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On Polycosecant Numbers

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Abstract

We introduce and study a "level two" generalization of the poly-Bernoulli numbers, which may also be regarded as a generalization of the cosecant numbers. We prove a

recurrence relation, two exact formulas, and a duality relation for negative upper-index numbers.

1 Introduction

The first named author [7] defined the *poly-Bernoulli numbers* A099594 and later Arakawa and the first named author [2] studied a slightly modified version. They are, denoted by $B_n^{(k)}$ and $C_n^{(k)}$ respectively, defined by using generating series, as follows. For an integer $k \in \mathbb{Z}$, let $(B_n^{(k)})_{n\geq 0}$ and $(C_n^{(k)})_{n\geq 0}$ be the sequences of rational numbers given respectively by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} \tag{1}$$

and

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}, \tag{2}$$

where $\mathrm{Li}_k(z)$ is the polylogarithm function (or rational function when $k \leq 0$) defined by

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (|z| < 1). \tag{3}$$

Since $\mathrm{Li}_1(z) = -\log(1-z)$, the generating functions on the left-hand sides of (1) and (2) when $k = 1$ become

$$\frac{te^t}{e^t-1} \quad \text{and} \quad \frac{t}{e^t-1}$$

respectively. Hence $B_n^{(1)}$ and $C_n^{(1)}$ represent the standard Bernoulli numbers A027641, A027642, the only difference being $B_1^{(1)} = 1/2$ and $C_1^{(1)} = -1/2$ and otherwise $B_n^{(1)} = C_n^{(1)}$.

Several properties of poly-Bernoulli numbers have been found including the following results:

$$B_n^{(k)} = (-1)^n \sum_{i=0}^{n} \frac{(-1)^i i! \left\{ {n \atop i} \right\}}{(i+1)^k}, \quad C_n^{(k)} = (-1)^n \sum_{i=0}^{n} \frac{(-1)^i i! \left\{ {n+1 \atop i+1} \right\}}{(i+1)^k},$$

where k is an integer, n a non-negative integer, and let $\left\{ {n \atop i} \right\}$ denote the Stirling numbers of the second kind A008277. Moreover, their dualities

$$B_n^{(-k)} = B_k^{(-n)}, \tag{4}$$
$$C_n^{(-k-1)} = C_k^{(-n-1)} \tag{5}$$

$(k, n \in \mathbb{Z}_{\geq 0})$ are derived in [7, Theorems 1 and 2] or in [8, Section 2]. For combinatorial applications, see [3].

In this paper, we study the following 'level 2' analog of poly-Bernoulli numbers, denoted by $D_n^{(k)}$, which we call the *polycosecant numbers*. For each $k \in \mathbb{Z}$, define $D_n^{(k)}$ by

$$\frac{\mathrm{A}_k(\tanh(t/2))}{\sinh t} = \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!}, \tag{6}$$

where $\mathrm{A}_k(z)$ is the series

$$\mathrm{A}_k(z) = 2\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^k} \tag{7}$$

and $\tanh(z)$ and $\sinh(z)$ are the usual hyperbolic tangent and sine functions respectively. Since $\mathrm{A}_k(z)$, $\tanh(z)$ and $\sinh(z)$ are all odd functions, we immediately see that $D_{2n+1}^{(k)} = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Note that $\mathrm{A}_1(z) = 2\tanh^{-1}(z)$, and thus

$$\sum_{n=0}^{\infty} D_n^{(1)} \frac{t^n}{n!} = \frac{t}{\sinh t} = \frac{it}{\sin(it)} \quad (i = \sqrt{-1}).$$

Hence $D_n^{(1)}$ is the cosecant number D_n which Nörlund [14, p. 27 (39) and p. 32 (52)] first introduced A001896, A001897. It should be noted that the terminology 'cosecant number' was not used by Nörlund. Apparently Kowalenko [11] was the first to use the terminology, but he adopted the name for $D_n/n!$ instead of D_n. He gave many applications, and studied a generalization together with interesting number-theoretical applications [5, 11, 12, 13].

