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New Partition Function Recurrences

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Abstract

We present Euler-type recurrence relations for some partition functions. Some of our results provide new recurrences for $p(n)$ the number of unrestricted partitions of n. Others establish recurrences for partition functions not yet considered.

1 Introduction

A partition of an integer n is a finite set of positive integers $\{\lambda_1, \ldots, \lambda_s\}$ such that $n = \lambda_1 + \cdots + \lambda_s$. The λ_i are called the parts of the partition. The number of partitions of n is usually denoted by $p(n)$ [9, A000041], with $p(0) = 1$ by convention. For example, we have $p(4) = 5$ since there are five partitions of 4, namely

$$4, 3+1, 2+2, 2+1+1, 1+1+1+1.$$

The generating function of $p(n)$, due to Euler [1, Eq. (1.1.6)], is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}. \tag{1}$$

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So, one can obtain the values of $p(n)$ by expanding the right-hand side of (1) and extracting the coefficient of q^n. Another way to obtain $p(n)$ was found by Euler after he proved the following identity (known as Euler's pentagonal number theorem):

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = \prod_{k=1}^{\infty}(1-q^k). \tag{2}$$

Indeed, multiplying (1) and (2) we obtain

$$\sum_{n=0}^{\infty} p(n)q^n \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = 1,$$

from which the following recurrence for $p(n)$ is derived after extracting the coefficient of q^n from both sides:

$$\begin{aligned}
&p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15)\\
&+\cdots + (-1)^j p(n - j(3j-1)/2) + (-1)^j p(n - j(3j+1)/2) + \cdots\\
&= \begin{cases} 1, & \text{if } n = 0;\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

The numbers $j(3j \pm 1)/2$ are the pentagonal numbers [9, A001318].

Some subsequent works brought new recurrence relations for $p(n)$ and other partition functions. Ewell [4, Theorem 2], for instance, presented the following recurrence for $p(n)$ involving the triangular numbers [9, A000217]

$$\begin{aligned}
&p(n) - p(n-1) - p(n-3) + p(n-6) + p(n-10) - p(n-15) - p(n-21)\\
&+\cdots + (-1)^j p(n - j(2j-1)) + (-1)^j p(n - j(2j+1)) + \cdots\\
&= \begin{cases} 0, & \text{if } n \text{ is odd;}\\ p_d(n/2), & \text{if } n \text{ is even,}\end{cases}
\end{aligned}$$

where $p_d(n)$ denotes the number of partitions of n into distinct parts [9, A000009]. Merca [6] derived two new recurrence relations for $p(n)$, which allowed him to obtain a more efficient method to compute the parity of $p(n)$. Ono, Robbins, and Wilson [8] presented recurrence relations for some partition functions, including $p_d(n)$, $qq(n)$ (the number of partitions into distinct odd parts [9, A000700]), $p_E(n)$ (the number of partitions into an even number of parts [9, A027187]), and $p_O(n)$ (the number of partitions into an odd number of parts [9, A027193]). Recently, Choliy, Kolitsch, and Sills [2] found a number of new recurrences for $p(n)$, including

$$\begin{aligned}
&p(n) - p(n-1) - p(n-2) + p(n-4) + p(n-8) - p(n-9) - p(n-18)\\
&+\cdots + (-1)^j p(n - j^2) + (-1)^j p(n - 2j^2) + \cdots\\
&= \begin{cases} 0, & \text{if } n \text{ is odd;}\\ qq(n), & \text{if } n \text{ is even,}\end{cases}
\end{aligned}$$

and

$$p(n) - 2p(n-1) + 2p(n-4) - 2p(n-9) + 2p(n-16) + \cdots + (-1)^j 2p(n-j^2) + \cdots = (-1)^n qq(n).$$

Additional recurrence relations for partition functions can be found in [2, 4, 5, 6, 7, 8].

In this paper, using some classical identities and generating function manipulations, we provide a number of new recurrence relations for $p(n)$, $qq(n)$, $\overline{p}(n)$ the number of overpartitions of n [9, A015128], $p_o(n)$ the number of partitions of n into odd parts [9, A000009], and the two-parameter function $p_m^c(n)$ (the number of partitions of n into parts congruent to $\pm c$ modulo m). For some of these functions, it is the first time that recurrence relations are presented.

