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Generalized Alternating Sums of Multiplicative Arithmetic Functions

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Abstract

In this paper, given a finite set of primes Q, we derive asymptotic formulas for generalized alternating sums of the form $\sum_{n\le x} t_Q(n)f(n)$ and $\sum_{n\le x} t_Q(n)\frac{1}{f(n)}$, where f is a multiplicative arithmetic function, and $t_Q(n)$ equals -1 if n is divisible by some prime $q \in Q$, and 1 otherwise. In particular, these results are applicable to known functions, such as Euler's totient function, the sum of divisors function, the divisor function, and others. In the particular case of $Q = \{2\}$, we generalize various results obtained by Tóth, even improving one of his results proposed as open problem.

1 Introduction

Throughout this paper, we let $\mathbb{P}$ denote the set of all prime numbers, and $Q = \{q_1, q_2, \ldots, q_r\}$ be any finite set of prime numbers. We use the notation $q_{\min} := \min\{q_i\}$ and $q_{\max} := \max\{q_i\}$. The letter p will always stand for a prime number.

Alternating sums appear in various topics of mathematics, including number theory. For example, Bordellès and Cloitre [1] established asymptotic formulas with error terms for alternating sums of the form

$$\sum_{n\le x}(-1)^{n-1}\frac{1}{g(n)},$$

where g belongs to a class of multiplicative functions, including Euler's totient function φ, the sum-of-divisors function σ and the Dedekind function ψ.

Tóth [11] established some general results for alternating sums of the form

$$\sum_{n\le x}(-1)^{n-1}f(n) \quad \text{or} \quad \sum_{n\le x}(-1)^{n-1}\frac{1}{f(n)},$$

where f belongs to a broader class of multiplicative arithmetic functions than those considered by Bordellès and Cloitre [1], extending their results to a whole new kind of multiplicative functions, such as the divisor function τ, the gcd-sum function P, the square free kernel κ, the square free numbers μ^2 function, the number of abelian groups $a(n)$, the sum-of-unitary-divisor function σ^*, the unitary-Euler function φ^*, the unitary-squarefree kernel κ^*, the powerful part of a number, and the sum-of-bi-unitary-divisors function.

In the last part of his paper, Tóth proposes a generalization for alternating sums, and also finds an asymptotic result for the generalized sum

$$\sum_{n\le x}t_Q(n)\sigma(n),$$

where

$$t_Q(n)=\begin{cases}1, & \text{if } q\nmid n \text{ for all } q\in Q;\\ -1, & \text{otherwise,}\end{cases}$$

is defined for a finite set of prime numbers Q, and $\sigma(n)=\sum_{d|n}d$ is the sum of divisors function.

Let

$$D_Q(f,s):=\sum_{n=1}^{\infty}t_Q(n)\frac{f(n)}{n^s},$$

be the Dirichlet series for the multiplicative function f, with generalized alternating signs depending on Q. For example, if $Q=\{2,3\}$, we have

$$D_{\{2,3\}}(f,s)=\frac{f(1)}{1^s}-\frac{f(2)}{2^s}-\frac{f(3)}{3^s}-\frac{f(4)}{4^s}+\frac{f(5)}{5^s}-\frac{f(6)}{6^s}+\frac{f(7)}{7^s}+\cdots$$

Tóth [11, Prop. 56] proved that

$$D_Q(f,s)=D(f,s)\left(2\prod_{q\in Q}\left(\sum_{v=0}^{\infty}\frac{f(q^v)}{q^{vs}}\right)^{-1}-1\right), \tag{1}$$

where $D(f,s)$ is the Dirichlet series associated with the multiplicative arithmetic function f.

For $q \in Q$, let us consider the formal power series

$$S_{f,q}(x) := 1 + \sum_{v=1}^{\infty} f(q^v)x^v,$$

and its inverse formal series

$$\overline{S}_{f,q}(x) := 1 + \sum_{v=1}^{\infty} b_{v,q}x^v.$$

From (1) we have, by convolution, that

$$\sum_{n\le x} t_Q(n)f(n) = \sum_{d\le x} h_{f,Q}(d) \sum_{j\le x/d} f(j),$$

where

$$h_{f,Q}(n) = \begin{cases} 2b_{v_1,q_1}\cdots b_{v_r,q_r}, & \text{if } n = q_1^{v_1}\cdots q_r^{v_r};\\ 1, & \text{if } n = 1;\\ 0, & \text{otherwise.}\end{cases}$$

From this expression we can derive asymptotic formulas for

$$\sum_{n\le x} t_Q(n)f(n), \tag{2}$$

as long as asymptotic formulas are known for $\sum_{n\le x} f(n)$, and the coefficients $b_{v,q}$ are adequately estimated.

For example, Tóth [11, Teo. 57] proved that

$$\sum_{n\le x} t_Q(n)\sigma(n) = \frac{\pi^2}{12}\left(2\prod_{q\in Q}\left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^2}\right)-1\right)x^2 + O(x(\log x)^{2/3}). \tag{3}$$

In this paper, we obtain asymptotic expressions for (2), among whose applications we obtain the result (3) and others.