Here, we give a table of $D_n^{(k)}$ for small k and n. A table for $k < 0$ will be given in §3.

k \ n	0	2	4	6	8	10
0	1	0	0	0	0	0
1	1	$-\frac{1}{3}$	$\frac{7}{15}$	$-\frac{31}{21}$	$\frac{127}{15}$	$-\frac{2555}{33}$
2	1	$-\frac{4}{9}$	$\frac{176}{225}$	$-\frac{6464}{2205}$	$\frac{3328}{175}$	$-\frac{1037312}{5445}$
3	1	$-\frac{13}{27}$	$\frac{3103}{3375}$	$-\frac{859939}{231525}$	$\frac{12761501}{496125}$	$-\frac{63453851}{232925}$
4	1	$-\frac{40}{81}$	$\frac{49184}{50625}$	$-\frac{98447744}{24310125}$	$\frac{4519218688}{156279375}$	$-\frac{6868861044736}{21791298375}$
5	1	$-\frac{121}{243}$	$\frac{751927}{759375}$	$-\frac{10665916999}{2552563125}$	$\frac{1488186370469}{49228003125}$	$-\frac{25213417199300173}{75506848869375}$

Table 1: $D_n^{(k)}$ $(0 \leq k \leq 5,\ 0 \leq n \leq 10, \text{even})$

We should also mention that our $D_n^{(k)}$ is (if slightly modified) a special case of a generalization of the poly-Bernoulli number which Sasaki [15, Definition 5] introduced. The numbers $D_n^{(k)}$ are closely connected to the 'multiple Hurwitz zeta functions' and the 'level 2' multiple zeta functions. We shall explore these connections in an ongoing project, which will be stated in the forthcoming paper.

2 Recurrence and explicit formulas for polycosecant numbers

In this section, we obtain a recurrence and explicit formulas for polycosecant numbers.

We begin by deriving a recurrence relation.

Proposition 1. *For every integer k and $n \geq 0$, the polycosecant numbers obey the recurrence relation of*

$$D_n^{(k-1)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k)}. \tag{8}$$

Proof. First, differentiate (6), which yields

$$\frac{\mathrm{A}_{k-1}(\tanh(t/2))}{\sinh t} = \cosh t \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} + \sinh t \sum_{n=1}^{\infty} D_n^{(k)} \frac{t^{n-1}}{(n-1)!}.$$

From the above result we have

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n^{(k-1)} \frac{t^n}{n!} &= \sum_{m=0}^{\infty} \frac{t^{2m}}{(2m)!} \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} + \sum_{m=0}^{\infty} \frac{t^{2m+1}}{(2m+1)!} \sum_{n=1}^{\infty} D_n^{(k)} \frac{t^{n-1}}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} D_{n-2m}^{(k)} \frac{t^n}{(2m)!(n-2m)!} + \sum_{n=1}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} D_{n-2m}^{(k)} \frac{t^n}{(2m+1)!(n-2m-1)!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m} D_{n-2m}^{(k)} \frac{t^n}{n!} + \sum_{n=1}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m+1} D_{n-2m}^{(k)} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k)} \frac{t^n}{n!}.
\end{aligned}$$

Since t is arbitrary, we can equate like powers of t on both sides, thereby obtaining the desired result. □

Since $\mathrm{A}_0(\tanh(t/2)) = \sinh(t)$, we observe $D_0^{(0)} = 1$ and $D_n^{(0)} = 0$ for $n \geq 1$. Hence equation (8) can be used to compute $D_n^{(k)}$ for $k < 0$ recursively starting from $D_n^{(0)}$. For $k > 0$, we rewrite (8) as

$$(n+1) D_n^{(k)} = D_n^{(k-1)} - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k)}$$

in order to compute $D_n^{(k)}$ recursively. We observe that $D_0^{(k)} = 1$ for all $k \in \mathbb{Z}$.

We continue by presenting two formulas for polycosecant numbers. Before doing so, we require the following lemma.

Lemma 2. *For $n \geq 1$ we have,*

$$x^n \left(\frac{d}{dx}\right)^n = \sum_{m=1}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} \left(x \frac{d}{dx}\right)^m.$$

Here, we let $\begin{bmatrix} n \\ m \end{bmatrix}$ denote the Stirling numbers of the first kind.

