *The elements in the left-most column of*

$$L = \left(\frac{1-\lambda x-\mu x^2}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)^{-1}$$

are the moments corresponding to the family of orthogonal polynomials with coefficient array L^{-1}.

We have in fact the following proposition, which characterizes those orthogonal polynomials that can be defined by Riordan arrays in terms of the Chebyshev polynomials of the second kind.

Proposition 52. *The Riordan array $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ is the coefficient array of the modified Chebyshev polynomials of the second kind given by*

$$P_n(x) = (\sqrt{s})^n U_n\left(\frac{x-r}{2\sqrt{s}}\right), \quad n = 0, 1, 2, \ldots$$

Corollary 53. *The Riordan array $\left(\frac{1-\lambda x-\mu x^2}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ is the coefficient array of the generalized Chebyshev polynomials of the second kind given by*

$$Q_n(x) = (\sqrt{s})^n U_n\left(\frac{x-r}{2\sqrt{s}}\right) - \lambda(\sqrt{s})^{n-1} U_{n-1}\left(\frac{x-r}{2\sqrt{s}}\right) - \mu(\sqrt{s})^{n-2} U_{n-2}\left(\frac{x-r}{2\sqrt{s}}\right), \quad n = 0, 1, 2, \ldots$$

The Hankel transform [13] of a sequence a_n is the sequence h_n of determinants $|a_{i+j}|_{0\leq i,j\leq n}$. For instance, the Hankel transform of the Catalan numbers is given by the all 1's sequence

$$1, 1, 1, 1, 1, \ldots.$$

The Hankel transform can have combinatorial significance; for instance, the Hankel transform of the ternary numbers begins

$$1, 2, 11, 170, 7429, 920460, 323801820, 323674802088, \ldots.$$

This sequence (A051255) counts the number of cyclically symmetric transpose complement plane partitions in a $(2n+2) \times (2n+2) \times (2n+2)$ box.

If the sequence a_n has a generating function $g(x)$, then the bivariate generating function of the Hankel matrix $|a_{i+j}|_{i,j\geq 0}$ is given by

$$\frac{xg(x) - yg(y)}{x - y}.$$

In the case that a sequence a_n has g.f. $g(x)$ expressible in the continued fraction form [23]

$$g(x) = \cfrac{a_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}$$

then we have the Heilermann formula [12]

$$h_n = a_0^{n+1} \beta_1^n \beta_2^{n-1} \cdots \beta_{n-1}^2 \beta_n = a_0^{n+1} \prod_{k=1}^{n} \beta_k^{n+1-k}. \tag{3}$$

Note that this independent from α_n.

We note that α_n and β_n are in general not integers (even if both a_n and h_n are integer valued). It is clear also that a Hankel transform has an infinite number of pre-images, since we are free to assign values to the α_n coefficients. For instance, a sequence a_n and its binomial transform $\sum_{k=0}^{n} \binom{n}{k}$ have the same Hankel transform, and the expansion of the INVERT transform of $g(x)$ given by $\frac{g(x}{1-xg(x)}$ and that of $g(x)$ will have the same Hankel transform.

Somos-4 sequences are most commonly associated with the x-coordinate of rational points on an elliptic curve [10, 20, 21]. The link between these sequences and Hankel transforms is made explicit in Theorem 7.1.1 of [20], for instance. Letting nP denote the n-fold sum $P + \cdots + P$ of points on an elliptic curve E, this result implies the following: if $P = (\bar{x}, \bar{y})$ and $Q = (x_0, y_0)$ are two distinct non-singular rational points on an elliptic curve E, denote, for all $n \in \mathbb{Z}$ such that $Q + nP \neq \mathcal{O}$ (the point at infinity on E), by (x_n, y_n) the coordinates of the point $Q + nP$. Then under these circumstances the numbers determined by

$$s_n = (-1)^{\binom{n+1}{2}} (x_{n-1} - \bar{x})(x_{n-2} - \bar{x})^2 \cdots (x_1 - \bar{x})^{n-1} (x_0 - \bar{x})^n s_0 \left(\frac{s_0}{s_{-1}} \right)^n$$

are elements of a Somos-4 sequence (given appropriate $s_0, s_{-1} \neq 0$). We can re-write this as

$$s_n = s_0 \left(\frac{s_0}{s_{-1}}\right)^n \prod_{k=0}^{n-1} (\bar{x} - x_k)^{n-k},$$

and we see that this is in the form of a Hankel transform.

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