Journal of Integer Sequences, Vol. 23 (2020),
Article 20.2.5

The Lie Bracket and the Arithmetic Derivative

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Abstract

We apply the Lie bracket approach to characterize the semi-derivations on the positive integers. The approach is motivated by the Stroock-Lie bracket identity commonly used in Malliavin calculus.

1 Introduction

The Malliavin derivative D, the Skorohod integral δ, and the associated Malliavin calculus are powerful tools for the analysis of stochastic processes. The Malliavin calculus, named after P. Malliavin, is also called the *stochastic calculus of variations* [8, p. VII, p. 1]. The definition of the Malliavin derivative and the Skorohod integral can be found, for example, in [2, p. 25], [3, pp. 20, 27], and [10, pp. 25, 40], respectively. The Stroock-Lie bracket type identity $[D,\delta] = D\delta - \delta D = I$ is a common notion in the Malliavin calculus [6, p. 355], which is also referred to as the fundamental theorem of calculus [3, Thm. 3.18, p. 37]. The Malliavin derivative and the Skorohod integral (the adjoint operator) act in the space of random processes, which are treated as functions of a Gaussian process.

On the one hand, many random structures satisfy the functional Gaussian approximation [9]. On the other hand, the Lie bracket is a powerful tool in the study of differential equations, in particular, quantum stochastic calculus [11]. This motivated us to apply the Lie bracket in a totally different environment. More exactly, we treat D and δ as operators acting on integer sequences or dynamical systems on the natural numbers $\mathbb{N}$. Following Barbeau [1], Ufnarovski and Åhlander [14], and Kovič [7], we study a partial number derivative operator D_A. The operator is introduced and characterized as a solution to the modified Stroock-Lie bracket type identity. The Barbeau arithmetic derivative D is then characterized as the dynamics on the positive integers, which satisfies the Stroock-Lie bracket identity $[D, \ell] = I$, and which holds for all linear functions $\ell = \ell_p = pn$, where p is a prime number. Moreover, the Stroock-Lie bracket characterization is illustrated by examples on several commutative rings without zero divisors. Finally, arithmetic type differential equations driven by D_A are briefly analyzed.

2 Lie bracket analysis of linear functions of positive integers

Definition 1. For two functions $F, U : \mathbb{Z}_+ \to \mathbb{Z}_+$, we define the *Lie bracket* or the *commutator* $[F, U]$ by

$$[F, U] = F \circ U - U \circ F, \text{ where } F \circ U(n) = F(U(n)).$$

Hence $[F, U](n) = F(U(n)) - U(F(n))$.

Let $\mathcal{L} = \{\ell_x : \ell_x(n) = nx,\ n \in \mathbb{Z}_+\}$ denote the set of linear functions on $\mathbb{Z}_+$. Notice that $\mathcal{L}$ is a commutative semiring with unity $\ell_1 = I$ and zero $\ell_0 = 0$ with respect to the multiplication and addition operations. Then $\ell_x \circ \ell_y = \ell_{xy}$, $\ell_x + \ell_y = \ell_{x+y}$, $\ell_x \circ \ell_1 = \ell_1 \circ \ell_x = \ell_x$, and $\ell_x + \ell_0 = \ell_0 + \ell_x = \ell_x$. By construction, the semiring $\mathcal{L}$ is isomorphic to the semiring $(\mathbb{Z}_+, +, \cdot)$ of non-negative integers.

Definition 2. Consider $D : \mathbb{Z}_+ \to \mathbb{Z}_+$ and define the *Lie bracket linearity set* of D by

$$W_D = \{x : \text{ there exists } y = y_x \text{ such that } [D, \ell_x] = \ell_y\}. \tag{1}$$

Lemma 3. *W_D is a multiplicative semigroup in $(\mathbb{Z}_+, \cdot)$ that includes 1.*

Proof. Let $x, z \in W_D$. By the Lie bracket definition and algebraic manipulations,

$$[D, \ell_{xz}](n) = D \circ \ell_{xz}(n) - \ell_{xz} \circ D(n) = [D, \ell_x](zn) + \ell_x \circ [D, \ell_z](n).$$

Now we apply to the last line, first, the definition of W_D and, then, the semiring properties of linear functions. This leads to $[D, \ell_{xz}](n) = \ell_{y_x} \circ \ell_z(n) + \ell_x \circ \ell_{y_z}(n) = \ell_{y_x z + x y_z}(n)$. □

Lemma 4. *The following statements are equivalent:*

(i) D satisfies the Leibnitz rule $D(mn) = mD(n) + nD(m)$;

(ii) $D(1) = 0$ *and* $W_D = \mathbb{Z}_+$.