2 Preliminaries

We recall Ramanujan's theta functions

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \text{ for } |ab| < 1, \tag{3}$$

and

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{4}$$

In the proofs of some of our results, we will need Jacobi triple product identity [1, Theorem 1.3.3] given by

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty,$$

where we use the following standard q-series notation:

$$(a;q)_0 = 1,$$
$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \forall n \geq 1,$$

and

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n, |q| < 1.$$

Using (3), we can rewrite Jacobi triple product identity in the form

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{5}$$

An important consequence of (5) is following identity (see [1, Eq. (1.3.14)])

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{6}$$

We also recall the well-known Euler's pentagonal number theorem [1, Corollary 1.3.5]:

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \tag{7}$$

3 Main results

In what follows, we let t_j^e (resp., t_j^o) denote the j-th even (resp., odd) triangular number [9, A014494] (resp., [9, A014493]). So, $t_1^e = 0$, $t_1^o = 1$, $t_2^e = 6$, $t_2^o = 3$, $t_3^e = 10$, $t_3^o = 15$, etc.

Theorem 1. *For all even integer $n \geq 0$, we have*

$$\begin{aligned} &p(n/2) + p((n-6)/2) + p((n-10)/2) + p((n-28)/2) + p((n-36)/2) + \\ &p((n-66)/2) + p((n-78)/2) + \cdots + p((n-t_j^e)/2) + \cdots = p_d(n), \end{aligned} \tag{8}$$

where $p_d(n)$ denotes the number of partitions of n into distinct parts. For all odd integer $n \geq 0$, we have

$$\begin{aligned} &p((n-1)/2) + p((n-3)/2) + p((n-15)/2) + p((n-21)/2) + \\ &p((n-45)/2) + p((n-55)/2) + \cdots + p((n-t_j^o)/2) + \cdots = p_o(n), \end{aligned} \tag{9}$$

where $p_o(n) = p_d(n)$ denotes the number of partitions of n into odd parts

Proof. Initially, we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} p_d(n) q^n &= \frac{1}{2}\left(\sum_{n=0}^{\infty} p_d(n) q^n + \sum_{n=0}^{\infty} p_d(n)(-q)^n\right) \\ &= \frac{1}{2}\left(\prod_{k=1}^{\infty}(1+q^k) + \prod_{k=1}^{\infty}(1+(-1)^k q^k)\right) \\ &= \frac{1}{2}\left(\prod_{k=1}^{\infty}(1+q^k) + \prod_{k=1}^{\infty}(1+q^{2k})(1-q^{2k-1})\right) \end{aligned}$$

and

$$\begin{aligned} \prod_{k=1}^{\infty}(1+q^k) &= \prod_{k=1}^{\infty}(1+q^k)\frac{(1-q^k)}{(1-q^k)}\frac{(1-q^{2k-1})}{(1-q^{2k-1})} \\ &= \prod_{k=1}^{\infty}\frac{(1-q^{2k-1})(1-q^{2k})}{(1-q^k)(1-q^{2k-1})} = \prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\psi(q). \end{aligned}$$

We also have

$$\prod_{k=1}^{\infty}(1+q^{2k})(1-q^{2k-1}) = \prod_{k=1}^{\infty}(1+q^{2k})(1-q^{2k-1})\frac{(1-q^{2k})}{(1-q^k)(1+q^k)}$$
$$= \prod_{k=1}^{\infty}\frac{(1-q^{2k-1})(1-q^{2k})}{(1-q^k)(1+q^{2k-1})} = \prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\psi(-q).$$

It follows that

$$\sum_{n=0}^{\infty}\frac{1+(-1)^n}{2}p_d(n)q^n = \frac{1}{2}\left(\prod_{k\geq 1}\frac{1}{1-q^{2k}}\psi(q) + \prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\psi(-q)\right)$$
$$= \left(\prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\right)\frac{1}{2}\big(\psi(q)+\psi(-q)\big)$$
$$= \left(\prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\right)\frac{1}{2}\left(\sum_{j=0}^{\infty}q^{\frac{j(j+1)}{2}} + \sum_{j=0}^{\infty}(-q)^{\frac{j(j+1)}{2}}\right).$$

