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Fermat Padovan And Perrin Numbers

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Abstract

In this paper, we determine all the Padovan and Perrin numbers that are also Fermat numbers.

1 Introduction

The Padovan sequence $\{P_m\}_{m\geq 0}$ is defined by

$$P_{m+3} = P_{m+1} + P_m, \tag{1}$$

for $m \geq 0$, where $P_0 = P_1 = P_2 = 1$. This is the sequence A000931 in the *On-Line Encyclopedia of Integer Sequences* (OEIS). A few terms of this sequence are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, \cdots$$

Let $\{E_m\}_{m\geq 0}$ be the Perrin sequence given by

$$E_{m+3} = E_{m+1} + E_m, \tag{2}$$

for $m \geq 0$, where $E_0 = 3$, $E_1 = 0$, and $E_2 = 2$. Its first few terms are

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, 367, 486, 644, 853, \cdots$$

It is the sequence A001608 in the OEIS.

Let us also recall that a Fermat number is a number of the form

$$\mathcal{F}_m = 2^{2^m} + 1,$$

where m is a nonnegative integer. The first elements of its list are

$$3, 5, 17, 257, 65537, 4294967297, 18446744073709551617,$$

$$340282366920938463463374607431768211457,$$

$$115792089237316195423570985008687907853269984665640564039457584007913129639937, \ldots$$

This is the sequence A019434 in the OEIS.

In a recent paper, Bravo and Herrera [2] found all k-Fibonacci and k-Lucas numbers that are also Fermat numbers. So the aim of this paper is to find all the Padovan and Perrin numbers that are also Fermat numbers. The proofs of the results that we obtained are mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport in [1]. Here, we use a version due to de Weger [7]. So in Section 2, we will recall some results based on Baker's method, the Baker-Davenport reduction method (de Weger's version), and some properties of Padovand and Perrin numbers. They are very useful for the proofs of our main results. In the last section, we will determine all Padovan and Perrin numbers that are Fermat numbers and show that these numbers are the only.

2 The tools

2.1 Linear forms in logarithms

We need some results from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. We start by recalling [3, Theorem 9.4], which is a modified version of a result of Matveev [6]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $d_1, \ldots, d_l$ be nonzero integers. We put

$$D = \max\{|d_1|, \ldots, |d_l|\},$$

and

$$\Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for} \quad j = 1, \ldots l,$$

where for an algebraic number η of minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(k)}) \in \mathbb{Z}[X]$$

over the integers with positive a_0. We write $h(\eta)$ for its Weil height given by

$$h(\eta) = \frac{1}{k}\left(\log a_0 + \sum_{j=1}^{k} \max\{0, \log |\eta^{(j)}|\}\right).$$

The following consequence of Matveev's theorem is [3, Theorem 9.4].

Theorem 1. *If* $\Gamma \neq 0$ *and* $\mathbb{L} \subseteq \mathbb{R}$, *then*

$$\log |\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \cdots A_l.$$

2.2 The Baker-Davenport reduction method

Here, we present a variant of the reduction method of Baker and Davenport due to de Weger [7].

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{3}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{4}$$

$$X \le X_0. \tag{5}$$

When $\beta = 0$ in (3), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots],$$

and let the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We have the following results.

Lemma 2. *(See [7, Lemma 3.2]) Let*

$$A = \max_{0 \le k \le Y_0} a_{k+1},$$

where

$$Y_0 = -1 + \frac{\log(\sqrt{5}X_0 + 1)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

If (4) *and* (5) *hold for* x_1, x_2 *and* $\beta = 0$, *then*

$$Y < \frac{1}{\delta}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right). \tag{6}$$

When $\beta \neq 0$ in (3), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x, we let $\|x\| = \min\{|x-n|, n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 3. *(See [7, Lemma 3.3]) Suppose that*

$$\| q\psi \| > \frac{2X_0}{q}.$$

Then, the solutions of (4) *and* (5) *satisfy*

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2| X_0}\right).$$

2.3 Properties of Padovan and Perrin sequences

In this subsection we recall some facts and properties of the Padovan and the Perrin sequences that will be used later. For more details about the Padovan and Perrin sequences, see [8].