2 Main results

Theorem 1. *Let f be a multiplicative function and consider the following four conditions*

(i) there exists a constant C_f such that

$$\sum_{n\le x} f(n) = C_f x^2 + O(xR_f(x)),$$

where $1 \ll R_f(x)$ when $x\to\infty$ and $R_f(x)$ is an increasing function;

(ii) $S_{f,q_i}\left(\frac{1}{q_i^2}\right)$ *converges for all* i*;*

(iii) *the sequence* $(b_{v,q_i})_{v\geq 0}$ *satisfies* $b_{v,q_i} \ll 1$ *for all* i*;*

(iv) *the sequence* $(b_{v,q_i})_{v\geq 0}$ *satisfies* $|b_{v,q_i}| \ll (r_i)^v$ *with* $1 \leq r_i \leq \frac{q_i^2}{q_{\max}}$ *for all* i*.*

Assume that conditions (i) and (ii) hold and that one of the two conditions (iii) or (iv) also holds. Then

$$\sum_{n\leq x} t_Q(n)f(n) = C_f x^2\left(\frac{2}{S_{f,q_1}(1/q_1^2)\cdots S_{f,q_n}(1/q_n^2)} - 1\right) + O(xR_f(x)).$$

Theorem 2. *Let* f *be a multiplicative function, and let us suppose that*

(i) *there exist constants* D_f *and* E_f *such that*

$$\sum_{n\leq x}\frac{1}{f(n)} = D_f(\log x + E_f) + O(x^{-1}R_{1/f}(x)),$$

where $1 \ll R_{1/f}(x) = o(x)$ *if* $x\to\infty$ *and* $R_{1/f}(x)$ *is an increasing function;*

(ii) *the radius of convergence of* $S_{1/f,q_i}(x)$ *is* $r_{1/f,q_i} > 1$*, for all* i*;*

(iii) *the coefficients of* b_{v,q_i} *satisfy* $b_{v,q_i} \ll M_i^v$ *if* $v\to+\infty$*, for all* i *and where* $M_i < \frac{q_{\min}}{q_i}$*.*

Then

$$\sum_{n\leq x} t_Q(n)\frac{1}{f(n)}$$
$$= D_f\left(\left(\frac{2}{\prod_i S_{1/f,q_i}(1)} - 1\right)(\log x + E_f) + \frac{2}{\prod_i S_{1/f,q_i}(1)}\cdot\sum_{i=1}^{r}\frac{\log(q_i)S'_{1/f,q_i}(1)}{S_{1/f,q_i}(1)}\right) + O(T_{1/f,Q}(x)),$$

where

$$T_{1/f,Q}(x) =$$
$$\begin{cases} x^{-1}R_{1/f}(x), & \text{if } \max(q_iM_i) < 1;\\ x^{-1}R_{1/f}(x)(\log x)^r, & \text{if } \max(q_iM_i) = 1;\\ (\log x)^{r-1}\max\{\log x\cdot x^{\log M_{\max}/\log q_{\min}}, x^{\log(\max(q_iM_i))/\log q_{\min}}\cdot x^{-1}R_{1/f}(x)\}, & \text{if } \max(q_iM_i) > 1.\end{cases}$$

3 Proofs of the main results

3.1 Proof of Theorem 1

Proof. Under the hypothesis of the theorem, we have that

$$\sum_{n\le x} t_Q(n)f(n) = \sum_{d\le x} h_{f,Q}(d) \sum_{j\le x/d} f(j) = \sum_{d\le x} h_{f,Q}(d) \left(C_f \frac{x^2}{d^2} + O\left(\frac{x}{d}R_f(x/d)\right)\right)$$
$$= C_f x^2 \cdot \sum_{d\le x} \frac{h_{f,Q}(d)}{d^2} + O\left(xR_f(x) \sum_{d\le x} \frac{|h_{f,Q}(d)|}{d}\right).$$

On one hand, for some $\delta' < 1$,

$$\sum_{d\le x} \frac{|h_{f,Q}(d)|}{d} \le \sum_{q_1^{v_1}\cdots q_r^{v_r}\le x} \frac{2|b_{v_1,q_1}\cdots b_{v_r,q_r}|}{q_1^{v_1}\cdots q_r^{v_r}} + 1$$
$$\ll \sum_{q_1^{v_1}\cdots q_r^{v_r}\le x} \frac{r_1^{v_1}\cdots r_r^{v_r}}{q_1^{v_1}\cdots q_r^{v_r}} \ll \sum_{v_1+\cdots+v_r\le \frac{\log x}{\log q_{\min}}} (\delta')^{v_1+\cdots+v_r}$$
$$\ll \sum_{n\le \frac{\log x}{\log q_{\min}}} (\delta')^n \binom{n+r-1}{r-1} = \sum_{n\le \frac{\log x}{\log q_{\min}}} (\delta')^n \frac{(n+r-1)\cdots(n+2)\cdot(n+1)}{(r-1)!}$$
$$\ll \sum_{n\le \frac{\log x}{\log q_{\min}}} (\delta')^n (n+r-1)^{r-1} \ll 1.$$

On the other hand, setting $s = 2$ in (1),

$$\sum_{d\le x} \frac{|h_{f,Q}(d)|}{d^2} = \sum_{d=1}^{\infty} \frac{|h_{f,Q}(d)|}{d^2} - \sum_{d>x} \frac{|h_{f,Q}(d)|}{d^2} = \frac{2}{S_{f,q_1}(1/q_1^2)\cdots S_{f,q_r}(1/q_r^2)} - 1,$$

and

$$x^2 \sum_{d>x} \frac{|h_{f,Q}(d)|}{d^2} \ll x^2 \sum_{q_1^{v_1}\cdots q_r^{v_r}>x} \frac{b_{v_1,q_1}\cdots b_{v_r,q_r}}{q_1^{2v_1}\cdots q_r^{2v_r}}.$$

Case (iii) of Theorem 1:

$$x^2 \sum_{d>x} \frac{|h_{f,Q}(d)|}{d^2} \ll x^2 \sum_{d>x} \frac{1}{d^2} \ll x \ll xR_f(x).$$

Case (iv) of Theorem 1:

Let us define $\delta := \max_i \left(\frac{r_i}{q_i^2}\right) < \frac{1}{q_{\max}}$. Then

$$x^2 \sum_{d>x} \frac{|h_{f,Q}(d)|}{d^2} \ll x^2 \sum_{v_1+\cdots+v_r > \frac{\log x}{\log q_{\max}}} \left(\frac{r_1}{q_1^2}\right)^{v_1} \cdots \left(\frac{r_r}{q_r^2}\right)^{v_r} \ll x^2 \sum_{v_1+\cdots+v_r > \frac{\log x}{\log q_{\max}}} (\delta)^{v_1+\cdots+v_r}$$
$$\ll x^2 \sum_{n > \frac{\log x}{\log q_{\max}}} \delta^n n^{r-1} \ll x^2 \delta^{\frac{\log x}{\log q_{\max}}} (\log x)^{r-1} = x^{2+\frac{\log \delta}{\log q_{\max}}} (\log x)^{r-1} \ll x R_f(x).$$

□

3.2 Proof of Theorem 2

Proof. From the hypothesis of the theorem, we have that

$$\sum_{n\le x} t_Q(n) \frac{1}{f(n)} = \sum_{d\le x} h_{1/f,Q}(d) \sum_{j\le x/d} \frac{1}{f(j)}$$
$$= \sum_{d\le x} h_{1/f,Q}(d) \left(D_f \left(\log \frac{x}{d} + E_f\right) + O\left(\left(\frac{x}{d}\right)^{-1} R_{1/f}(x/d)\right)\right)$$
$$= D_f(\log x + E_f) \sum_{d\le x} h_{1/f,Q}(d) - D_f \sum_{d\le x} h_{1/f,Q}(d) \log d + O\left(x^{-1} R_{1/f}(x) \cdot \sum_{d\le x} d |h_{1/f,Q}(d)| \right)$$
$$= D_f(\log x + E_f) \sum_{d=1}^{\infty} h_{1/f,Q}(d) + O\left(\log x \sum_{d>x} |h_{1/f,Q}(d)| \right)$$
$$- D_f \sum_{d=1}^{\infty} h_{1/f,Q}(d) \log d + O\left(\sum_{d>x} |h_{1/f,Q}(d)| \log d \right) + O\left(x^{-1} R_{1/f}(x) \sum_{d\le x} d \cdot |h_{1/f,Q}(d)| \right).$$

In particular, by (1),

$$\sum_{d=1}^{\infty} \frac{h_{1/f,Q}(d)}{d^s} = \frac{2}{S_{1/f,q_1}(1/q_1^s) \cdots S_{1/f,q_r}(1/q_r^s)} - 1,$$

then we see that

$$\sum_{d=1}^{\infty} h_{1/f,Q}(d) = \frac{2}{S_{1/f,q_1}(1) \cdots S_{1/f,q_r}(1)} - 1,$$

and

$$\sum_{d=1}^{\infty} h_{1/f,Q}(d) \log d = -2 \cdot \left(\frac{\log q_1 \cdot S'_{1/f,q_1}(1)}{S^2_{1/f,q_1}(1) \cdots S_{1/f,q_r}(1)} + \cdots + \frac{\log q_r \cdot S'_{1/f,q_r}(1)}{S_{1/f,q_1}(1) \cdots S^2_{1/f,q_r}(1)} \right).$$

We also have that

$$
\sum_{d>x} |h_{1/f,Q}(d)| \le 1 + \sum_{q_1^{v_1}\cdots q_r^{v_r}>x} 2|b_{v_1,q_1}\cdots b_{v_r,q_r}| \ll \sum_{q_1^{v_1}\cdots q_r^{v_r}>x} M_1^{v_1}\cdots M_r^{v_r} \ll \sum_{q_1^{v_1}\cdots q_r^{v_r}>x} M_{\max}^{v_1+\cdots+v_r}
$$

$$
\ll \sum_{v_1+\cdots+v_r>\frac{\log x}{\log q_{\max}}} M_{\max}^{v_1+\cdots+v_r} \ll \sum_{n>\frac{\log x}{\log q_{\max}}} M_{\max}^{n}(n+r-1)^{r-1} \ll \left(\frac{\log x}{\log q_{\max}}\right)^{r-1} M_{\max}^{\frac{\log x}{\log q_{\max}}}
$$

$$
\ll (\log x)^{r-1} x^{\frac{\log M_{\max}}{\log q_{\max}}}.
$$

Similarly,

$$
\begin{aligned}
&\sum_{d>x} |h_{1/f,Q}(d)| \log d \ll \sum_{q_1^{v_1}\cdots q_r^{v_r}>x} |b_{v_1,q_1}\cdots b_{v_r,q_r}|(v_1\log q_1 + \cdots + v_r \log q_r) \\
&\ll \sum_{q_1^{v_1}\cdots q_r^{v_r}>x} (M_{\max})^{v_1+\cdots+v_r} \log q_{\max}\cdot(v_1+\cdots+v_r) \ll \sum_{n>\frac{\log x}{\log q_{\max}}} (M_{\max})^{n}(n+r-1)^{r-1}\cdot n \\
&\ll \sum_{n>\frac{\log x}{\log q_{\max}}} (M_{\max})^{n} n^{r} \ll (\log x)^r x^{\log M_{\max}/\log q_{\max}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\sum_{d\le x} d|h_{1/f,Q}(d)| \ll \sum_{q_1^{v_1}\cdots q_r^{v_r}\le x} q_1^{v_1}\cdots q_r^{v_r} M_1^{v_1}\cdots M_r^{v_r} = \sum_{q_1^{v_1}\cdots q_r^{v_r}\le x} (q_1M_1)^{v_1}\cdots(q_rM_r)^{v_r} \\
&\ll \sum_{v_1+\cdots+v_r\le\frac{\log x}{\log q_{\min}}} (\max(q_iM_i))^{v_1+\cdots+v_r} \ll \sum_{n\le\frac{\log x}{\log q_{\min}}} (\max(q_iM_i))^{n}\cdot n^{r-1} \\
&\ll \begin{cases} 1, & \text{if } \max(q_iM_i) < 1; \\ (\log x)^r, & \text{if } \max(q_iM_i) = 1; \\ x^{\log\max(q_iM_i)/\log q_{\min}}\cdot(\log x)^{r-1}, & \text{if } \max(q_iM_i) > 1. \end{cases}
\end{aligned}
$$

Case 1: If $\max(q_iM_i) < 1$, then

$$
\begin{aligned}
&(\log x)^r \cdot x^{\log M_{\max}/\log q_{\max}} \ll x^{-1} R_{1/f}(x) \\
&\Leftrightarrow (\log x)^r x^{1+\log M_{\max}/\log q_{\max}} \ll R_{1/f}(x),
\end{aligned}
$$

since $q_{\max}M_{\max} < 1$, which implies that $\log q_{\max} + \log M_{\max} < 0$.