The parity of the exponent $\frac{j(j+1)}{2}$ is given by

$$\frac{4i(4i+1)}{2} = 8i^2+2i, \tag{10}$$
$$\frac{(4i-1)(4i-1+1)}{2} = 8i^2-2i, \tag{11}$$
$$\frac{(4i-2)(4i-2+1)}{2} = 8i^2-6i+1, \tag{12}$$
$$\frac{(4i-3)(4i-3+1)}{2} = 8i^2-10i+3. \tag{13}$$

The even triangular numbers are given by (10) and (11), while (12) and (13) represent the odd triangular numbers. Thus

$$\sum_{j=0}^{\infty}q^{\frac{j(j+1)}{2}} = \sum_{i=0}^{\infty}q^{8i^2+2i}+q^{8i^2-2i}+q^{8i^2-6i+1}+q^{8i^2-10i+3} \tag{14}$$

and

$$\sum_{j=0}^{\infty}(-q)^{\frac{j(j+1)}{2}} = \sum_{i=0}^{\infty}q^{8i^2+2i}+q^{8i^2-2i}-q^{8i^2-6i+1}-q^{8i^2-10i+3}, \tag{15}$$

which yields

$$\sum_{n=0}^{\infty}\frac{1+(-1)^n}{2}p_d(n)q^n = \left(\prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\right)\frac{1}{2}\sum_{i=0}^{\infty}2q^{8i^2+2i}+2q^{8i^2-2i}$$
$$= \sum_{k=0}^{\infty}p(k)q^{2k}\sum_{j=0}^{\infty}q^{t_j^e} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p(k)q^{2k+t_j^e}.$$

Now we extract the coefficient of q^n on both sides of the above equation to obtain

$$\sum_{n=0}^{\infty} p_d(n)q^n = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty} p((n-t_j^e)/2)\right)q^n,$$

which completes the proof of (8).

In order to prove (9), we begin with

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1-(-1)^n}{2}p_o(n)q^n &= \frac{1}{2}\left(\sum_{n=0}^{\infty}p_o(n)q^n - \sum_{n=0}^{\infty}p_o(n)(-q)^n\right)\\ &= \frac{1}{2}\left(\prod_{k=1}^{\infty}\frac{1}{(1-q^{2k-1})} - \prod_{k=1}^{\infty}\frac{1}{(1-(-q)^{2k-1})}\right)\\ &= \frac{1}{2}\left(\prod_{k=1}^{\infty}\frac{1}{(1-q^{2k-1})} - \prod_{k=1}^{\infty}\frac{1}{(1+q^{2k-1})}\right).\end{aligned}$$

We note that

$$\prod_{k=1}^{\infty}\frac{1}{(1-q^{2k-1})} = \prod_{k=1}^{\infty}\frac{1}{(1-q^{2k-1})}\frac{(1-q^{2k})}{(1-q^{2k})} = \prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\psi(q)$$

and

$$\prod_{k=1}^{\infty}\frac{1}{(1+q^{2k-1})} = \prod_{k=1}^{\infty}\frac{1}{(1+q^{2k-1})}\frac{(1-q^{2k})}{(1-q^{2k})} = \prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\psi(-q).$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1-(-1)^n}{2}p_o(n)q^n &= \frac{1}{2}\left(\prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\psi(q) - \prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\psi(-q)\right)\\ &= \left(\prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\right)\frac{1}{2}\big(\psi(q)-\psi(-q)\big)\\ &= \left(\prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\right)\frac{1}{2}\left(\sum_{j=0}^{\infty}q^{\frac{j(j+1)}{2}} - \sum_{j=0}^{\infty}(-q)^{\frac{j(j+1)}{2}}\right).\end{aligned}$$