Proof. This result can be proved in the same manner as [1, Proposition 2.6 (4)]. Hence we omit here. □

Theorem 3. *For $k \in \mathbb{Z}$ and $n \geq 0$, the following results hold.*

(1)

$$D_n^{(k)} = 4 \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=1}^{2m+1} \sum_{q=0}^{n-2m} (2^{p+q+1} - 1) \binom{n}{q} \begin{bmatrix} 2m+1 \\ p \end{bmatrix} \begin{Bmatrix} n-q \\ 2m \end{Bmatrix} \frac{B_{p+q+1}}{p+q+1},$$

where B_n $(= C_n^{(1)})$ are the Bernoulli numbers, and

(2)

$$D_n^{(k)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=2m}^{n} \frac{(-1)^p (p+1)!}{2^p} \binom{p}{2m} \begin{Bmatrix} n+1 \\ p+1 \end{Bmatrix}.$$

Proof. We may express (6) as

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} &= \frac{\mathrm{A}_k(\tanh(t/2))}{\sinh t} \\ &= 2 \sum_{m=0}^{\infty} \frac{(\tanh(t/2))^{2m+1}}{(2m+1)^k} \frac{1}{\sinh t} \\ &= 4 \sum_{m=0}^{\infty} \frac{1}{(2m+1)^k} \frac{e^t (e^t - 1)^{2m}}{(e^t + 1)^{2m+2}}. \end{aligned} \tag{9}$$

Since

$$\frac{1}{(x+1)^{n+1}} = \frac{(-1)^n}{n!} \left(\frac{d}{dx}\right)^n \frac{1}{x+1}, \tag{10}$$

we see that by setting $x = e^t$ and introducing Lemma 2,

$$\frac{e^{nt}}{(e^t+1)^{n+1}} = \frac{1}{n!}\sum_{p=1}^{n}(-1)^p \begin{bmatrix} n \\ p \end{bmatrix} \left(\frac{d}{dt}\right)^p \frac{1}{e^t+1}. \tag{11}$$

Moreover, from the generating functions

$$\frac{t}{e^t-1} = \sum_{q=0}^{\infty} B_q \frac{t^q}{q!},$$

and

$$\frac{1}{e^t+1} = \frac{1}{e^t-1} - \frac{2}{e^{2t}-1},$$

we find that

$$\frac{1}{e^t+1} = \sum_{q=0}^{\infty}(1-2^q)B_q \frac{t^{q-1}}{q!}.$$

Taking the p-th derivative on both sides yields

$$\left(\frac{d}{dt}\right)^p \left(\frac{1}{e^t+1}\right) = \sum_{q=p+1}^{\infty}(1-2^q)\frac{B_q}{q}\frac{t^{q-p-1}}{(q-p-1)!} = \sum_{q=0}^{\infty}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}.$$

Now we substitute the above result into (11) to obtain

$$\begin{aligned}\frac{e^{nt}}{(e^t+1)^{n+1}} &= \frac{1}{n!}\sum_{p=1}^{n}(-1)^p \begin{bmatrix} n \\ p \end{bmatrix} \sum_{q=0}^{\infty}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!} \\ &= \frac{1}{n!}\sum_{q=0}^{\infty}\sum_{p=1}^{n}(-1)^p \begin{bmatrix} n \\ p \end{bmatrix} (1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\frac{e^t}{(e^t+1)^{2m+2}} &= \frac{e^{-(2m+1)t}}{(e^{-t}+1)^{2m+2}} \\ &= \frac{1}{(2m+1)!}\sum_{q=0}^{\infty}\sum_{p=1}^{2m+1}(-1)^{p+q} \begin{bmatrix} 2m+1 \\ p \end{bmatrix} (1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}.\end{aligned}$$

With the aid of the generating function given by [1, Proposition 2.6 (7)] and noting that $\begin{Bmatrix} s \\ 2m \end{Bmatrix} = 0$ if $s < 2m$ and

$$(e^t-1)^{2m} = (2m)!\sum_{s=0}^{\infty} \begin{Bmatrix} s \\ 2m \end{Bmatrix} \frac{t^s}{s!},$$

we arrive at

$$\begin{aligned}
&\frac{e^t(e^t-1)^{2m}}{(e^t+1)^{2m+2}}\\
&=\frac{1}{2m+1}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\begin{bmatrix}2m+1\\p\end{bmatrix}\begin{Bmatrix}s\\2m\end{Bmatrix}\frac{B_{p+q+1}}{p+q+1}\frac{t^{q+s}}{q!s!}\\
&=\frac{1}{2m+1}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\binom{n}{q}\begin{bmatrix}2m+1\\p\end{bmatrix}\begin{Bmatrix}n-q\\2m\end{Bmatrix}\frac{B_{p+q+1}}{p+q+1}\frac{t^{n}}{n!}.
\end{aligned}$$