Case 2: If $\max(q_iM_i) = 1$, then

$$
\begin{aligned}
&(\log x)^r \cdot x^{\log M_{\max}/\log q_{\max}} \ll (\log x)^r x^{-1} R_{1/f}(x) \\
&\Leftrightarrow x^{1+\log M_{\max}/\log q_{\max}} \ll R_{1/f}(x).
\end{aligned}
$$

Case 3: If $\max(q_i M_i) > 1$,

$$\begin{aligned}&(\log x)^r x^{\log M_{\max}/\log q_{\max}} \ll x^{-1} R_{1/f}(x) x^{\frac{\log \max(q_i M_i)}{\log q_{\min}}} (\log x)^{r-1} \\ &\Leftrightarrow (\log x) \cdot x^{1+\frac{\log M_{\max}}{\log q_{\max}} - \frac{\log \max(q_i M_i)}{\log q_{\min}}} \ll R_{1/f}(x).\end{aligned}$$

Since all three cases have been considered, the proof of Theorem 2 is complete.

□

4 Applications

For the applications to various multiplicative functions shown in this section, it is sufficient to verify that conditions (i), (ii), (iii) and (iv) of Theorems 1 and 2 hold.

4.1 Euler's totient function $\varphi(n)$

Let us consider Euler's totient function $\varphi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)$.

Theorem 3.

$$\sum_{n\le x} t_Q(n)\varphi(n) = \frac{3}{\pi^2}x^2\left(\frac{2}{\prod_{q\in Q}\left(1+\frac{1}{q}\right)} - 1\right) + O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right), \qquad (4)$$

and

$$\begin{aligned}\sum_{n\le x} t_Q(n)\frac{1}{\varphi(n)} &= A\Bigg(\Bigg(2\prod_{q\in Q}\frac{(q-1)^2}{q^2-q+1} - 1\Bigg)(\log x + \gamma - B) \\ &+ 2\prod_{q\in Q}\frac{(q-1)^2}{q^2-q+1}\cdot\sum_{q\in Q}\frac{q^2\log q}{(q-1)(q^2-q+1)}\Bigg) + O(R_{1/\varphi}(x)),\end{aligned} \qquad (5)$$

where $R_{1/\varphi}(x) = x^{-1}(\log x)^{2/3}$ if $2 \notin Q$, and $R_{1/\varphi}(x) = x^{-1}(\log x)^{2/3+r}$ if $2\in Q$, γ is Euler's constant, and constants A and B are defined by

$$A = \frac{\zeta(2)\zeta(3)}{\zeta(6)}, \qquad B = \sum_{p\in\mathbb{P}}\frac{\log p}{p^2-p+1}.$$

Proof. Concerning (4), we know from Walfisz [12, p. 144] that

$$\sum_{n\le x}\varphi(n) = \frac{3}{\pi^2}x^2 + O(x(\log x)^{2/3}(\log\log x)^{4/3}).$$

Then we have $R_\varphi(x) = (\log x)^{2/3}(\log\log x)^{4/3}$, so condition (i) of Theorem 1 is satisfied.

On the other hand, we can see that

$$S_{f,q}(x) = 1 + \sum_{v=1}^{\infty} \varphi(q^v)x^v = 1 + \sum_{v=1}^{\infty}(q^v - q^{v-1})x^v$$
$$= 1 + (1 - \frac{1}{q})\left(\frac{qx}{1-qx}\right) = \frac{1-x}{1-qx}, \quad |x| < 1/q,$$

and condition (ii) of Theorem 1 is satisfied.

Then we conclude that

$$\overline{S}_{\varphi,q}(x) = \frac{1-qx}{1-x} = 1 + \frac{(1-q)x}{(1-x)} = 1 + (1-q)x\sum_{v=0}^{\infty} x^v \qquad (|x| < 1),$$

so $b_{v,q} = (1-q) \ll 1$, and condition (iii) of Theorem 1 is satisfied. Furthermore, we see that $S_{\varphi,q}(1/q^2) = \frac{1-1/q^2}{1-q\cdot 1/q^2} = \frac{q+1}{q}$.

Concerning (5), we know from Landau [2, Thm. 1.1] and Sitaramachandraro [4] that

$$\sum_{n\le x} \frac{1}{\varphi(n)} = A(\log x + \gamma - B) + O(x^{-1}(\log x)^{2/3}).$$

Then we have that $R_{1/\varphi}(x) = (\log x)^{2/3}$, impliying that condition (i) of Theorem 2 is satisfied.

On the other hand, we see that

$$S_{1/\varphi,q}(x) = 1 + \sum_{v=1}^{\infty} \frac{1}{\varphi(q^v)}x^v = 1 + \sum_{v=1}^{\infty} \frac{x^v}{(q^v - q^{v-1})}$$
$$= 1 + \frac{q}{(q-1)}\frac{x}{(q-x)} = \frac{x + q(q-1)}{(q-1)(q-x)} \qquad (|x| < q),$$

so condition (ii) of Theorem 2 is satisfied.