By (14) and (15), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1-(-1)^n}{2}p_o(n)q^n &= \left(\prod_{k=1}^{\infty}\frac{1}{1-q^{2k}}\right)\frac{1}{2}\left(\sum_{i=0}^{\infty}2q^{8i^2-6i+1} + 2q^{8i^2-10i+3}\right)\\ &= \sum_{k=0}^{\infty}p(k)q^{2k}\sum_{j=0}^{\infty}q^{t_j^o} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p(k)q^{2k+t_j^o}.\end{aligned}$$

Extracting the coefficient of q^n in the identity above, we obtain

$$\sum_{n=0}^{\infty} p_o(n)q^n = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty} p((n-t_j^o)/2)\right)q^n,$$

from which (9) follows. □

We recall that Corteel and Lovejoy [3] introduced the overpartitions of n, which are partitions in which the first occurrence of a number may be overlined. For instance, there are eight overpartitions of 3, namely

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

We let $\overline{p}(n)$ denote the number of overpartitions of n [9, A015128]. In the next three results, we present recurrence relations for $\overline{p}(n)$.

Theorem 2. *For all $n \geq 0$, we have*

$$\begin{aligned}&\overline{p}(n) - 2\overline{p}(n-1) + 2\overline{p}(n-4) - 2\overline{p}(n-9) + 2\overline{p}(n-16) - 2\overline{p}(n-25) + \\ &2\overline{p}(n-36) - \cdots + 2(-1)^j\overline{p}(n-j^2) + \cdots \\ &= \begin{cases} 1, & \text{if } n=0; \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. We recall from [3] that the generating function for overpartitions is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{k=1}^{\infty} \frac{(1+q^k)}{(1-q^k)}.$$

We note that

$$\begin{aligned}\prod_{k=1}^{\infty}(1-q^k) &= \prod_{k=1}^{\infty}(1-q^k)\frac{(1-q^{2k-1})}{(1-q^{2k-1})}\frac{(1-q^{2k})}{(1-q^{2k})} \\ &= \prod_{k=1}^{\infty}\frac{(1-q^k)(1-q^{2k-1})(1-q^{2k})}{(1-q^{2k})(1-q^{2k-1})} \\ &= \prod_{k=1}^{\infty}\frac{(1-q^{2k-1})^2(1-q^{2k})}{(1-q^{2k-1})} \\ &= \varphi(-q)\prod_{k=1}^{\infty}\frac{1}{(1-q^{2k-1})}.\end{aligned}$$

By Euler's identity, we have

$$\prod_{k=1}^{\infty}(1-q^k) = \varphi(-q)\prod_{k=1}^{\infty}(1+q^k),$$

and, then,

$$\prod_{k=1}^{\infty}\frac{1}{(1-q^k)} = \frac{1}{\varphi(-q)}\prod_{k=1}^{\infty}\frac{1}{(1+q^k)}.$$

Hence we obtain the following equivalent identities:

$$\begin{aligned}
\sum_{n=0}^{\infty}\overline{p}(n)q^n &= \frac{\prod_{k=1}^{\infty}(1+q^k)}{\varphi(-q)\prod_{k=1}^{\infty}(1+q^k)},\\
\varphi(-q)\sum_{n=0}^{\infty}\overline{p}(n)q^n &= 1,\\
\left(1+2\sum_{k=1}^{\infty}(-1)^k q^{k^2}\right)\sum_{n=0}^{\infty}\overline{p}(n)q^n &= 1,\\
\sum_{n=0}^{\infty}\left(\overline{p}(n)q^n + 2\sum_{k=1}^{\infty}(-1)^k\overline{p}(n)q^{k^2+n}\right) &= 1,\\
\sum_{n=0}^{\infty}\left(\overline{p}(n) + 2\sum_{k=1}^{\infty}(-1)^k\overline{p}(n-k^2)\right)q^n &= 1.
\end{aligned}$$

The result now follows from the last identity. □

Our second recurrence for $\overline{p}(n)$ involves $p_d(n)$.