Substituting the above result into (9), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}D_n^{(k)}\frac{t^n}{n!}\\
&=4\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{k+1}}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\\
&\quad\times\binom{n}{q}\begin{bmatrix}2m+1\\p\end{bmatrix}\begin{Bmatrix}n-q\\2m\end{Bmatrix}\frac{B_{p+q+1}}{p+q+1}\frac{t^{n}}{n!}\\
&=4\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{(2m+1)^{k+1}}\sum_{p=1}^{2m+1}\sum_{q=0}^{n-2m}(2^{p+q+1}-1)\binom{n}{q}\begin{bmatrix}2m+1\\p\end{bmatrix}\begin{Bmatrix}n-q\\2m\end{Bmatrix}\frac{B_{p+q+1}}{p+q+1}\frac{t^{n}}{n!}.
\end{aligned}$$

In obtaining the above result, we have used $B_{p+q+1}=0$ for $p+q\geq 1$ and even, while $\begin{Bmatrix}n-q\\2m\end{Bmatrix}=0$ for $n-q<2m$. By equating like powers, we arrive at the first result in the theorem.

To prove the second result, we require a formula from [4] for the higher order tangent numbers, $T_{n,m}$, whose generating function is

$$\frac{\tan^m t}{m!}=\sum_{n=m}^{\infty}T_{n,m}\frac{t^n}{n!}. \tag{12}$$

The formula is

$$T_{n,m}=\frac{i^{n-m}}{m!}\sum_{p=m}^{n}(-2)^{n-p}p!\binom{p-1}{m-1}\begin{Bmatrix}n\\p\end{Bmatrix}. \tag{13}$$

From (6),

$$\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} = \frac{\mathrm{A}_k(\tanh(t/2))}{\sinh t} = \frac{d}{dt}\mathrm{A}_{k+1}(\tanh(t/2))$$
$$= 2\frac{d}{dt}\sum_{m=0}^{\infty} \frac{(\tanh(t/2))^{2m+1}}{(2m+1)^{k+1}}. \tag{14}$$

By using $\tanh t = -i\tan(it)$ and equations (12) and (13), we can write

$$(\tanh(t/2))^m = (-i)^m m! \sum_{n=m}^{\infty} T_{n,m} \frac{i^n}{2^n}\frac{t^n}{n!}$$
$$= (-i)^m(-1)^{\frac{n-m}{2}} \sum_{n=m}^{\infty}\sum_{p=m}^{n} (-2)^{n-p} p! \binom{p-1}{m-1} \left\{ {n \atop p} \right\} \frac{i^n}{2^n}\frac{t^n}{n!}$$
$$= (-1)^m \sum_{n=m}^{\infty}\sum_{p=m}^{n} (-1)^p \frac{p!}{2^p} \binom{p-1}{m-1} \left\{ {n \atop p} \right\} \frac{t^n}{n!}.$$

Therefore, we find that

$$\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{k+1}} \sum_{n=2m+1}^{\infty}\sum_{p=2m+1}^{n} (-1)^{p+1}\frac{p!}{2^{p-1}} \binom{p-1}{2m} \left\{ {n \atop p} \right\} \frac{t^{n-1}}{(n-1)!}$$
$$= \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{k+1}} \sum_{n=2m}^{\infty}\sum_{p=2m}^{n} (-1)^p \frac{(p+1)!}{2^p} \binom{p}{2m} \left\{ {n+1 \atop p+1} \right\} \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=2m}^{n} \frac{(-1)^p(p+1)!}{2^p} \binom{p}{2m} \left\{ {n+1 \atop p+1} \right\} \frac{t^n}{n!}.$$

By equating like powers of t, we arrive at the second result in the theorem. □

3 Duality

We now turn our attention to the duality property of the polycosecant numbers. We shall present two different proofs using the same generating function. The first proof is based on a closed symmetric formula for the generating function, while the second is more indirect and complicated. However, we have decided to include the latter proof since it reveals fascinating results, especially regarding hyperbolic trigonometric functions.

Here is a table of $D_{2n}^{(-2k-1)}$ for small k and n.