Finally,

$$\overline{S}_{1/\varphi,q}(x) = (q-1)\left(-1 + \frac{q^2}{x + q(q-1)}\right) = (q-1)\Big(-1 + \frac{q}{q-1}\sum_{v=0}^{\infty} \frac{(-1)^v}{q^v(q-1)^v}x^v\Big)$$
$$(|x| < q(q-1)),$$

so $b_{v,q} = \frac{(-1)^v q}{q^v(q-1)^v(q-1)} \ll \left(\frac{1}{q(q-1)}\right)^v$, having that $M_q = \frac{1}{q(q-1)} < \frac{q_{\min}}{q}$, and condition (iii) of Theorem 2 is satisfied with $\max(M_i q_i) = 1/(q_{\min} - 1) = 1$ if $q_{\min} = 2$, and $\max(M_i q_i) < 1$ if $q_{\min} > 2$. Furthermore, we see that

$$S_{1/\varphi,q}(1) = \frac{q^2 - q + 1}{(q-1)^2} \quad \text{and} \quad S'_{1/\varphi,q}(1) = \frac{q^2}{(q-1)^3}.$$

□

4.2 Sum of divisors function

The sum of divisors functions is defined by $\sigma(n) = \sum_{d|n} d$.

Theorem 4.

$$\sum_{n\le x} t_Q(n)\sigma(n) = \frac{\pi^2}{12}x^2\left(2\prod_{q\in Q}\frac{(q-1)^2(q+1)}{q^3}-1\right) + O\left(x(\log x)^{2/3}\right), \tag{6}$$

and

$$\begin{aligned}\sum_{n\le x} t_Q(n)\frac{1}{\sigma(n)} &= E\Bigg(\Big(\frac{2}{\prod_i K_{q_i}}-1\Big)(\log x+\gamma+F)\\ &+\frac{2}{\prod_i K_{q_i}}\cdot\sum_{i=1}^{r}\frac{\log q_i\cdot K'_{q_i}}{K_{q_i}}\Bigg) + O(x^{-1}(\log x)^{2/3+r}(\log\log x)^{4/3}),\end{aligned} \tag{7}$$

where γ is Euler's constant, the constants K_q and K'_q are defined by

$$K_q = 1+(q-1)\sum_{v=1}^{\infty}\frac{1}{q^{v+1}-1} \qquad \textit{and} \qquad K'_q = (q-1)\sum_{v=1}^{\infty}\frac{v}{q^{v+1}-1},$$

(as a particular case, we have that $K_2 \doteq 1.606695$ is the Erdős-Borwein constant, which can be seen in the sequence A065442 of the Sloane's On-line Encyclopedia of Integer Sequences (OEIS) [8]), and the constants E and F are defined by

$$E = \prod_{p\in\mathbb{P}}\alpha(p), \qquad F = \sum_{p\in\mathbb{P}}\frac{(p-1)^2\beta(p)\log p}{p\alpha(p)},$$

with

$$\begin{aligned}\alpha(p) &= 1-\frac{(p-1)^2}{p}\sum_{j=1}^{\infty}\frac{1}{(p^j-1)(p^{j+1}-1)},\\ \beta(p) &= \sum_{j=1}^{\infty}\frac{j}{(p^j-1)(p^{j+1}-1)}.\end{aligned}$$

Proof. Concerning (6), we know from Walfisz [12, p. 99] that

$$\sum_{n\le x}\sigma(n) = \frac{\pi^2}{12}x^2 + O(x(\log x)^{2/3}).$$

Then we have that $R_\varphi(x) = (\log x)^{2/3}$, so condition (i) of Theorem 1 is satisfied.

On the other hand, we have that

$$S_{\sigma,q}(x) = 1 + \sum_{v=1}^{\infty} \sigma(q^v)x^v = 1 + \sum_{v=1}^{\infty} \frac{q^{v+1}-1}{q-1}x^v$$
$$= 1 + \frac{1}{q-1}\left(\frac{q^2x}{1-qx} - \frac{x}{1-x}\right) = \frac{1}{(1-qx)(1-x)}, \quad |x| < 1/q,$$

so condition (ii) of Theorem 1 is satisfied.

Then we see that

$$\overline{S}_{\sigma,q}(x) = (1-qx)(1-x) = 1 - (q+1)x + qx^2 \qquad (x \in \mathbb{R}),$$

so $b_{0,q} = 1, b_{1,q} = -(q+1), b_{2,q} = q$ and $b_{v,q} = 0$ if $v \geq 3$, and condition (iii) of Theorem 1 is satisfied. Furthermore, we see that $S_{\sigma,q}(1/q^2) = \frac{1}{(1-1/q)(1-1/q^2)} = \frac{q^3}{(q-1)^2(q+1)}$.

Concerning (7), we know from Sita Ramaiah and Suryanarayana [6, Cor. 4.1] that

$$\sum_{n \leq x} \frac{1}{\sigma(n)} = E(\log x + \gamma + F) + O(x^{-1}(\log x)^{2/3}(\log\log x)^{4/3}).$$

Then we conclude that $R_{1/\varphi}(x) = (\log x)^{2/3}(\log\log x)^{4/3}$, so condition (i) of Theorem 2 is satisfied.

On the other hand,

$$S_{1/\sigma,q}(x) = 1 + \sum_{v=1}^{\infty} \frac{1}{\sigma(q^v)}x^v = 1 + \sum_{v=1}^{\infty} \frac{(q-1)}{q^{v+1}-1}x^v, \qquad (|x| < q),$$

so condition (ii) of Theorem 2 is satisfied.

