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Parity Considerations for the Mex-Related Partition Functions of Andrews and Newman

Robson da Silva
Departamento de Ciência e Tecnologia
Universidade Federal de São Paulo
São José dos Campos, SP, 12247–014
Brazil
silva.robson@unifesp.br

James A. Sellers[1]
Department of Mathematics and Statistics
University of Minnesota, Duluth
Duluth, MN 55812
USA
jsellers@d.umn.edu

Abstract

In a recent paper, Andrews and Newman extended the mex-function to integer partitions and proved many partition identities connected with these functions. In this paper, we present parity considerations of one of the families of functions they studied, namely $p_{t,t}(n)$. Among our results, we provide complete parity characterizations of $p_{1,1}(n)$ and $p_{3,3}(n)$.

1 Introduction

Andrews and Newman [2] recently generalized the minimal excludant function (mex-function) to apply to integer partitions. Given a partition λ of n, they defined the mex-function

[1] Corresponding author.

$\mathrm{mex}_{A,a}(\lambda)$ to be the smallest integer congruent to a modulo A that is not part of λ. They denote the number of partitions λ of n satisfying

$$\mathrm{mex}_{A,a}(\lambda) \equiv a \pmod{2A}$$

by $p_{A,a}(n)$. For example, consider $n = 5$, $A = 3$, and $a = 2$. In the table below, we list the seven partitions λ of $n = 5$ and the corresponding values of $\mathrm{mex}_{3,2}(\lambda)$ for each λ:

Partition λ	$\mathrm{mex}_{3,2}(\lambda)$
5	2
4+1	2
3+2	5
3+1+1	2
2+2+1	5
2+1+1+1	5
1+1+1+1+1	2

Table 1: Values of $\mathrm{mex}_{3,2}(\lambda)$ for partitions λ of $n = 5$

We see that four of the partitions in Table 1 satisfy $\mathrm{mex}_{3,2}(\lambda) \equiv 2 \pmod 6$. Therefore, $p_{3,2}(5) = 4$.

Andrews and Newman proved [2, Lemma 9] that the generating function for $p_{t,t}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{t,t}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{tn(n+1)/2}, \tag{1}$$

where we use the following standard q-series notation:

$$(a;q)_0 = 1,$$
$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),\ \forall n \geq 1,$$

and

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n,\ |q| < 1.$$

The two functions $p_{1,1}(n)$ [7, A064428] and $p_{3,3}(n)$ [7, A260894] play an important role in [2]. Indeed, Andrews and Newman proved the following connections between these two functions and the enumeration of partitions according to their rank and their crank. (In 1944, Dyson [4] defined the *rank* of a partition as the largest part minus the number of parts. For instance, the rank of the partition $7+6+6+3+2$ is $7-5=2$. He also conjectured the existence of a statistic, the *crank* of a partition, which would provide a combinatorial proof of Ramanujan's congruence $p(11n+6) \equiv 0 \pmod{11}$. In 1988, Andrews and Garvan [1] presented the definition of the crank of a partition. Given a partition λ, we let $l(\lambda)$ denote the largest part of λ, $w(\lambda)$ denote the number of ones in λ, and $\mu(\lambda)$ denote the number of parts of λ larger than $w(\lambda)$. The crank $c(\lambda)$ of λ is given by

$$c(\lambda)=\begin{cases} l(\lambda), & \text{if } w(\lambda)=0; \\ \mu(\lambda)-w(\lambda), & \text{if } w(\lambda)>0. \end{cases}$$

For example, $c(7+6+6+3+2)=7$ and $c(7+6+6+3+2+1+1)=4-2=2$.)

Theorem 1 ([2], Theorem 2)**.** *$p_{1,1}(n)$ equals the number of partitions of n with non-negative crank.*

Theorem 2 ([2], Theorem 3)**.** *$p_{3,3}(n)$ equals the number of partitions of n with rank ≥ -1.*

In this paper, we present parity results for $p_{t,t}(n)$ for several odd values of t. In Section 2, we present complete parity characterizations for $p_{1,1}(n)$ and $p_{3,3}(n)$ which are available to us thanks to the lacunarity of the corresponding generating functions modulo 2. Unfortunately, the generating function for $p_{t,t}(n)$ modulo 2, for $t>3$, fails to be as lacunary. Even so, a set of congruences modulo 2 for $p_{t,t}(2kn+j)$, $t=5,7,11,13,17,19,23$, is presented in Section 3. All of the proof techniques required to prove these parity results are elementary, including classical generating function manipulations and well–known results of Euler and Jacobi.