Theorem 3. *For all $n \geq 0$, we have*

$$\begin{aligned}
&\overline{p}(n) - \overline{p}(n-1) - \overline{p}(n-2) + \overline{p}(n-5) + \overline{p}(n-7) - \cdots \\
&\qquad \cdots + (-1)^j\left(\overline{p}(n-j(3j-1)/2) + \overline{p}(n-j(3j+1)/2)\right) + \cdots = p_d(n).
\end{aligned}$$

Proof. We have

$$\begin{aligned}\prod_{k=1}^{\infty}(1+q^k) &= \prod_{k=1}^{\infty}\frac{(1+q^k)}{(1-q^k)}\prod_{k=1}^{\infty}(1-q^k)\\ &= \sum_{k=1}^{\infty}\overline{p}(k)q^k\sum_{k=-\infty}^{\infty}(-1)^j q^{\frac{j(3j-1)}{2}}\\ &= \sum_{k=0}^{\infty}\overline{p}(k)q^k\left(1+\sum_{j=1}^{\infty}(-1)^j q^{\frac{j(3j-1)}{2}}+\sum_{j=1}^{\infty}(-1)^j q^{\frac{j(3j+1)}{2}}\right)\\ &= \sum_{k=0}^{\infty}\left(\overline{p}(k)q^k+\sum_{j=1}^{\infty}(-1)^j\overline{p}(k)q^{k+\frac{j(3j-1)}{2}}+\sum_{j=1}^{\infty}(-1)^j\overline{p}(k)q^{k+\frac{j(3j+1)}{2}}\right).\end{aligned}$$

Therefore, we obtain

$$\sum_{n=0}^{\infty}p_d(n)q^n = \sum_{n=0}^{\infty}\left(\overline{p}(n)+\sum_{j=1}^{\infty}(-1)^j\left(\overline{p}(n-j(3j-1)/2)+\overline{p}(n-j(3j+1)/2)\right)\right)q^n,$$

from which the proof follows by comparing coefficients of q^n on both sides of the last equation. □

We let $\overline{p_d}(n)$ denote the number of overpartitions of n into distinct parts. Then we have the following recurrence for $\overline{p}(n)$.

Theorem 4. *For all $n \geq 0$, we have*

$$\begin{aligned}\overline{p}(n)-&\overline{p}(n-2)-\overline{p}(n-4)+\overline{p}(n-10)+\overline{p}(n-14)-\cdots\\ &\cdots+(-1)^j\big(\overline{p}(n-j(3j-1))+\overline{p}(n-j(3j+1))\big)+\cdots=\overline{p_d}(n).\end{aligned}$$

Proof. By (7) we have

$$\begin{aligned}\sum_{k=0}^{\infty}\overline{p_d}(n)q^n &= \prod_{k=1}^{\infty}(1+q^k)^2=\prod_{k=1}^{\infty}\frac{(1-q^{2k})(1+q^k)}{1-q^k}\\ &= (q^2;q^2)_\infty\sum_{k=0}^{\infty}\overline{p}(k)q^k=\sum_{k=0}^{\infty}\overline{p}(k)q^k\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j-1)}\\ &= \sum_{k=0}^{\infty}\overline{p}(k)q^k\left(1+\sum_{j=1}^{\infty}(-1)^j q^{j(3j-1)}+\sum_{j=1}^{\infty}(-1)^j q^{j(3j+1)}\right)\\ &= \sum_{n=0}^{\infty}\left(\overline{p}(n)+\sum_{j=1}^{\infty}(-1)^j\big(\overline{p}(n-j(3j-1))+\overline{p}(n-j(3j+1))\big)\right)q^n.\end{aligned}$$

Thus, the result follows from extracting the coefficient of q^n on both sides of this identity. □

Now we prove a recurrence relations satisfied by $qq(n)$, the number of partitions of n into distinct odd parts.

Theorem 5. *For all $n \geq 0$, we have*

$$\begin{aligned}&qq(n) - qq(n-4) - qq(n-8) + qq(n-20) + qq(n-28) - qq(n-48) - \\ &qq(n-60) + \cdots + (-1)^j \big(qq(n-2j(3j-1)) + qq(n-2j(3j+1))\big) \cdots \\ &= \begin{cases} 1, & \text{if } n \text{ is a triangular number;} \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Proof. It is easy to see that

$$\prod_{k=1}^{\infty}(1+q^{2k-1}) = \prod_{k=1}^{\infty}\frac{(1+q^k)}{(1+q^{2k})} = \prod_{k=1}^{\infty}\frac{(1+q^k)(1-q^{2k})}{(1-q^{4k})}.$$