The coefficients $\left(\frac{q-1}{q^{v+1}-1}\right)$ of this last power series form a log-convex sequence. Indeed,

$$\left(\frac{q-1}{q^{v+1}-1}\right)^2 \leq \left(\frac{q-1}{q^v-1}\right) \cdot \left(\frac{q-1}{q^{v+2}-1}\right), v \geq 0$$
$$\Leftrightarrow (q^v-1)(q^{v+2}-1) \leq (q^{v+1}-1)^2 \Leftrightarrow 2q^{v+1} \leq q^v + q^{v+2} \Leftrightarrow 2q \leq 1 + q^2.$$

By Kaluza's theorem (if a power series $\sum_{v=0}^{\infty} a_v$ satisfies the conditions $a_v > 0$ and $a_v^2 \leq a_{v-1}a_{v+1} (v \geq 1)$, then the coefficients of its reciprocal power series $\sum_{v\geq 0} b_v$ satisfy $-a_v/a_0^2 \leq b_v \leq 0,\ v \geq 1$) [11, Lem. 8], we have that $-\frac{q-1}{q^{v+1}-1} \leq b_{v,q} \leq 0$, and therefore, $b_{v,q} \ll \left(\frac{1}{q}\right)^v$. We conclude that $M_i = \frac{1}{q_i} < \frac{q_{\min}}{q_i}$, so condition (iii) of Theorem 2 is satisfied with $\max(q_i M_i) = 1$.

Furthermore, we see that

$$S_{1/\sigma,q}(1) = K_q \quad \text{and} \quad S'_{1/\sigma,q}(1) = K'_q.$$

□

4.3 Unitary divisor function

A natural number m is a unitary divisor of a number n if m is a divisor of n, and m and n/m are coprime. Let us define the arithmetic function $\sigma^*(n)$ as the sum of the unitary divisors of n (analogous to the sum of divisors function).

We have that σ^* is multiplicative and $\sigma^*(p^v) = p^v + 1$.

Theorem 5. *If $q_{\min} \geq q_{\max}^{2/3}$ (in particular, if Q consists of a single prime), then*

$$\sum_{n\leq x} t_Q(n)\sigma^*(n) = \frac{\pi^2}{12\zeta(3)}x^2\left(2\prod_{q\in Q}\frac{(q^2-1)}{(q^2+q+1)}-1\right) + O\left(x(\log x)^{5/3}\right), \tag{8}$$

and

$$\sum_{n\leq x} t_Q(n)\frac{1}{\sigma^*(n)} = E_Q^*\log x + F_Q^* + O(x^{-1}(\log x)^{5/3+r}(\log\log x)^{4/3}) \tag{9}$$

for some constant F_Q^*, and for E_Q^*, the latter being defined by

$$E_Q^* = B^*\left(\frac{2}{\prod_{q\in Q}R_q}-1\right) \quad \textit{with} \quad B^* = \prod_p\left(R_p\cdot\left(1-\frac{1}{p}\right)\right) \quad \textit{and} \quad R_q := 1+\sum_{v=1}^{\infty}\frac{1}{q^v+1}.$$

Proof. We first prove (8). We know from Sitaramachandrarao and Suryanarayana [3, Eq. 1.4] that

$$\sum_{n\leq x}\sigma^*(n) = \frac{\pi^2}{12\zeta(3)}x^2 + O(x(\log x)^{5/3}),$$

so $R_{\sigma^*}(x) = (\log x)^{5/3}$, and condition (i) of Theorem 1 is satisfied.

Similarly, we have that

$$S_{\sigma^*,q}(x) = 1+\sum_{v=1}^{\infty}\sigma^*(q^v)x^v = 1+\sum_{v=1}^{\infty}(q^v+1)x^v = \frac{1-qx^2}{(1-qx)(1-x)}$$

for $|x| < 1/q$, so condition (ii) of Theorem 1 is satisfied.

We conclude that the reciprocal of the power series is given by

$$\begin{aligned}\overline{S}_{\sigma^*,q}(x) &= \frac{1-(q+1)x+qx^2}{1-qx^2} = -1+\frac{2-(q+1)x}{1-qx^2}\\ &= -1+\left(\frac{1+1/2(q^{1/2}+q^{-1/2})}{1+\sqrt{q}x}+\frac{1-1/2(q^{1/2}+q^{-1/2})}{1-\sqrt{q}x}\right)\\ &= -1+\left(1+1/2(q^{1/2}+q^{-1/2})\right)\sum_{v=0}^{\infty}(-1)^v\sqrt{q}^v x^v+\left(1-1/2(q^{1/2}+q^{-1/2})\right)\sum_{v=0}^{\infty}\sqrt{q}^v x^v,\end{aligned}$$

with $|x| < \frac{1}{\sqrt{q}}$. Then

$$b_{v,q} = \left(1 + 1/2(q^{1/2} + q^{-1/2})\right)(-1)^v \sqrt{q}^v + \left(1 - 1/2(q^{1/2} + q^{-1/2})\right)\sqrt{q}^v \ll \sqrt{q}^v.$$

We see that $r_i = q_i^{0.5}$ and $r_i \leq q_i^2/q_{\max}$, so condition (iv) of Theorem 2 is satisfied. Furthermore, we see that

$$S_{\sigma^*,q}(\frac{1}{q^2}) = \frac{1 - 1/q^3}{(1 - 1/q)(1 - 1/q^2)} = \frac{q^2 + q + 1}{q^2 - 1}.$$

Now, we prove (9). We know from Sita Ramaiah and Suryanarayana [7, p. 1352] that

$$\sum_{n \leq x} \frac{1}{\sigma^*(n)} = B^* \log x + D^* + O(x^{-1}(\log x)^{5/3}(\log\log x)^{4/3}).$$

Then we have $R_{1/\sigma^*} = (\log x)^{5/3}(\log\log x)^{4/3}$, so condition (i) of Theorem 2 is satisfied.

On the other hand,

$$S_{1/\sigma^*,q}(x) = 1 + \sum_{v=1}^{\infty} \frac{1}{\sigma^*(q^v)} x^v = 1 + \sum_{v=1}^{\infty} \frac{1}{q^v + 1} x^v \qquad (|x| < q),$$

so condition (ii) of Theorem 2 is satisfied.

