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Mean Value Inequalities for Motzkin Numbers

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Abstract

Following up on results of Aigner, we present some inequalities for Motzkin numbers M_n. In particular, we prove that the sequence $(1/M_n)_{n\geq 1}$ is strictly convex.

1 Introduction

The Motzkin numbers, named after the American mathematician Theodore S. Motzkin (1908–1970), are defined by

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1} \binom{2k}{k} \binom{n}{2k} \quad (n = 0, 1, 2, \ldots).$$

They satisfy the recurrence relation

$$M_n = \frac{2n+1}{n+2} M_{n-1} + \frac{3(n-1)}{n+2} M_{n-2} \quad (n \geq 2). \tag{1}$$

The generating function is given by

$$\frac{1}{2x^2}\Big(1 - x - \sqrt{1-2x-3x^2}\Big) = \sum_{n=0}^{\infty} M_n x^n.$$

Here are the first few Motzkin numbers (see the sequence A001006 in [3]):

$$1, \quad 1, \quad 2, \quad 4, \quad 9, \quad 21, \quad 51, \quad 127, \quad 323, \quad 835.$$

The Motzkin numbers have interesting applications in number theory and geometry, and they play an important role in various counting problems. For instance, M_n is the number of paths from $(0,0)$ to $(n,0)$ in the integer plane $\mathbb{Z} \times \mathbb{Z}$ which never dip below the x-axis and use only steps $(1,0)$, $(1,1)$ and $(1,-1)$. Donaghey and Shapiro [2] presented a selection of 14 situations where these numbers occur; also see Stanley [4].

This note was inspired by an interesting paper published by Aigner [1] in 1998. He used tools from linear algebra to prove the inequalities

$$M_n < 3M_{n-1} \quad (n \geq 1), \tag{2}$$

$$M_n^2 \leq M_{n-1} M_{n+1} \quad (n \geq 1), \tag{3}$$

and the limit relation

$$\lim_{n\to\infty} \frac{M_{n-1}}{M_n} = \frac{1}{3}.$$

A combinatorial proof of (3) was given by Sun and Wang [5]. From (3) we conclude that the sequence $(M_n)_{n\geq 0}$ is log-convex.

Here, we present counterparts of (3). In particular, we show that the sequence $(1/M_n)_{n\geq 1}$ is strictly convex. This result and (3) lead to the double-inequality

$$\frac{2}{1/M_{n-1} + 1/M_{n+1}} < M_n \leq \sqrt{M_{n-1} M_{n+1}} \quad (n \geq 2).$$

Therefore, we see that if $n \geq 2$, then M_n separates the harmonic and geometric means of M_{n-1} and M_{n+1}.

We introduce the following notation. Let a and b be positive real numbers. The weighted harmonic and geometric means of a and b are defined by

$$H_t(a,b) = \frac{1}{t/a + (1-t)/b} \quad \text{and} \quad G_t(a,b) = a^t b^{1-t} \quad (0 \leq t \leq 1),$$

respectively. For $t = 1/2$ we obtain the unweighted harmonic and geometric means of a and b as follows:

$$H(a,b) = \frac{2ab}{a+b} \quad \text{and} \quad G(a,b) = \sqrt{ab}.$$

Moreover, let

$$F_t(a,b) = tG(a,b) + (1-t)H(a,b) \quad (t \in \mathbb{R}).$$

We remark that there is a connection between $F_t(a,b)$ and the so-called Heron means,

$$K_t(a,b) = tA(a,b) + (1-t)G(a,b),$$

where $A(a,b) = (a+b)/2$ denotes the arithmetic mean of a and b. We have

$$A(a,b)F_t(a,b) = G(a,b)K_t(a,b).$$

2 Inequalities

First, we offer lower and upper bounds for M_n in terms of weighted harmonic and geometric means of M_{n-1} and M_{n+1}.

Theorem 1. *Let $v, w \in (0,1)$. The inequalities*

$$H_v(M_{n-1}, M_{n+1}) \le M_n \le G_w(M_{n-1}, M_{n+1}) \tag{4}$$

hold for all $n \ge 2$ if and only if $v \ge 5/14 = 0.35714\ldots$ and $w \le 1/2$.

Proof. We assume that (4) is valid for all $n \ge 2$. Then,

$$H_v(M_2, M_4) \le M_3 \quad \text{and} \quad M_2 \le G_w(M_1, M_3).$$

This leads to

$$\frac{1}{v/2 + (1-v)/9} \le 4 \quad \text{and} \quad 2 \le 4^{1-w}.$$

It follows that $v \ge 5/14$ and $w \le 1/2$.