The characteristic equation

$$x^3 - x - 1 = 0$$

has roots $\alpha, \beta, \gamma = \overline{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12}$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \text{ and } r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Let

$$\begin{aligned} c_\alpha &= \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} = \frac{1+\alpha}{-\alpha^2+3\alpha+1}, \\ c_\beta &= \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)} = \frac{1+\beta}{-\beta^2+3\beta+1}, \\ c_\gamma &= \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \frac{1+\gamma}{-\gamma^2+3\gamma+1} = \overline{c_\beta}. \end{aligned} \tag{7}$$

Binet's formula for P_n is

$$P_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n, \text{ for all } n \geq 0, \tag{8}$$

and Binet's formula for E_n is

$$E_n = \alpha^n + \beta^n + \gamma^n, \text{ for all } n \geq 0. \tag{9}$$

Numerically, we have

$$\begin{aligned} 1.32 < \alpha < 1.33, \\ 0.86 < |\beta| = |\gamma| < 0.87, \\ 0.72 < c_\alpha < 0.73, \\ 0.24 < |c_\beta| = |c_\gamma| < 0.25. \end{aligned} \tag{10}$$

It is easy to check that

$$|\beta| = |\gamma| = \alpha^{-1/2}.$$

Further, using induction, one can prove that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1}, \quad \text{holds for all } n \geq 4 \tag{11}$$

and

$$\alpha^{n-2} \leq E_n \leq \alpha^{n+1}, \quad \text{holds for all } n \geq 2, \tag{12}$$

see [5].

3 Proofs of the main results

In this section, we set and prove the two main results of our paper.

3.1 Our first main result

We will prove our first main result in this subsection.

Theorem 4. *The only Fermat numbers in the Padovan sequence are* $P_5 = 3$ *and* $P_7 = 5$.

Proof. Let us consider the Diophantine equation

$$P_n = 2^m + 1. \tag{13}$$

A quick computation with Maple reveals that the solutions of the Diophantine equation (13) in the interval $[0, 150]$ are P_3, P_4, P_5, P_7, P_9, and P_{16}. It is easy to see that the only Fermat numbers are P_5 and P_7.

From now, we assume that $n > 150$. Then by (11), we have

$$\alpha^{n-2} < P_n = 2^m + 1 < 2^{m+1}$$

and

$$2^m < 2^m + 1 = P_n < \alpha^{n-1}.$$

Thus we get

$$(n-2)c_1 - 1 < m < (n-1)c_1, \quad \text{where } c_1 := \log\alpha/\log 2.$$

In particular, we have $m < n/2$. So to solve equation (13), it suffices to get a good upper bound for n.

By (8), equation (13) can be expressed as

$$2^m - c_\alpha\alpha^n = c_\beta\beta^n + c_\gamma\gamma^n - 1,$$

which we rewrite as

$$|2^m - c_\alpha\alpha^n| = |c_\beta\beta^n + c_\gamma\gamma^n - 1| < \frac{3}{2}.$$

Multiplying through by $c_\alpha^{-1}\alpha^{-n}$, we obtain

$$\left|2^m c_\alpha^{-1}\alpha^{-n} - 1\right| < 2.1\alpha^{-n}. \tag{14}$$

Now, we apply Matveev's theorem by taking

$$\Gamma := 2^m c_\alpha^{-1}\alpha^{-n} - 1$$

and

$$\eta_1 := 2, \quad \eta_2 := c_\alpha, \quad \eta_3 := \alpha, \quad b_1 := m, \quad b_2 := -1, \quad b_3 := -n.$$

The algebraic numbers η_1, η_2 and η_3 belong to $\mathbb{L} = \mathbb{Q}(\alpha)$ for which $d_{\mathbb{L}} = 3$. Since $m < n/2$, therefore we can take $D := n = \max\{1, m, n\}$. Furthermore, we have

$$h(\eta_1) = \log 2 \quad \text{and} \quad h(\eta_3) = \frac{\log \alpha}{3}.$$

In this case we choose

$$\max\{3h(\eta_1), |\log \eta_1|, 0.16\} < 2.1 := A_1$$

and

$$\max\{3h(\eta_3), |\log \eta_3|, 0.16\} = \log \alpha := A_3.$$

On the other hand, the minimal polynomial of c_α is

$$23x^3 - 23x^2 + 6x - 1$$

and has roots c_α, c_β and c_γ. Since $|c_\alpha| < 1$ and $|c_\beta| = |c_\gamma| < 1$, then we get

$$h(\eta_2) = \frac{\log 23}{3}.$$

So we can take

$$\max\{3h(\eta_2), |\log \eta_2|, 0.16\} < 3.2 := A_3.$$

To apply Matveev's theorem we will prove that $\Gamma \neq 0$. Suppose the contrary i.e $\Gamma = 0$, so we get

$$2^m = c_\alpha \alpha^n.$$

Conjugating the above relation using the $\mathbb{Q}$-automorphism of Galois σ defined by $\sigma = (\alpha\beta)$ and taking the absolute value we obtain

$$1 < 2^m = |c_\beta| |\beta|^n < 1,$$

which is a contradiction. Thus one can see that $\Gamma \neq 0$.