Thus

$$\sum_{j=0}^{\infty} qq(j)q^j = \prod_{k=1}^{\infty}\frac{(1+q^k)(1-q^{2k})}{(1-q^{4k})},$$

which can be rewritten as

$$\prod_{k=1}^{\infty}(1-q^{4k})\sum_{j=0}^{\infty} qq(j)q^j = \sum_{i=0}^{\infty} p_d(i)q^i \prod_{k=1}^{\infty}(1-q^{2k}).$$

That is to say

$$(q^4;q^4)_\infty \sum_{j=0}^{\infty} qq(j)q^j = (q^2;q^2)_\infty \sum_{i=0}^{\infty} p_d(i)q^i.$$

Then, by (7), we have

$$\begin{aligned}&(q^4;q^4)_\infty \sum_{j=0}^{\infty} qq(j)q^j \\ &\quad = \left(1 + \sum_{k=1}^{\infty}(-1)^k q^{2j(3j-1)} + \sum_{k=1}^{\infty}(-1)^k q^{2j(3j+1)}\right)\sum_{j=0}^{\infty} qq(j)q^j \\ &\quad = \sum_{n=0}^{\infty}\left(qq(n) + \sum_{k=0}^{\infty}(-1)^k\big(qq(n-2j(3j-1)) + qq(n-2j(3j+1))\big)\right)q^n.\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
(q^2;q^2)_\infty & \sum_{i=0}^{\infty} p_d(i)q^i \\
&= \sum_{i=0}^{\infty} p_d(i)q^i \left(1+\sum_{k=1}^{\infty}(-1)^k q^{k(3k-1)} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)}\right) \\
&= \sum_{n=0}^{\infty}\left(p_d(n)+\sum_{k=1}^{\infty}(-1)^k\big(p_d(n-k(3k-1))+p_d(n-k(3k+1))\big)\right)q^n.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(qq(n)+\sum_{k=1}^{\infty}(-1)^k\big(qq(n-2j(3j-1))+qq(n-2j(3j+1))\big)\right)q^n \\
&= \sum_{n=0}^{\infty}\left(p_d(n)+\sum_{k=1}^{\infty}(-1)^k\big(p_d(n-k(3k-1))+p_d(n-k(3k+1))\big)\right)q^n.
\end{aligned}
$$

The result follows from extracting the coefficient of q^n on both sides of the last equation and using Theorem 1 of [8]. □

We let $p_o(n)$ denote the number of partitions into odd parts. The next theorem presents a recurrence for $p_o(n)$.

Theorem 6. *For all $n \geq 0$, we have*

$$
\begin{aligned}
p_o(n) - & p_o(n-1) - p_o(n-5) + p_o(n-8) + p_o(n-16) - \cdots \\
\cdots + & (-1)^j\big(p_o(n-j(3j-2)) + p_o(n-j(3j+2))\big) + \cdots \\
&= \begin{cases} 1, & \text{if } n \text{ is 3 times a triangular number;} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Proof. Setting $a=-q$ and $b=-q^5$ in (3) and (5) we obtain

$$
(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty = f(-q,-q^5) = \sum_{j=-\infty}^{\infty}(-q)^{\frac{j(j+1)}{2}}(-q^5)^{\frac{j(j-1)}{2}},
$$

from which it follows that

$$
\begin{aligned}
\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j-2)} &= \prod_{k=1}^{\infty}(1-q^{6k-5})(1-q^{6k-1})(1-q^{6k}) \qquad (16) \\
&= \prod_{k=1}^{\infty}(1-q^{6k-5})(1-q^{6k-1})(1-q^{6k})\frac{(1-q^{6k-3})}{(1-q^{6k-3})} \\
&= \prod_{k=1}^{\infty}\frac{(1-q^{2k-1})(1-q^{6k})}{(1-q^{6k-3})}.
\end{aligned}
$$