Theorem 4. *For $n, k \in \mathbb{Z}_{\geq 0}$, the polycosecant numbers possess the duality property of*

$$D_{2n}^{(-2k-1)} = D_{2k}^{(-2n-1)}. \tag{15}$$

$k \backslash n$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	13	121	1093	9841	88573
2	1	121	4081	111721	2880481	72799321
3	1	1093	111721	7256173	403087441	20966597653
4	1	9841	2880481	403087441	42931692481	4032800405041
5	1	88573	72799321	20966597653	4032800405041	638704166793133

Table 2: $D_{2n}^{(-2k-1)}$ $(0 \le k \le 5,\ 0 \le n \le 5)$

First proof. We show that the generating function of $D_{2n}^{(-2k-1)}$,

$$F(x,y) := \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} D_{2n}^{(-2k-1)} \frac{x^{2n}}{(2n)!}\frac{y^{2k}}{(2k)!}, \tag{16}$$

is symmetric in x and y. This is ensured by the following closed formula for $F(x,y)$. $\square$

Proposition 5. *Let*

$$G(x,y) = \frac{e^{x+y}}{(1+e^x+e^y-e^{x+y})^2}.$$

Then one finds

$$F(x,y) = G(x,y) + G(x,-y) + G(-x,y) + G(-x,-y).$$

Proof. We first compute the generating function of all $D_n^{(-k)}$,

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} D_n^{(-k)} \frac{x^n}{n!}\frac{y^k}{k!}. \tag{17}$$

We claim that the formula

$$f(x,y) = \frac{e^x(e^y-1)}{1+e^x+e^y-e^{x+y}} + \frac{e^{-x}(e^y-1)}{1+e^{-x}+e^y-e^{-x+y}} \tag{18}$$

holds. To prove this, we first observe that, by definition,

$$\begin{aligned} f(x,y) &= \sum_{k=0}^{\infty} \frac{\mathrm{A}_{-k}(\tanh(x/2))}{\sinh x}\frac{y^k}{k!} \\ &= \frac{2}{\sinh x}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(2n+1)^k(\tanh(x/2))^{2n+1}\frac{y^k}{k!}. \end{aligned}$$

Noting that

$$2\sum_{n=0}^{\infty}(2n+1)^k t^{2n+1} = 2\left(t\frac{d}{dt}\right)^k \frac{t}{1-t^2} = \left(t\frac{d}{dt}\right)^k \left(\frac{1}{1-t} - \frac{1}{1+t}\right), \tag{19}$$

and by the standard formula (*cf.*, *e.g.*, [1, Proposition 2.6 (4)])

$$\left(t\frac{d}{dt}\right)^k = \sum_{m=1}^{k} \left\{ {k \atop m} \right\} t^m \left(\frac{d}{dt}\right)^m,$$

we find that the right-hand side of (19) becomes

$$\begin{aligned}
&\sum_{m=1}^{k} \left\{ {k \atop m} \right\} t^m \left(\frac{d}{dt}\right)^m \left(\frac{1}{1-t} - \frac{1}{1+t}\right) \\
&\quad = \sum_{m=1}^{k} \left\{ {k \atop m} \right\} m! \left(\frac{t^m}{(1-t)^{m+1}} - \frac{(-t)^m}{(1+t)^{m+1}}\right).
\end{aligned}$$

Therefore, by setting $t = \tanh(x/2)$ and noting $t/(1-t) = (e^x-1)/2$, $-t/(1+t) = (e^{-x}-1)/2$, $(\sinh x)(1-t) = e^{-x}(e^x-1)$, $(\sinh x)(1+t) = e^x - 1$, we arrive at

$$\begin{aligned}
f(x,y) &= \frac{1}{\sinh x}\sum_{k=0}^{\infty}\sum_{m=1}^{k} \left\{ {k \atop m} \right\} m! \left(\frac{t^m}{(1-t)^{m+1}} - \frac{(-t)^m}{(1+t)^{m+1}}\right)\frac{y^k}{k!} \\
&= \sum_{k=0}^{\infty}\sum_{m=1}^{k} \left\{ {k \atop m} \right\} m! \left(\frac{e^x}{e^x-1}\left(\frac{e^x-1}{2}\right)^m - \frac{1}{e^x-1}\left(\frac{e^{-x}-1}{2}\right)^m\right)\frac{y^k}{k!} \\
&= \sum_{m=1}^{\infty}(e^y-1)^m \left(\frac{e^x}{e^x-1}\left(\frac{e^x-1}{2}\right)^m - \frac{1}{e^x-1}\left(\frac{e^{-x}-1}{2}\right)^m\right) \\
&= \frac{e^x}{e^x-1}\cdot\frac{(e^y-1)(e^x-1)}{2-(e^y-1)(e^x-1)} - \frac{1}{e^x-1}\cdot\frac{(e^y-1)(e^{-x}-1)}{2-(e^y-1)(e^{-x}-1)} \\
&= \frac{e^x(e^y-1)}{1+e^x+e^y-e^{x+y}} + \frac{e^{-x}(e^y-1)}{1+e^{-x}+e^y-e^{-x+y}}.
\end{aligned}$$