Using Matveev's theorem, we get

$$\begin{aligned}\log |\Gamma| &> -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1 + \log 3)(1 + \log n) \cdot 2.1 \cdot \log \alpha \cdot 3.2 \\ &> -1.82 \cdot 10^{13} \cdot \log \alpha \cdot (2 \log n) = 3.64 \cdot 10^{13} \cdot \log \alpha \cdot \log n.\end{aligned}$$

The last inequality together with (14) leads to

$$n < 3.65 \cdot 10^{13} \log n.$$

Thus we obtain

$$n < 1.3 \cdot 10^{15}. \tag{15}$$

Now, we will use Lemma 3 to reduce the upper bound (15) of n.

Define

$$\Lambda := m \log 2 - n \log \alpha + \log(1/c_\alpha).$$

Clearly, we have $e^\Lambda - 1 = \Gamma$. Since $\Gamma \neq 0$, then $\Lambda \neq 0$. If $\Lambda > 0$, then we get

$$0 < \Lambda < e^\Lambda - 1 = \left|e^\Lambda - 1\right| = |\Gamma| < 2.1\alpha^{-n}.$$

If $\Lambda < 0$, then we have $1 - e^\Lambda = \left|e^\Lambda - 1\right| = |\Gamma| < 1/2$, because $n > 150$. This implies that $e^{|\Lambda|} < 2$. Thus we have

$$0 < |\Lambda| < e^{|\Lambda|} - 1 = e^{|\Lambda|}\,|\Gamma| < 4.2\alpha^{-n}.$$

From both cases, we deduce that

$$0 < |n(-\log \alpha) + m \log 2 + \log(1/c_\alpha)| < 4.2 \exp(-n \log \alpha).$$

The inequality (15) implies that we take $X_0 := 1.3 \cdot 10^{13}$. Further, we choose

$$c := 4.2, \quad \delta := \log \alpha, \quad \psi := \frac{\log(1/c_\alpha)}{\log 2}$$

$$\vartheta := \frac{\log \alpha}{\log 2}, \quad \vartheta_1 := -\log \alpha, \quad \vartheta_2 := \log 2, \quad \beta := \log(1/c_\alpha).$$

Using Maple, we see that

$$q_{41} = 2263631680285337$$

satisfies the hypotheses of Lemma 3. Furthermore, Lemma 3 implies that

$$n < \frac{1}{\log \alpha} \log\left(\frac{2263631680285337^2 \cdot 4.2}{\log 2 \cdot 1.3 \cdot 10^{13}}\right) \leq 150. \tag{16}$$

This contradicts the assumption that $n > 150$. Therefore, the theorem is proved. □

3.2 Our second main result

In this subsection we will prove the following result.

Theorem 5. *The only Fermat numbers in the Perrin sequence are* $E_0 = E_3 = 3$, $E_5 = E_6 = 5$, *and* $E_{10} = 17$.

Proof. Let us consider the Diophantine equation

$$E_n = 2^m + 1. \tag{17}$$

A quick computation in Maple reveals that the solutions of Diophantine equation (13) in the interval $[0, 150]$ are E_0, E_2, E_3, E_4, E_5, E_6 and E_{10}. It is easy to see that the only Fermat numbers are E_0, E_3, E_5, E_6 and E_{10}.

From now, we assume that $n > 150$, then by (12) we have

$$\alpha^{n-2} < E_n = 2^m + 1 < 2^{m+1}$$

and

$$2^m < 2^m + 1 = E_n < \alpha^{n+1}.$$

Then, we get

$$(n-2)c_1 - 1 < m < (n+1)c_1, \quad \text{where } c_1 := \log\alpha/\log 2.$$

In particularly we have $m < n/2$. To solve equation (17), it suffices to get a good upper bound for n.