Then, by (4), we have

$$
\begin{aligned}
\psi(q^3) &= \prod_{k=1}^{\infty} \frac{1}{(1-q^{2k-1})} \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-2)} \\
&= \sum_{i=0}^{\infty} p_o(i) q^i \left(1 + \sum_{j=1}^{\infty} (-1)^j q^{j(3j-2)} + \sum_{j=1}^{\infty} (-1)^j q^{j(3j+2)}\right) \\
&= \sum_{n=0}^{\infty} \left(p_o(n) + \sum_{j=1}^{\infty} (-1)^j \left(p_o(n-j(3j-2)) + p_o(n-j(3j+2))\right)\right) q^n.
\end{aligned}
$$

The result follows by comparing the coefficients of q^n on both sides of the last expression. □

Let ℓ be a positive integer. A partition of n having no part divisible by ℓ is called an ℓ-regular partition of n. We let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. The generating function of $b_\ell(n)$ is

$$
\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{(q^\ell; q^\ell)_\infty}{(q;q)_\infty}.
$$

Our next result is a recurrence relation for $p(n)$ involving $b_\ell(n)$.

Theorem 7. *Let $\ell \geq 1$. For all $n \geq 0$, we have*

$$
\begin{aligned}
&p(n) - p(n-\ell) - p(n-2\ell) + p(n-5\ell) + p(n-7\ell) - \cdots \\
&\cdots + (-1)^j \big(p(n-\ell j(3j-1)/2) + p(n-\ell j(3j+1)/2)\big) + \cdots = b_\ell(n).
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} b_\ell(n) q^n \\
&= \prod_{k=1}^{\infty} \frac{(1-q^{\ell k})}{(1-q^k)} \\
&= \sum_{n=0}^{\infty} p(n) q^n \left(\sum_{j=-\infty}^{\infty} (-1)^j q^{\ell \frac{j(3j-1)}{2}} \right) \\
&= \sum_{n=0}^{\infty} p(n) q^n \left(1 + \sum_{j=1}^{\infty} (-1)^j q^{\ell \frac{j(3j-1)}{2}} + \sum_{j=1}^{\infty} (-1)^j q^{\ell \frac{j(3j+1)}{2}} \right) \\
&= \sum_{n=0}^{\infty} \left(p(n) + \sum_{j=1}^{\infty} (-1)^j \big(p(n)(n-\ell j(3j-1)/2) + p(n-\ell j(3j+1)/2)\big)\right) q^n,
\end{aligned}
$$

from which the result follows. □

We close this section with a recurrence relation for the number of partitions of n having parts congruent to $\pm c \pmod{m}$.

Theorem 8. *Given integers a and $m \geq 1$, we let $p_m^c(n)$ denote the number of partitions of n having parts congruent to $\pm c$ modulo m. Then, for all $n \geq 0$,*

$$\begin{aligned} p_m^c(n) - &p_m^c(n-(m-c)) - p_m^c(n-c) + p_m^c(n-(3m-2c)) \\ &+ p_m^c(n-(m+2c)) + \cdots + (-1)^j p_m^c(n-(mj^2+(m-2c)j)/2) \\ &+ (-1)^j p_m^c(n-(mj^2-(m-2c)j)/2) + \cdots \\ &= \begin{cases} 1, & \text{if } n = mk_j^e; \\ -1, & \text{if } n = mk_j^o; \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where k_j^e (resp., k_j^o) is the j-th even (resp., odd) pentagonal number [9, A014633] (resp., [9, A014632]).

Proof. Setting $c = -q^{m-c}$ and $b = -q^c$ in (3) and (5), we obtain

$$\begin{aligned} (q^{m-c}; q^m)_\infty (q^c; q^m)_\infty (q^m; q^m)_\infty &= f(-q^{m-c}, -q^c) \\ &= \sum_{j=-\infty}^{\infty} (-q^{m-c})^{\frac{j(j+1)}{2}} (-q^c)^{\frac{j(j-1)}{2}}, \end{aligned}$$

which yields

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{mj^2+(m-2c)j}{2}} = \prod_{k=1}^{\infty} (1-q^{mk-c})(1-q^{mk-(m-c)})(1-q^{mk}). \tag{17}$$

The generating function for p_m^c is given by

$$\sum_{i=0}^{\infty} p_m^c(i) q^i = \prod_{k=1}^{\infty} \frac{1}{(1-q^{mk-c})(1-q^{mk-(m-c)})}.$$