Next, let $w \le 1/2$ and $\lambda_n = M_{n-1}/M_n$. From (3) we obtain $\lambda_n \le \lambda_2 = 1/2$ for $n \ge 2$. Thus,

$$\begin{aligned} G_w(M_{n-1}, M_{n+1}) &= (\lambda_n \lambda_{n+1})^w M_{n+1} \\ &\ge (\lambda_n \lambda_{n+1})^{1/2} M_{n+1} \\ &= G_{1/2}(M_{n-1}, M_{n+1}) \ge M_n. \end{aligned}$$

The left-hand side of (4) is equivalent to

$$Q_n \le v,$$

where

$$Q_n = \frac{M_{n-1}(M_{n+1} - M_n)}{M_n(M_{n+1} - M_{n-1})} = \frac{\lambda_n(1 - \lambda_{n+1})}{1 - \lambda_n\lambda_{n+1}}.$$

It remains to show that for $n \geq 2$,

$$Q_n \leq \frac{5}{14}. \tag{5}$$

We have $Q_2 = 1/3$ and $Q_3 = 5/14$. Let $n \geq 4$. Inequality (5) can be written as

$$0 \leq \frac{5}{\lambda_n} + 9\lambda_{n+1} - 14.$$

Applying (1) gives

$$\frac{1}{\lambda_n} = \frac{2n + 1 + 3(n-1)\lambda_{n-1}}{n+2} \quad \text{and} \quad \lambda_{n+1} = \frac{n+3}{2n + 3 + 3n\lambda_n}. \tag{6}$$

From (3) we conclude that $\lambda_n \leq \lambda_{n-1}$, so that we get

$$\lambda_{n+1} \geq \frac{n+3}{2n + 3 + 3n\lambda_{n-1}}. \tag{7}$$

Using (6) and (7) yields

$$\begin{aligned} \frac{5}{\lambda_n} + 9\lambda_{n+1} - 14 \quad &\geq \quad \frac{5(2n + 1 + 3(n-1)\lambda_{n-1})}{n+2} + \frac{9(n+3)}{2n + 3 + 3n\lambda_{n-1}} - 14 \\ &= \quad \frac{S_n}{(n+2)(2n + 3 + 3n\lambda_{n-1})}, \end{aligned}$$

where

$$S_n = n^2 - 13n - 15 + 9(2n^2 - 6n - 5)\lambda_{n-1} + 45(n^2 - n)\lambda_{n-1}^2.$$

From (2) we obtain $\lambda_{n-1} > 1/3$. Thus,

$$S_n > n^2 - 13n - 15 + 9(2n^2 - 6n - 5)\frac{1}{3} + 45(n^2 - n)\frac{1}{9} = 6(2n^2 - 6n - 5) > 0.$$

This implies that (5) holds. The proof of Theorem 1 is now complete. □

Our second result yields sharp bounds for M_n in terms of $F_t(M_{n-1}, M_{n+1})$. The following companion to (4) is valid.

Theorem 2. *Let $\alpha, \beta \in \mathbb{R}$. The inequalities*

$$F_\alpha(M_{n-1}, M_{n+1}) \leq M_n \leq F_\beta(M_{n-1}, M_{n+1}) \tag{8}$$

hold for all $n \geq 2$ if and only if

$$\alpha \leq \frac{8}{3(11\sqrt{2} - 12)} = 0.74983\ldots \quad \textit{and} \quad \beta \geq 1.$$

Proof. First, we assume that (8) is valid for all $n \geq 2$. Then we have

$$F_\alpha(M_2, M_4) \leq M_3 \quad \text{and} \quad M_2 \leq F_\beta(M_1, M_3),$$

and hence,

$$\sqrt{18}\alpha + \frac{36}{11}(1-\alpha) \leq 4 \quad \text{and} \quad 2 \leq 2\beta + \frac{8}{5}(1-\beta).$$

Therefore, it follows that

$$\alpha \leq \frac{8}{3(11\sqrt{2}-12)} \quad \text{and} \quad 1 \leq \beta.$$

Applying (3) and the fact that $F_t(a,b)$ is increasing with respect to t we obtain for $\beta \geq 1$ and $n \geq 2$,

$$M_n \leq F_1(M_{n-1}, M_{n+1}) \leq F_\beta(M_{n-1}, M_{n+1}).$$

Let

$$\alpha_0 = \frac{8}{3(11\sqrt{2}-12)}.$$

We have to show that for $n \geq 2$,

$$F_{\alpha_0}(M_{n-1}, M_{n+1}) \leq M_n,$$

or, equivalently,

$$\alpha_0 \leq R_n,$$

where

$$R_n = \frac{M_n - H(M_{n-1}, M_{n+1})}{G(M_{n-1}, M_{n+1}) - H(M_{n-1}, M_{n+1})}.$$