Moreover, $[D, \ell_m] = \ell_{D(m)}$.

Proof. (i) $\Longrightarrow$ (ii) follows by direct calculations.

(ii) $\Longrightarrow$ (i): Fix m. Then, for any $n \in \mathbb{Z}_+$, we have

$$y_m n = \ell_{y_m}(n) = [D, \ell_m](n) = D \circ \ell_m(n) - \ell_m \circ D(n) = D(mn) - mD(n).$$

Finally, take $n = 1$. Then $y_m = D(m) - mD(1) = D(m)$. Therefore, $[D, \ell_m] = \ell_{D(m)}$ and $D(mn) = mD(n) + nD(m)$, proving the lemma. □

Remark 5. Let x be a linear function l_x, and let the function composition $\circ$ be replaced by the usual multiplication. Then the main Lie bracket characteristic becomes the Pincherle derivative $f' = f \cdot x - x \cdot f = [f, l_x]$, as introduced in [12]. Tempesta [13] applied the Pincherle derivative and the associated Lie bracket approach in quantum calculus.

We now apply Lemmas 3 and 4 to characterize the arithmetic type derivative D_A as the dynamics on the positive integers, which satisfies the Stroock-Lie bracket type identity $[D_A, \ell_x] = y_x I$.

Let $\mathcal{P}$ denote the set of all primes. We say that sets A, B are orthogonal, if for any $x \in A$ and $y \in B$, we have $\gcd(x, y) = 1$, i.e., the sets A and B do not have any common divisors. Notice that disjoint subsets in $\mathcal{P}$ are orthogonal.

Lemma 6. *Consider the nonempty subset of primes* $A \subset \mathcal{P}$. *Let* $D_A : \mathbb{Z}_+ \to \mathbb{Z}_+$ *be such that* $D_A(1) = 0$. *The following properties are equivalent:*

(i) $[D_A, \ell_p] = I$, *for* $p \in A$, *and* $[D_A, \ell_p] = 0$, *for* $p \in \bar{A} = \mathcal{P} - A$;

(ii) $D_A = \sum_{p \in A} D_p$, *where* $D_p(n) = jp^{j-1}m$, *for* $n = p^j m$ *with* $m \perp p$. *Moreover,* D_A *satisfies the Leibniz rule and has a representation*

$$D_A(n) = n \sum_{p \in A} \frac{n_p}{p}, \text{ where } n = \prod_{p \in \mathcal{P}} p^{n_p}. \tag{2}$$

Proof. (ii) $\Longrightarrow$ (i) follows by direct calculations.

(i) $\Longrightarrow$ (ii): Consider the linearity set W_{D_A}. Notice that $\{0, 1\} \subseteq W_{D_A}$ and the smallest multiplicative semigroup, containing all primes, is $\mathbb{Z}_+ - \{0, 1\}$. By Lemma 3, it then follows that $W_{D_A} = \mathbb{Z}_+$. Thus, by Lemma 4, the function D satisfies the Leibniz rule. Moreover, $D_A(p) = 1$, for $p \in A$, and $D_A(p) = 0$, for $p \notin A$. Now we apply the argument from Ufnarovski and Åhlander [14]. Consider the log transform of D_A defined by $L_A(n) = D_A/n$. By the Leibniz rule, we see that L_A is a homomorphism of the multiplicative semigroup to the additive semigroup on $\mathbb{Z}_+$, which shows that

$$L_A(n) = \sum_{p \in \mathcal{P}} \frac{n_p}{p} D_A(p) = \sum_{p \in A} \frac{n_p}{p}, \text{ where } n = \prod_{p \in \mathcal{P}} p^{n_p}.$$

This implies representation (2). Take $A = \{p\}$. We then derive $D_p(p^j m) = jp^{j-1}m$, for $m \perp p$. This proves the representation $D_A = \sum_{p\in A} D_p$. □