Hence, we can rewrite (17) as

$$\prod_{k=1}^{\infty} \frac{1}{(1-q^{mk-c})(1-q^{mk-(m-c)})} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{mj^2+(m-2c)j}{2}} = \prod_{k=1}^{\infty} (1-q^{mk}),$$

or, equivalently,

$$\sum_{i=0}^{\infty} p_m^c(i) q^i \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{mj^2+(m-2c)j}{2}} = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{mj(3j-1)}{2}}.$$

This last identity yields

$$\sum_{i=0}^{\infty} p_m^c(i) q^i \left(1 + \sum_{j=1}^{\infty}(-1)^j q^{\frac{mj^2+(m-2c)j}{2}} + \sum_{j=1}^{\infty}(-1)^j q^{\frac{mj^2-(m-2c)j}{2}}\right)$$

$$= \sum_{n=0}^{\infty}\Bigg(p_m^c(n) + \sum_{j=1}^{\infty}(-1)^j p_m^c(n-(mj^2+(m-2c)j)/2)$$

$$+(-1)^j p_m^c(n-(mj^2-(m-2c)j)/2)\Bigg) q^n.$$

Therefore

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{mj(3j-1)}{2}} = \sum_{n=0}^{\infty}\Bigg(p_m^c(n) + \sum_{j=1}^{\infty}(-1)^j p_m^c(n-(mj^2+(m-2c)j)/2)$$

$$+(-1)^j p_m^c(n-(mj^2-(m-2c)j)/2)\Bigg) q^n,$$

which completes the proof. □

As special cases of Theorem 8, we have the following corollaries which provide recurrence relations for the number of partitions that appear in the well-known Rogers-Ramanujan's identities.

Corollary 9. *Let $p_{R1}(n)$ denote the number of partitions of n whose parts are congruent to ± 1 modulo 5 and let $p_{R2}(n)$ denote the number of partitions of n whose parts are congruent to ± 2 modulo 5. Then, for all $n \geq 0$, we have*

$$p_{R1}(n) - p_{R1}(n-1) - p_{R1}(n-4) + p_{R1}(n-7) + p_{R1}(n-13) - \cdots$$
$$\cdots + (-1)^j \big(p_{R1}(n-j(5j-3)/2) + p_{R1}(n-j(5j+3)/2)\big) + \cdots$$
$$= \begin{cases} 1, & \text{if } n = 5h_j^e; \\ -1, & \text{if } n = 5h_j^o; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$p_{R2}(n) - p_{R2}(n-1) - p_{R2}(n-2) + p_{R2}(n-5) + p_{R2}(n-7) - \cdots$$
$$\cdots + (-1)^j \big(p_{R2}(n-j(3j-1)/2) + p_{R2}(n-j(3j+1)/2)\big) + \cdots$$
$$= \begin{cases} 1, & \text{if } n = k_j^e; \\ -1, & \text{if } n = k_i^o; \\ 0, & \text{otherwise,} \end{cases}$$

where h_j^e (resp., h_j^o) is the j-th heptagonal number [9, A085787] with j even (resp., odd).

Corollary 10. *Let $s_1(n)$ denote the number of partitions of n having congruent to ± 1 modulo 6 and let $s_2(n)$ denote the number of partitions of n whose parts are congruent to ± 2 modulo 6. Then, for all $n \geq 0$,*

$$\begin{aligned} &s_1(n) - s_1(n-1) - s_1(n-5) + s_1(n-8) + s_1(n-16) - \cdots \\ &\quad \cdots + (-1)^j \big(s_1(n - j(3j-2)) + s_1(n - j(3j+2))\big) + \cdots \\ &= s_2(n) - s_2(n-2) - s_2(n-4) + s_2(n-10) + s_2(n-14) - \cdots \\ &\quad \cdots + (-1)^j \big(s_2(n - j(3j-1)) + s_2(n - j(3j+1))\big) + \cdots \\ &= \begin{cases} 1, & \text{if } n = 6k_j^e; \\ -1, & \text{if } n = 6k_i^o; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

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