By direct computation we find

$$R_2 = 1, \quad R_3 = \alpha_0, \quad R_4 = 0.932\ldots, \quad R_5 = 0.930\ldots, \quad R_6 = 0.958\ldots,$$

$$R_7 = 0.966\ldots, \quad R_8 = 0.974\ldots, \quad R_9 = 0.979\ldots, \quad R_{10} = 0.983\ldots.$$

Let $n \geq 11$. It suffices to prove that $R_n > 3/4$, or, equivalently,

$$3G(M_{n-1}, M_{n+1}) + H(M_{n-1}, M_{n+1}) < 4M_n. \tag{9}$$

As before, let $\lambda_n = M_{n-1}/M_n$. Then, (9) can be written as

$$0 < U_n - V_n,$$

where

$$U_n = 4 - 2\Big(\frac{1}{\lambda_n} + \lambda_{n+1}\Big)^{-1} \quad \text{and} \quad V_n = 3\sqrt{\lambda_n \frac{1}{\lambda_{n+1}}}.$$

Using (6) gives

$$U_n = 4 - 2\Big(\frac{1}{\lambda_n} + \frac{n+3}{2n+3+3n\lambda_n}\Big)^{-1}$$
$$= \frac{2\big(4n+6+3(n+1)\lambda_n + 3n\lambda_n(1-\lambda_n)\big)}{2n+3+(4n+3)\lambda_n}$$

and

$$V_n = 3\sqrt{\lambda_n \frac{2n+3+3n\lambda_n}{n+3}}.$$

Now we have

$$\begin{aligned}&(U_n^2 - V_n^2)(n+3)\big(2n+3+(4n+3)\lambda_n\big)^2 \\ &= P_1(\lambda_n)n^3 + P_2(\lambda_n)n^2 + P_3(\lambda_n)n + P_4(\lambda_n)\end{aligned} \tag{10}$$

with

$$\begin{aligned}P_1(t) &= -396t^4 - 864t^3 - 348t^2 + 120t + 64,\\ P_2(t) &= -540t^4 - 2340t^3 - 1260t^2 + 636t + 384,\\ P_3(t) &= -243t^4 - 1512t^3 - 1503t^2 + 810t + 720,\\ P_4(t) &= -243t^3 - 378t^2 + 189t + 432.\end{aligned}$$

Since $P_j''(t) < 0$ $(j = 1, 2, 3, 4)$ for $t \geq 0$, we conclude that the functions P_1, P_2, P_3, and P_4 are concave on $[0, \infty)$. It follows that, for $t \in [0, 0.38]$,

$$P_j(t) \geq m_j \quad (j = 1, 2, 3, 4),$$

where

$$m_j = \min\{P_j(0), P_j(0.38)\}.$$

We have

$$m_1 = 3.68\ldots, \quad m_2 = 304.07\ldots, \quad m_3 = 720, \quad m_4 = 432.$$

Thus, it follows that $P_j(t) > 0$ $(j = 1, 2, 3, 4)$. Since $(\lambda_n)_{n\geq 1}$ is decreasing, we obtain $\lambda_n \leq \lambda_{11} = 0.377\ldots$. From (10) we conclude that $U_n^2 - V_n^2 > 0$, so that $U_n + V_n > 0$ leads to $U_n - V_n > 0$. This completes the proof of Theorem 2. □

Remark 3.

(i) Computer calculations reveal that for $n = 4, 5, \ldots, 500$ the lower bound given in (8) with $\alpha = 8/(33\sqrt{2} - 36)$ is greater than the lower bound in (4) with $v = 5/14$.

(ii) Aigner's inequality (3) can be written as

$$\frac{M_n - M_{n-1}}{M_{n+1} - M_n} \leq \frac{M_n}{M_{n+1}}. \tag{11}$$

It is possible to show that the following counterpart of (11) holds.

Theorem 4. *For all $n \geq 2$ we have*

$$c_0 \frac{M_n}{M_{n+1}} \leq \frac{M_n - M_{n-1}}{M_{n+1} - M_n} \leq c_1 \frac{M_n}{M_{n+1}}$$

with the best possible constant factors $c_0 = 9/10$ and $c_1 = 1$.

This result can be proved by using the same method as in the proof of Theorem 2. So we omit the details.

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References

[1] M. Aigner, Motzkin numbers, *European J. Combin.* **19** (1998), 663–675.

[2] R. Donaghey and L. W. Shapiro, Motzkin numbers, *J. Combin. Theory Ser. A* **23** (1977), 291–301.

[3] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, 2020. Available at https://oeis.org.

[4] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999.

[5] H. Sun and Y. Wang, A combinatorial proof of the log-convexity of Catalan-like numbers, *J. Integer Sequences* **17** (2014), Article 14.5.2.

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