Corollary 7. *Let $D : \mathbb{N} \to \mathbb{N}$ such that $D(1) = 0$. Assume that for each linear function $\ell = \ell_p = pn$, where p is a fixed prime, the following Stroock-Lie bracket identity*

$$[D, \ell] = I, \text{ i.e., } D\ell(n) = n + \ell D(n), \tag{3}$$

holds for all $n \in \mathbb{N}$. Then D is an arithmetic derivative, i.e.,

$$D(n) = n' = n \sum_{i=1}^{k} \frac{n_i}{p_i}, \text{ where } n = \prod_{i=1}^{k} p_i^{n_i}.$$

Moreover, $[D, \ell_m] = D(m)I$. In particular, we have the following characterization of the Stroock-Lie bracket identity

$$[D, \ell_m] = I \text{ if and only if } m \text{ is a prime.} \tag{4}$$

Proof. Let us prove the last statement. The Stroock-Lie bracket equation states that

$$D\ell_m(n) = n + \ell_m D(n) = n + mD(n) = n + mn'.$$

The left-hand side of the former equation is computed by

$$D\ell_m(n) = (mn)' = D(mn) = m'n + mn'.$$

Hence, by equating it to the right-hand side of the same equation, we derive $m'n + mn' = n + mn'$. Clearly the last equation holds if and only if $m' = 1$. Therefore, m is a prime, as proved in [14]. □

3 Lie bracket properties

3.1 Lie bracket properties for the arithmetic derivative

From Corollary 7, we derive

Corollary 8. *Let $m_1 + \cdots + m_k = m$. Then $[D, \ell_{m_1} + \cdots + \ell_{m_k}] = I$ holds if and only if m is a prime.*

Remark 9. (i) In particular, $[D, \ell_p + \ell_2] = I$ if and only if p and $p + 2$ are twin primes.

(ii) According to the Goldbach weak conjecture, every prime number greater than 5 can be expressed as the sum of three primes. Then, for each such triple of primes p_1, p_2, p_3 with $p_1 + p_2 + p_3$ being a prime, we have $[D, \ell_{p_1} + \ell_{p_2} + \ell_{p_3}] = I$.

Barbeau [1] proved that if the natural number n is not a prime or unity, then $n' \geqslant 2\sqrt{n}$. The equality holds if and only if $n = p^2$, where p is a prime. In particular, the equation $m' = 2$ does not have solutions in positive integers. Therefore, we obtain the following lemmas.

Lemma 10. *For any primes p_1 and p_2,*

$$[D, \ell_{p_1} + \ell_{p_2}] \neq [D, \ell_{p_1}] + [D, \ell_{p_2}] .$$

Proof. By definition, the right-hand side of the former equation equals to $2n$, $n \in \mathbb{N}$. The left-hand side of the equation is $[D, \ell_{p_1} + \ell_{p_2}](n) = D((p_1 + p_2)n) - (p_1 + p_2) D(n) = (p_1 + p_2)' n$. It remains to notice that the equation $(p_1 + p_2)' = 2$ does not have solutions. □

Lemma 11. *For any primes p_1 and p_2,*

$$[D + \ell_{p_1}, \ell_{p_2}] = I.$$

Proof. The left-hand side of the former equation is $[D + \ell_{p_1}, \ell_{p_2}] = [D, \ell_{p_2}] + [\ell_{p_1}, \ell_{p_2}] = I + [\ell_{p_1}, \ell_{p_2}]$. Clearly $[\ell_{p_1}, \ell_{p_2}] = 0$, which completes the proof. □

Next lemma shows that there are infinitely many non-linear dynamics or integer sequences satisfying the Stroock-Lie bracket lemma for the arithmetic derivative.