By (9), equation (17) can be rewritten into the form

$$2^m - \alpha^n = \beta^n + \gamma^n - 1.$$

So we deduce that

$$|2^m - \alpha^n| = |\beta^n + \gamma^n - 1| < 2.8.$$

Dividing both sides by α^{-n}, we get

$$\left|2^m\alpha^{-n} - 1\right| < 2.8\alpha^{-n}. \tag{18}$$

Now, we apply Matveev's theorem by taking

$$\Gamma' := 2^m\alpha^{-n} - 1$$

and

$$\eta_1 := 2, \quad \eta_2 := \alpha, \quad b_1 := m, \quad b_2 := -n.$$

The algebraic numbers η_1 and η_2 belong to $\mathbb{L} := \mathbb{Q}(\alpha)$ for which $d_{\mathbb{L}} = 3$. Since $m < n/2$, therefore we take $D := n = \max\{1, m, n\}$. As seen before, we choose

$$A_1 := 2.1 \quad \text{and} \quad A_2 := \log\alpha.$$

We can prove that $\Gamma' \neq 0$ using the same method as above to show that $\Gamma \neq 0$.

Matveev's theorem gives

$$\begin{aligned}\log|\Gamma'| &> -1.4 \cdot 30^5 \cdot 2^{4.5} \cdot 3^2(1+\log 3)(1+\log n) \cdot 2.1 \cdot \log\alpha \\ &> -3.06 \cdot 10^{10} \cdot \log\alpha \cdot (2\log n) = -6.12 \cdot 10^{10} \cdot \log\alpha \cdot \log n.\end{aligned}$$

Comparing the last inequality with (18) yields

$$n < 6.13 \cdot 10^{10} \log n.$$

Consequently, we obtain

$$n < 1.8 \cdot 10^{12}. \tag{19}$$

Now, to reduce the upper bound (19) of n, we will use Lemma 2.

Consider

$$\Lambda' := m \log 2 - n \log \alpha.$$

Clearly, we have $e^{\Lambda'} - 1 = \Gamma'$. Since $\Gamma' \neq 0$, then $\Lambda' \neq 0$. If $\Lambda' > 0$, then we get

$$0 < \Lambda' < e^{\Lambda'} - 1 = \left| e^{\Lambda'} - 1 \right| = |\Gamma'| < 2.8\alpha^{-n}.$$

If $\Lambda' < 0$, then we have $1 - e^{\Lambda'} = \left| e^{\Lambda'} - 1 \right| = |\Gamma'| < 1/2$, because $n > 150$. Thus $e^{|\Lambda'|} < 2$. Therefore, we obtain

$$0 < |\Lambda'| < e^{|\Lambda'|} - 1 = e^{|\Lambda'|} |\Gamma'| < 5.6\alpha^{-n}.$$

In both cases we have

$$0 < |n(-\log \alpha) + m \log 2| < 5.6 \exp(-\log \alpha \cdot n).$$

The inequality (15) implies that we can take $X_0 := 1.8 \cdot 10^{12}$, thus we get $Y_0 = 59.3134\ldots$. Further, we choose

$$c := 5.6, \quad \delta := \log \alpha, \quad \vartheta := \frac{\log \alpha}{\log 2}, \quad \vartheta_1 := -\log \alpha, \quad \vartheta_2 := \log 2.$$

We use Maple to find that

$$A := \max_{0 \leq k \leq 59} a_{k+1} = 80.$$

So Lemma 2 tells us

$$n < \frac{1}{\log \alpha} \log \left(\frac{5.6 \cdot (80 + 2) \cdot 1.8 \cdot 10^{12}}{\log 2} \right) \leq 126. \tag{20}$$

This contradicts the assumption that $n > 150$. Therefore, the theorem is proved. □

References

[1] A. Baker and H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J. Math. Oxford Ser. (2)* **20** (1969), 129–137.

[2] J. J. Bravo and J. L. Herrera, Fermat k-Fibonacci and k-Lucas numbers, *Math. Bohem.* **145** (2020), 19–32.

[3] Y. Bugeaud, M. Maurice, and S. Siksek, Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers, *Ann. Math.* **163** (2006), 969–1018.

[4] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser. (2)* **49** (1998), 291–306.

[5] A. C. G. Lomelí and S. H. Hernández, Repdigits as sums of two Padovan numbers, *J. Integer Sequences* **22** (2019), Article 19.2.3.

[6] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, II, *Izv. Math.* **64** (2000), 1217–1269.

[7] B. M. M. de Weger, *Algorithms for Diophantine Equations*, Stichting Mathematisch Centrum, 1989.

[8] B. M. M. de Weger, Padua and Pisa are exponentially far apart, *Publ. Sec. Mat. Univ. Autònoma Barcelona* **41** (1997), 631–651.

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Return to Journal of Integer Sequences home page.