This proves the identity (18). From (18) we see that $f(x,y)$ is even in x, and so we have

$$\frac{f(x,y)-f(x,-y)}{2} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} D_{2n}^{(-2k-1)}\frac{x^{2n}}{(2n)!}\frac{y^{2k+1}}{(2k+1)!}.$$

Our generating function $F(x,y)$ is the derivative of this relation with respect to y, and Proposition 5 follows from a straightforward calculation, and by the symmetry of $F(x,y)$ in x and y, Theorem 4 is proved.

□

Before presenting the second proof of Theorem 4, we require several lemmas.

Lemma 6. *The function $F(x,y)$ defined by (16) can be expressed as*

$$F(x,y) = 2\sum_{n=0}^{\infty} \frac{\partial}{\partial x}\left(\tanh^{2n+1}(x/2)\right)\cosh((2n+1)y).$$

Proof. By (6) and (16), we have

$$\begin{aligned}
F(x,y) &= 2\sum_{k=0}^{\infty} \frac{\mathrm{A}_{-2k-1}(\tanh(x/2))}{\sinh(x)}\frac{y^{2k}}{(2k)!} \\
&= \frac{2}{\sinh(x)}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(2n+1)^{2k+1}\tanh^{2n+1}(x/2)\frac{y^{2k}}{(2k)!} \\
&= \frac{2}{\sinh(x)}\sum_{n=0}^{\infty}(2n+1)\tanh^{2n+1}(x/2)\cosh((2n+1)y) \\
&= \frac{1}{\sinh(x/2)\cosh(x/2)}\sum_{n=0}^{\infty}(2n+1)\tanh^{2n}(x/2)\frac{\sinh(x/2)}{\cosh(x/2)}\cosh((2n+1)y) \\
&= 2\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\left(\tanh^{2n+1}(x/2)\right)\cosh((2n+1)y).
\end{aligned}$$

This completes the proof. □

We write

$$F(x,y) = \sum_{m=0}^{\infty} g_m(x)\frac{y^{2m}}{(2m)!} = \sum_{m=0}^{\infty} h_m(y)\frac{x^{2m}}{(2m)!}.$$

If one can show that $g_m(x) = h_m(x)$ for all $m \geq 0$, then the second proof will be complete.

First, we consider $g_m(x)$. Expanding $\cosh((2n+1)y)$ in Lemma 6 and equating like powers of y, we obtain

$$g_m(x) = \left(\frac{\partial}{\partial y}\right)^{2m} F(x,y)\bigg|_{y=0} = 2\frac{d}{dx}\sum_{n=0}^{\infty}(2n+1)^{2m}\tanh^{2n+1}(x/2).$$

Next we note that

$$\sum_{n=0}^{\infty}(2n+1)^{2m}t^{2n+1} = \left(t\frac{d}{dt}\right)^{2m}\sum_{n=0}^{\infty}t^{2n+1} = \left(t\frac{d}{dt}\right)^{2m}\frac{t}{1-t^2}. \tag{20}$$

Setting $t = \tanh(x/2)$ and noting

$$dt = \frac{1}{2}\frac{1}{\cosh^2(x/2)}dx, \qquad \frac{t}{1-t^2} = \frac{\tanh(x/2)}{1-\tanh^2(x/2)} = \frac{1}{2}\sinh x,$$

we have

$$t\frac{d}{dt} = \tanh(x/2)\cdot 2\cosh^2(x/2)\frac{d}{dx} = \sinh x\,\frac{d}{dx}.$$

Therefore we obtain

$$g_m(x) = \frac{d}{dx}\left(\sinh x\,\frac{d}{dx}\right)^{2m}\sinh x. \tag{21}$$

We can explicitly write down the right-hand side by using the following lemma.

For $m \in \mathbb{Z}_{\geq 0}$, we define sequences $(a_i^{(m)})_{0\leq i\leq m} \subset \mathbb{Q}$ inductively by

$$\begin{aligned} &a_0^{(0)} = 1, \\ &a_i^{(m)} = \frac{1}{2}\left(i(2i-1)a_{i-1}^{(m-1)} - (2i+1)^2 a_i^{(m-1)} + (i+1)(2i+3)a_{i+1}^{(m-1)}\right) \quad (m \geq 1), \end{aligned} \tag{22}$$

where we formally interpret $a_i^{(m)} = 0$ for $i < 0$ or $i > m$.