Let us choose $a_n := \frac{1}{q^n+1}$, if $n \geq 1$, and $a_0 = 1$, as the coefficients for this formal power series. We also define b_n as the coefficients of the associated reciprocal series. Then

$$\begin{aligned}
a_n b_{n+1} &= \sum_{k=1}^{n-1} b_k (a_{n+1}a_{n-k} - a_n a_{n+1-k}) + b_n(a_{n+1} - a_n a_1) \qquad (n \geq 2),\\
\frac{1}{q^n+1} b_{n+1} &= \sum_{k=1}^{n-1} b_k \cdot \left(\frac{1}{q^{n+1}+1} \cdot \frac{1}{q^{n-k}+1} - \frac{1}{q^n+1} \cdot \frac{1}{q^{n+1-k}+1}\right)\\
&\quad + b_n \left(\frac{1}{q^{n+1}+1} - \frac{1}{q^n+1}\frac{1}{q+1}\right)\\
&= \sum_{k=1}^{n-1} b_k \frac{((q^n+1)(q^{n+1-k}+1) - (q^{n+1}+1)(q^{n-k}+1))}{(q^{n+1}+1)(q^{n-k}+1)(q^n+1)(q^{n+1-k}+1)} + b_n \frac{(q^n+1)(q+1) - (q^{n+1}+1)}{(q^{n+1}+1)(q^n+1)(q+1)}.
\end{aligned}$$

Then

$$\begin{aligned}
b_{n+1} &= \frac{q^n(q-1)}{q^{n+1}+1} \cdot \sum_{k=1}^{n-1} \frac{b_k(q^{-k}-1)}{(q^{n-k}+1)(q^{n+1-k}+1)} + \frac{b_n}{q+1} \cdot \frac{(q^n+q)}{(q^{n+1}+1)}\\
&= \frac{(q-1)q^n}{q^{n+1}+1} \cdot \sum_{k=1}^{n-1} b_k \frac{q^k - q^{2k}}{(q^n+q^k)(q^{n+1}+q^k)} + \frac{b_n}{q+1} \cdot \frac{(q^n+q)}{(q^{n+1}+1)}.
\end{aligned} \tag{10}$$

Let us suppose that $|b_i| \le \frac{C}{q^i}$ for some constant C and for all $i = 0, 1, 2, \ldots, n$. We prove by induction that $|b_{n+1}| \le \frac{C}{q^{n+1}}$. Indeed, we have that

$$|b_{n+1}| \le C \cdot \left(\frac{(q-1)q^n}{q^{n+1}+1} \cdot \sum_{k=1}^{n-1} \frac{q^k}{(q^k+q^n)^2} + \frac{q^n+q}{(q+1)\cdot q^n \cdot (q^{n+1}+1)} \right). \qquad (11)$$

Let us define the function $f(n) := \frac{q^{n+1}\cdot(q^n+q)}{(q+1)\cdot q^n \cdot (q^{n+1}+1)} = \frac{q^{n+1}+q^2}{(q+1)(q^{n+1}+1)} \to \frac{1}{q+1}$ if $n \to \infty$.

Furthermore, $f(n) = \frac{1}{q+1}\left(1 + \frac{q^2-1}{q^{n+1}+1}\right)$ is an increasing function on n, therefore $f(n) \le \frac{1.01}{q+1}$ for n large enough.

Similarly, we define the function $g(n) := \frac{(q-1)q^n q^{n+1}}{q^{n+1}+1} \cdot \sum_{k=1}^{n-1} \frac{q^k}{(q^k+q^n)^2}$. We have that

$$\begin{aligned}
g(n) &= \frac{(q-1)q^{2n+1}}{q^{2n+1}+q^n} \cdot \sum_{k=1}^{n-1} \frac{q^k}{(q^{n/2}+q^{k-n/2})^2} = \frac{q-1}{1+q^{-1-n}} \cdot \sum_{k=1}^{n-1} \frac{1}{(q^{(n-k)/2}+q^{-(n-k)/2})^2} \\
&= \frac{1}{4}\frac{(q-1)}{(1+q^{-1-n})} \cdot \sum_{k=1}^{n-1} \frac{1}{\cosh^2(\frac{n-k}{2}\log q)} = \frac{(q-1)}{4(1+q^{-1-n})} \cdot \sum_{j=1}^{n-1} \frac{1}{\cosh^2(j \log q/2)} \\
&\le (q-1)\sum_{j=1}^{\infty} \frac{1}{(e^{j\log q/2}+e^{-j\log q/2})^2} = (q-1)\cdot \sum_{j=1}^{\infty} \frac{1}{(q^{j/2}+q^{-j/2})^2} \\
&= (q-1)\sum_{j=1}^{\infty} \frac{q^{-j}}{(1+q^{-j})^2} \le (q-1)\sum_{j=1}^{\infty} q^{-j}(1-q^{-j}+q^{-2j})^2 \\
&= (x^{-1}-1)\sum_{j=1}^{\infty} x^j \cdot (1-x^j+x^{2j})^2,
\end{aligned}$$

where we set $x := q^{-1}$. We have that

$$\begin{aligned}
g(n) &\le (q-1)\cdot \sum_{j=1}^{\infty} (x^j - 2x^{2j} + 3x^{3j} - 2x^{4j} + x^{5j}) \\
&= (q-1)\left(\frac{x}{1-x} - \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} - \frac{2x^4}{1-x^4} + \frac{x^5}{1-x^5} \right) \\
&= \frac{q^9+q^8+5q^7+5q^6+7q^5+7q^4+8q^3+4q^2+3q+1}{(q^4+q^3+q^2+q+1)(q^2+q+1)(q^2+1)(q+1)} =: s(q).
\end{aligned}$$

Continuing with (10), we have that, for all n sufficiently large,

$$|b_{n+1}| \le \frac{C}{q^{n+1}}\left(s(q) + \frac{1.01}{q+1}\right) \le \frac{C}{q^{n+1}},$$

since $s(q) + 1.01/(q+1) < 1$ for all primes q. Indeed, the real function $s(x) + 1.01/(x+1)$ decreases in the interval $[2, \delta]$, for some δ, and increases in $[\delta, \infty)$, but it is always less than 1. Then condition condition (iii) of Theorem 2 is satisfied with $M_i = \frac{1}{q_i}$. □

The result (9) improves the result (51) by Tóth [11], thus solving open problem 41 of that publication.