Lemma 12. *Let p_1, p_2 and q be three different prime numbers. And let $\sigma(q^q) = p_1 q^q$, $\sigma(n) = p_2 n$, for $n \neq q^q$. Then $[D, \sigma] = I$ and σ is not linear.*

Proof. For $n \neq q^q$,

$$D\sigma(n) = (p_2 n)' = n + p_2(n') = n + \sigma(n).$$

On the other hand, by definition we have $D(q^q) = q^q$ if q is a prime. Hence, for $n = q^q$, $D\sigma(q^q) = (p_1 q^q)' = q^q + p_1(q^q)' = q^q + \sigma(q^q)$, proving that $[D, \sigma] = I$. Clearly σ is not linear, since $\sigma(2q^q) = p_2(2q^q) \neq 2q^q = 2\sigma(q^q)$. □

3.2 Lie bracket properties for other derivatives

Following Ufnarovski and Åhlander [14], we define the generalized arithmetic derivative by

$$D(x) = x \sum_{i=1}^{k} \frac{x_i D(p_i)}{p_i}, \text{ where } x = \prod_{i=1}^{k} p_i^{x_i}.$$

Note that from this definition $D(p) = 1$ is no longer true for prime p, in general. Then the restriction on the linear function ℓ_m in (4) can be weakened to m such that $D(m) = 1$.

3.2.1 Partial arithmetic derivative D_p

The function D_p, defined in Lemma 6, is originated in Kovič [7] and referred to as the partial arithmetic derivative. Notice that $D_p = D_{\{p\}} = D$ is a generalized arithmetic derivative defined by $D(p) = 1$ and $D(q) = 0$ for all other primes q.

Lemma 13. *Let p, q be two different primes. Then*

(i) $[D_p, \ell_p] = I$;

(ii) $[D_p, \ell_q] = 0$;

(iii) $[D_p, \ell_p + \ell_q] = 0$;

(iv) $[D_p + D_q, \ell_p + \ell_q] = 0$.

Proof. The first two properties follow from Lemma 6 (i). By the Leibnitz rule for D_p and since $p \nmid p + q$, we have $[D_p, \ell_p + \ell_q] = 0$. This shows that D_p is not linear and proves (iii). Finally, the last property (iv) follows by the linearity of the Lie bracket. □

3.2.2 General arithmetic derivative D_A

Now consider the general arithmetic derivative D_A defined in Lemma 6. Notice that for $A \neq \emptyset$, the derivative D_A is not linear. Motivated by Haukkanen et al. [5], we derive the following properties on D_A.

Lemma 14. *Let A, A_i, $i = 1, 2, \ldots$ be nonempty subsets of primes. Then*

(i) $[D_{A_1}, D_{A_2}] = 0$ *if and only if* $A_1 = A_2$;

(ii) Let $n = \prod_{p \in \cap A_i} p^{n_p}$. *Then* $D_{A_i}(n) = D_{A_j}(n)$, *for all* i, j.

(iii) Let $n = p^j$ *where* p *is a prime. Assume that all prime divisors of* $2, 3, \ldots, j$ *are not in* $\bigcup_i A_i$. *Then, for any positive integers* k *and* $i_1, \ldots, i_k$, *we have*

$$D_{A_{i_1}} \cdots D_{A_{i_k}}(n) = D_p^k(n).$$

Proof. It follows by direct calculations. □

By inspecting the proof of Lemma 13, we extend it to the general arithmetic derivatives.

Lemma 15. *Let $p \in A$, $q \in B$ and $A \perp B$. Then*

(i) $[D_A, \ell_p] = I$;

(ii) $[D_A, \ell_q] = 0$;

(iii) $[D_A, \ell_p + \ell_q] = 0$;

(iv) $[D_A + D_B, \ell_p + \ell_q] = 0$.

4 Extensions of the Stroock-Lie bracket lemma to other rings

Motivated by Ufnarovski and Åhlander [14] and Haukkanen et al. [4], we discuss the extensions of Lemma 6 to commutative rings with the unique factorization property.

4.1 Extensions to polynomial rings

Consider a polynomial ring $K[\mathbb{C}]$, which is a unique factorization domain. By F_i denote monic irreducible polynomials, i.e., single-variable polynomials with leading coefficients 1. Any $F \in K[\mathbb{C}]$ admits the unique factorization $F = z\prod_{i=1}^{k} F_i^{n_i}$, where $n_i \in \mathbb{N}$ and $z \in \mathbb{C}$. Following Ufnarovski and Åhlander [14], define the derivative of polynomials as