Lemma 7. *For $m \in \mathbb{Z}_{\geq 0}$,*

$$\left(\sinh x\,\frac{d}{dx}\right)^{2m}\sinh x = \sum_{i=0}^{m} a_i^{(m)}\sinh((2i+1)x). \tag{23}$$

Proof. We give the proof by induction on m. For $m = 0$, the identity trivially holds. We assume

$$\left(\sinh x\,\frac{d}{dx}\right)^{2(m-1)}\sinh x = \sum_{i=0}^{m-1} a_i^{(m-1)}\sinh((2i+1)x).$$

Using

$$\cosh(kx)\sinh(x) = \frac{1}{2}\left(\sinh((k+1)x) - \sinh((k-1)x)\right),$$

we have

$$\left(\sinh x\,\frac{d}{dx}\right)^{2m-1}\sinh x = \frac{1}{2}\sum_{i=0}^{m-1}(2i+1)a_i^{(m-1)}\left(\sinh((2i+2)x) - \sinh(2ix)\right),$$

and

$$\begin{aligned}
&\left(\sinh x\,\frac{d}{dx}\right)^{2m}\sinh x\\
&=\sum_{i=0}^{m-1}(2i+1)a_i^{(m-1)}\biggl(\frac{i+1}{2}\left(\sinh((2i+3)x)-\sinh((2i+1)x)\right)\\
&\qquad\qquad -\frac{i}{2}\left(\sinh((2i+1)x)-\sinh((2i-1)x)\right)\biggr)\\
&=\frac{1}{2}\sum_{i=1}^{m} i(2i-1)a_{i-1}^{(m-1)}\sinh((2i+1)x)\\
&\quad-\frac{1}{2}\sum_{i=0}^{m-1}(2i+1)^2a_i^{(m-1)}\sinh((2i+1)x)\\
&\quad+\frac{1}{2}\sum_{i=0}^{m-2}(i+1)(2i+3)a_{i+1}^{(m-1)}\sinh((2i+1)x).
\end{aligned}$$

Thus we observe that the coefficients of $\sinh((2i+1)x)$ are in accordance with (22), thereby completing this proof by induction. □

Using this lemma, we obtain

$$g_m(x)=\sum_{i=0}^{m}(2i+1)a_i^{(m)}\cosh((2i+1)x). \tag{24}$$

Secondly, we compute $h_m(y)$. Again by using Lemma 6, we have

$$\begin{aligned}
h_m(y)&=\left(\frac{\partial}{\partial x}\right)^{2m}F(x,y)\bigg|_{x=0}\\
&=2\sum_{n=0}^{\infty}\left(\frac{d}{dx}\right)^{2m+1}\left(\tanh^{2n+1}(x/2)\right)\cosh((2n+1)y)\bigg|_{x=0}\\
&=2\sum_{n=0}^{m}\left(\frac{d}{dx}\right)^{2m+1}\tanh^{2n+1}(x/2)\bigg|_{x=0}\cdot\cosh((2n+1)y) \qquad (25)
\end{aligned}$$

because

$$\tanh^{2n+1}(x/2)=\frac{x^{2n+1}}{2^{2n+1}}+O(x^{2n+2})\quad(x\to 0).$$

We write down the right-hand side of (25) by using the following lemma.

Lemma 8. *For $n, l \in \mathbb{Z}_{\geq 0}$, there exist sequences $(b_j^{(n,l)})_{0\leq j\leq l} \subset \mathbb{Q}$ such that*

$$\left(\frac{d}{dx}\right)^l \tanh^{2n+1}(x/2) = \sum_{j=0}^{l} b_j^{(n,l)} \tanh^{2n+1-l+2j}(x/2), \tag{26}$$

where $b_j^{(n,l)} = 0$ if $2n+1-l+2j < 0$. In particular,

$$\left(\frac{d}{dx}\right)^{2m+1} \tanh^{2n+1}(x/2)\Big|_{x=0} = b_{m-n}^{(n,2m+1)}. \tag{27}$$

Proof. For each n, we can immediately obtain the form (26) by induction on l, using the relation

$$\frac{d}{dx}\tanh^{2n+1}(x/2) = \frac{2n+1}{2}\left(\tanh^{2n}(x/2) - \tanh^{2n+2}(x/2)\right).$$