4.4 Dedekind ψ function

Recall that the Dedekind function $\psi(n)$ is defined as $\psi(n) = n \prod_{p|n}(1 + \frac{1}{p})$.

Theorem 6. *We have that*

$$\sum_{n\le x} t_Q(n)\psi(n) = \frac{15}{2\pi^2}x^2 \left(2\prod_{q\in Q}\left(\frac{q(q-1)}{q^2+1}\right) - 1\right) + O(x(\log x)^{2/3}) \tag{12}$$

and

$$\sum_{n\le x} t_Q(n)\frac{1}{\psi(n)} = C\Bigg((\log x + \gamma + D)\cdot\left(2\prod_{q\in Q}\frac{q^2-1}{q^2+q-1} - 1\right) + 2\prod_{q\in Q}\frac{q^2-1}{q^2+q-1}\cdot\sum_{q\in Q}\frac{q^2\log q}{(q-1)(q^2+q-1)}\Bigg) + O(x^{-1}(\log x)^{2/3}(\log\log x)^{4/3}), \tag{13}$$

where

$$C = \prod_p\left(1 - \frac{1}{p(p+1)}\right) \quad \text{and} \quad D = \sum_p \frac{\log p}{p^2+p-1}.$$

(The constant $C \doteq 0.704442$ is sometimes called the carefree constant, and its digits form the sequence A065463 in OEIS [8].)

Proof. The proof of (12) is quite similar to that of (4). We know from Walfisz [12, p. 100] that

$$\sum_{n\le x}\psi(n) = \frac{15}{2\pi^2}x^2 + O(x(\log x)^{2/3}),$$

and we obtain that

$$S_{\psi,q}(x) := 1 + \sum_{v=1}^{\infty}\psi(q^v)x^v = \frac{1+x}{1-qx} \qquad (|x| < \frac{1}{q}),$$

thus concluding that

$$\overline{S}_{\psi,q}(x) = \frac{1-qx}{1+x} = 1 + (q+1)x\sum_{v=1}^{\infty}(-1)^v x^v.$$

Therefore, $b_{v,q} = (-1)^v(q+1) \ll 1$.

The proof of (13) is quite similar to that of (5). We know from Sita Ramaiah and Suryanarayana [7, Cor. 4.2] that

$$\sum_{n\leq x}\frac{1}{\psi(n)} = C(\log x+\gamma+D)+O(x^{-1}(\log x)^{2/3}(\log\log x)^{4/3}).$$

and, for the reciprocal power series, we obtain that

$$S_{1/\psi,q}(x) = 1+\sum_{v=1}^{\infty}\frac{1}{\psi(q^v)}x^v = \frac{q^2+q-x}{(q+1)(q-x)},$$

and

$$\overline{S}_{1/\psi,q}(x) = \frac{(q+1)(q-x)}{q^2+q-x} = 1-q\sum_{v=1}^{\infty}(\frac{1}{q^2+q})^v x^v.$$

Then we have that $b_{v,q} = -q\left(\frac{1}{q^2+q}\right)^v$ and $M_i = 1/(q_i^2+q_i)$. □

4.5 Euler's unitary function

We now consider an analogue of the Euler totient function, namely the multiplicative function φ^* defined on the prime powers p^v by $\varphi^*(p^v) = p^v-1$.

Theorem 7. *If $q_{\min}\geq q_{\max}^{2/3}$, we have that*

$$\sum_{n\leq x} t_Q(n)\varphi^*(n) = \frac{C}{2}x^2\left(2\prod_{q\in Q}\left(\frac{q^2-1}{q^2+q-1}\right)-1\right)+O(x(\log x)^{5/3}(\log\log x)^{4/3}), \tag{14}$$

where C is defined as in Theorem 6.

Proof. The proof is quite similar to that of (8). We know from Sitaramachandrarao and Suryanarayana [3] that

$$\sum_{n\leq x}\varphi^*(n) = \frac{C}{2}x^2+O(x(\log x)^{5/3}(\log\log x)^{4/3}),$$

and we obtain that

$$S_{\varphi^*,q}(x) = 1+\sum_{v=1}^{\infty}\varphi^*(q^v)x^v = \frac{1-2x+qx^2}{(1-qx)(1-x)}$$

thus concluding that

$$\overline{S}_{\varphi^*,q}(x) = \frac{(1-qx)(1-x)}{qx^2-2x+1} = 1+\frac{\sqrt{q-1}}{2}i\left(-\frac{1}{\omega}\sum_{v=0}^{\infty}(x/\omega)^{v+1}+\frac{1}{\overline{\omega}}\sum_{v=0}^{\infty}(x/\overline{\omega})^{v+1}\right),$$

with $\omega = \frac{1}{q}\cdot(1+i\sqrt{q-1})$. Consequently, $b_{v,q} = -\frac{\sqrt{q-1}}{2}i(-\frac{1}{\omega}\omega^{-v}-\frac{1}{\overline{\omega}}\overline{\omega}^{-v}) \ll (q^{1/2})^v$. □

Note: Knowing, from [5], that for certain constants L^* and M^*,

$$\sum_{n\le x}\frac{1}{\varphi^*(n)} = L^*\log x + M^* + O(x^{-1}(\log x)^{5/3}),$$

the author conjectures that a general result can be established for

$$\sum_{n\le x} t_Q(n)\frac{1}{\varphi^*(n)}$$

by proceeding in a manner similar to the one used in the proof of (9).

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