$$DF = F\sum_{i=1}^{k}\frac{n_i}{F_i}$$

and $D(z) = 0$, for $z \in \mathbb{C}$. Since $D(zF) = zD(F) = zF\sum_{i=1}^{k}(n_i/F_i)$, it follows that D satisfies the Leibnitz rule. Notice that $D(G) = 1$ if and only if G is a monic irreducible polynomial. Consider a linear functional $\ell = \ell_G(H) = GH$, where $G, H \in K[\mathbb{C}]$. Then, by direct calculations similar to (4), the Stroock-Lie bracket identity holds $[D, \ell_G] = I$ if and only if $D(G) = 1$, i.e., G is the monic irreducible polynomial. Moreover, the sum of k monic irreducible polynomials is an irreducible polynomial with the leading coefficient k. Therefore, similar to Corollary 8, we then derive the following result.

Corollary 16. *Let $F_i, i = 1, \ldots, k$, be the monic irreducible polynomials in the polynomial ring $K[\mathbb{C}]$. Then $[D, \ell_{F_1} + \cdots + \ell_{F_k}] = kI$.*

4.2 Extensions to integers and rational numbers

Following Ufnarovski and Åhlander [14], we define

$$D(x) = x\sum_{i=1}^{k}\frac{x_i}{p_i},$$

where $0 < x = \prod_{i=1}^{k} p_i^{x_i} \in \mathbb{Q}$, p_i are different primes, and x_i are integers. Then, for $0 > x \in \mathbb{Q}$, we take $D(x) = -D(x)$.

The function D is a map from $\mathbb{Q}$ to $\mathbb{Q}$. As proved in Ufnarovski and Åhlander [14], the map D satisfies the Leibnitz rule. In particular, the Stroock-Lie bracket identity $[D, \ell_x] = I$ holds if and only if $D(x) = 1$. In this case, x does not need to be a prime, for example one can take $x = -5/4$ (see [14]).

5 Arithmetic type differential equations

5.1 First order linear arithmetic type differential equations

We begin by considering several cases of the arithmetic type differential equations.

Lemma 17. *Let A be a nonempty set of primes and x be a positive integer. Then*

(i) $D_A(x) = 0$ *if and only if* $x \perp A$*;*

(ii) $D_A(x) = 1$ *if and only if* x *is a prime in* A*;*

(iii) $D_A(x) = x$ *if and only if* $x = p^p k$ *for* $p \in A$ *and a positive integer* $k \perp A$*;*

(iv) $pD_A(x) = x$*, where* $p \in A$*, if and only if* $x = pk$ *for a positive integer* $k \perp A$*.*

Proof. (i) Notice that $D_A = \sum_{p\in A} D_p$, $D_p(x) \geq 0$. Haukkanen et al. [5, Theorem 1] proved that $D_p(x) = 0$ if and only if $p \nmid x$. Thus, $D_A = 0$ if and only if $x \perp A$.

(ii) For the above representation, $D_A(x) = 1$ if and only if there exists $p \in A$ such that $D_p(x) = 1$, which is equivalent to $x = p$.

(iii) We follow arguments in the proofs of [14, Theorem 4 and 5]. Assume that $x = p^j k$, where $p \nmid k$ and $p \in A$. Then $D_A(x) = p^{j-1}(jk + pD_A(k))$. And so, if $0 < j < p$, then $D_A(x) = p^{j-1}\tilde{k}$, where $p \nmid \tilde{k}$, implying that $D_A(x) \neq x$. On the other hand, assume that $x = p^p k$ with $k > 1$. Then

$$D_A(x) = p^p(k + D_A(k)) = p^p k$$

if and only if $D_A(k) = 0$, which holds by (i) if and only if $k \perp A$.

(iv) Clearly $p|x$. Let $x = np$, $n \in \mathbb{N}$. By assumption $p \in A$ and, hence, $D_A(p) = 1$. By the Leibnitz rule, we obtain

$$pD_A(x) = pD_A(np) = p(pD_A(n) + n) \geqslant pn = x,$$

where the equality holds if and only if $D_A(n) = 0$. By (i), then $n \perp A$, proving (iv). □

5.2 Lie bracket arithmetic type differential equations

Fix a nonempty subset A of primes and consider the following equation

$$x = [D_A, \ell_x](a) = D_A(x)a,$$

where a, x are positive integers.