□

Combining Lemma 8 and (25), we obtain

$$h_m(y) = 2\sum_{n=0}^{m} b_{m-n}^{(n,2m+1)} \cosh((2n+1)y). \tag{28}$$

Now we are going to show $2b_{m-n}^{(n,2m+1)} = (2i+1)a_i^{(m)}$, which implies $g_m(x) = h_m(x)$. For $m, n \in \mathbb{Z}_{\geq 0}$ with $n \leq m$, set $\widetilde{b}_n^{(m)} = 2b_{m-n}^{(n,2m+1)}$. Then, by (27), we have $\widetilde{b}_0^{(0)} = 1$. Furthermore the following lemma holds.

Lemma 9. *For $m \in \mathbb{Z}_{\geq 1}$, the $\widetilde{b}_n^{(m)}$ satisfy the recurrence relation given by*

$$\widetilde{b}_n^{(m)} = \frac{2n+1}{2}\left(n\widetilde{b}_{n-1}^{(m-1)} - (2n+1)\widetilde{b}_n^{(m-1)} + (n+1)\widetilde{b}_{n+1}^{(m-1)}\right) \quad (n \leq m), \tag{29}$$

where we interpret $b_i^{(k)} = 0$ for $i < 0$ or $i > k$.

Proof. It follows from (26) that

$$\left(\frac{d}{dx}\right)^{2m+1} \tanh^{2n+1}(x/2) = \sum_{j=0}^{2m+1} b_j^{(n,2m+1)} \tanh^{2n-2m+2j}(x/2). \tag{30}$$

Differentiating twice and using (26), we see that the left-hand side is equal to

$$\begin{aligned}
&\left(\frac{d}{dx}\right)^{2m}\left(\frac{2n+1}{2}\tanh^{2n}(x/2)-\tanh^{2n+2}(x/2)\right)\\
&=\frac{2n+1}{2}\left(\frac{d}{dx}\right)^{2m-1}\left(n\tanh^{2n-1}(x/2)-(2n+1)\tanh^{2n+1}(x/2)+(n+1)\tanh^{2n+3}(x/2)\right)\\
&=\frac{2n+1}{2}\Bigg(n\sum_{j=0}^{2m-1}b_j^{(n-1,2m-1)}\tanh^{2n-2m+2j}(x/2)\\
&\qquad-(2n+1)\sum_{j=0}^{2m-1}b_j^{(n,2m-1)}\tanh^{2n-2m+2+2j}(x/2)\\
&\qquad+(n+1)\sum_{j=0}^{2m-1}b_j^{(n+1,2m-1)}\tanh^{2n-2m+4+2j}(x/2)\Bigg).
\end{aligned}$$

If we let $x\to 0$, the above result goes to

$$\begin{aligned}
&\frac{2n+1}{2}\left(nb_{m-n}^{(n-1,2m-1)}-(2n+1)b_{m-n-1}^{(n,2m-1)}+(n+1)b_{m-n-2}^{(n+1,2m-1)}\right)\\
&=\frac{2n+1}{4}\left(n\widetilde{b}_{n-1}^{(m-1)}-(2n+1)\widetilde{b}_{n}^{(m-1)}+(n+1)\widetilde{b}_{n+1}^{(m-1)}\right).
\end{aligned}$$

On the other-hand, the right-hand side of equation (30) tends to $b_{m-n}^{(n,2m+1)}=\widetilde{b}_n^{(m)}/2$ as $x\to 0$. Thus we obtain (29). □

Second proof of Theorem 4. For $(a_i^{(m)})_{0\le i\le m}$ defined by (22), set $\widetilde{a}_i^{(m)}=(2i+1)a_i^{(m)}$. Then (22) can be written as $\widetilde{a}_0^{(0)}=1$ and

$$\widetilde{a}_i^{(m)}=\frac{2i+1}{2}\left(i\widetilde{a}_{i-1}^{(m-1)}-(2i+1)^2\widetilde{a}_i^{(m-1)}+(i+1)\widetilde{a}_{i+1}^{(m-1)}\right)$$

which has exactly the same form as the recurrence relation (29) for $\widetilde{b}_n^{(m)}$. Therefore one concludes $\widetilde{a}_n^{(m)}=\widetilde{b}_n^{(m)}$. Comparing (24) and (28), we obtain $g_m(x)=h_m(x)$. Thus we complete our second proof of Theorem 4. □

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