In the next lemma, we characterize those pairs (a, x) that satisfy the equation.

Lemma 18. *The arithmetic differential equation* $aD_A(x) = x$ *has a solution in natural numbers* a, x *if and only if one of the following statements is satisfied:*

(i) $a = p$ *and* $x = kp$*;*

(ii) $a = 1$ *and* $x = kp^p$*,*

where p *is a prime in* A *and* $k \perp A$*.*

Proof. To identify the pair (a, x), we first determine a, and then solve the equation $aD_A(x) = x$. First, assume $a = 1$. The equation becomes $D_A(x) = x$. According to property (iii) of Lemma 17, then $x = p^p k$, where p is a prime in A and $k \perp A$.

Next, assume $a = p$ with $p \in A$. Then the equation becomes $pD_A(x) = x$. By property (iv) of Lemma 17, we then have $x = pk$, where p is a prime in A and $k \perp A$.

Now assume that $a \neq 1$. We show that $a = p$, for some $p \in A$. Firstly, consider a subcase $a \perp A$. Then the equation becomes $aD_A(x) = x$. Hence,

$$x = aD_A(x) = aD_A[aD_A(x)] = a^2 D_A^2(x) \ .$$

For any j, we then derive $x = a^j D_A^j(x)$, which is not possible. Hence, $a = pc$, for some $p \in A$ and $c \geq 1$. The equation becomes $pcD_A(x) = x$ and it remains to show that $c = 1$. We proceed by absurd and assume that $c > 1$. Let $D_A(x) = b$. Then

$$pcD_A(x) = pcD_A(pcb) = pc(cb + pD_A(cb)) > pcb = x,$$

because $c > 1$ and $D_A(cb) \geq 0$. Thus, the statement $x = pcD_A(x)$ is not possible. Hence, $c = 1$ and the proof is complete. □

6 Acknowledgments

We are grateful to the Editor-in-Chief and the referee for their valuable comments and suggestions.

References

[1] E. J. Barbeau, Remark on an arithmetic derivative, *Canad. Math. Bull.* **4** (1961), 117–122.

[2] D. R. Bell, *The Malliavin Calculus*, Dover, 2007.

[3] G. Di Nunno, B. Oksendal, and F. Proske, *Malliavin Calculus for Levy Processes with Applications to Finance*, Springer-Verlag, 2009.

[4] P. Haukkanen, M. Mattila, J. K. Merikoski, and T. Tossavainen, Can the arithmetic derivative be defined on a non-unique factorization domain? *J. Integer Sequences* **16** (2013), Article 13.1.2.

[5] P. Haukkanen, J. K. Merikoski, and T. Tossavainen, On arithmetic partial differential equations, *J. Integer Sequences* **19** (2016), Article 16.8.6.

[6] Y. Hu, *Analysis on Gaussian Space*, World Scientific, 2017.

[7] J. Kovič, The arithmetic derivative and antiderivative, *J. Integer Sequences* **15** (2012), Article 12.3.8.

[8] P. Malliavin and A. Thalmaier, *Stochastic Calculus of Variations in Mathematical Finance*, Springer, 2005.

[9] F. Merlevede, M. Peligrad, and S. Utev, *Functional Gaussian approximation for dependent structures*, Vol. 6 of Oxford Studies in Probability, Oxford University Press, 2019.

[10] D. Nualart, *The Malliavin calculus and Related Topics*, 2nd edition, Springer, 2006.

[11] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, 2012.

[12] S. Pincherle, Operatori lineari e coefficienti di fattoriali, *Alti Accad. Naz. Lincei, Rend. Cl. Fis. Mat. Nat.* **18** (1933), 417–519.

[13] P. Tempesta, From symmetries to number theory, *Physics of Atomic Nuclei* **72** (2009), 866–874.

[14] V. Ufnarovski and B. Åhlander, How to differentiate a number, *J. Integer Sequences* **6** (2003), Article 03.3.4.

2000 Mathematics Subject Classification: Primary 11A25; Secondary 11A41, 11R27.

Keywords: arithmetic derivative, unique factorization, prime number, Leibnitz rule, integer sequence.

Received July 26 2019; revised version received February 19 2020. Published in *Journal of Integer Sequences*, February 19 2020.

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