2 Parity characterizations of $p_{1,1}(n)$ and $p_{3,3}(n)$

In order to prove the parity characterization for $p_{1,1}(n)$, we need the following well-known identities.

Lemma 3. *The following identities hold:*

$$(q;q)_\infty^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}, \tag{2}$$

$$(q;q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}. \tag{3}$$

Proof. Equation (2) is Jacobi's identity [3, Eq. (1.3.24)]. Equation (3) is Euler's pentagonal number theorem [3, Eq. (1.3.18)]. □

With the above in hand, we now present the characterization modulo 2 for $p_{1,1}(n)$.

Theorem 4. *For all $n\geq 1$, we have*

$$p_{1,1}(n)\equiv\begin{cases} 1 \pmod 2, & \textit{if } n=k(3k\pm 1) \textit{ for some } k; \\ 0 \pmod 2, & \textit{otherwise.} \end{cases}$$

Proof. Taking $k = 1$ in (1), we find that

$$\sum_{n=0}^{\infty} p_{1,1}(n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}$$
$$\equiv \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod{2}. \tag{4}$$

From Jacobi's identity (2), we have

$$(q;q)_\infty^3 \equiv \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod{2}.$$

Using this fact and (4) we obtain an expression for the generating function for $p_{1,1}(n)$ modulo 2:

$$\sum_{n=0}^{\infty} p_{1,1}(n)q^n \equiv (q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod{2}. \tag{5}$$

In order to complete the proof, we make use of Euler's pentagonal number theorem (3) to obtain

$$\sum_{n=0}^{\infty} p_{1,1}(n)q^n \equiv (q^2;q^2)_\infty \equiv \sum_{k=-\infty}^{\infty} q^{k(3k-1)} \pmod{2}$$
$$\equiv \sum_{k=1}^{\infty} q^{k(3k+1)} + \sum_{k=0}^{\infty} q^{k(3k-1)} \pmod{2},$$

which completes the proof. □

With Theorem 4 in hand, we can now prove a number of corollaries for specific arithmetic progressions.

Corollary 5. *For all $n \geq 0$, we have $p_{1,1}(2n+1) \equiv 0 \pmod{2}$.*

Proof. Note that $2n+1$ can never be represented as $k(3k \pm 1)$ because $k(3k \pm 1)$ is always even (it is twice a pentagonal number for any value k). Thus, by Theorem 4, the result follows. □

Corollary 6. *Let $p \geq 5$ be prime, and let r, $1 \leq r \leq p-1$, be such that $12r+1$ is a quadratic non–residue modulo p. Then, for all $n \geq 0$, we have $p_{1,1}(pn+r) \equiv 0 \pmod{2}$.*

Proof. Note that, if $pn + r = k(3k \pm 1)$ for some k, then $12(pn + r) + 1 = (6k \pm 1)^2$. Notice also that

$$12(pn + r) + 1 = 12pn + 12r + 1 \equiv 12r + 1 \pmod{p}$$

but $12r + 1$ is assumed to be a quadratic non–residue modulo p. Therefore, such a representation cannot exist. So by Theorem 4, we have $p_{1,1}(pn + r) \equiv 0 \pmod{2}$ for all $n \geq 0$. □

We discuss now the parity characterization of $p_{3,3}(n)$. By (1) and Ramanujan's theta function

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{6}$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{3,3}(n)q^n &\equiv \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{3n(n+1)/2} \pmod 2 \\
&= \frac{1}{(q;q)_\infty}\frac{(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty} \qquad \text{(by (6))} \\
&\equiv \frac{1}{(q;q)_\infty}\frac{(q^3;q^3)_\infty^4}{(q^3;q^3)_\infty} \pmod 2 \\
&= \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.
\end{aligned}$$

The generating function for the number of 3-core partitions [5, Theorem 1], denoted by $a_3(n)$ [7, A033687], is given by

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.$$

So, $p_{3,3}(n) \equiv a_3(n) \pmod 2$ for all $n \geq 0$. Thanks to the work of Hirschhorn and Sellers [5, Theorem 6] on a closed formula for $a_3(n)$, we have the following parity characterization for $p_{3,3}(n)$:

Theorem 7. *For all $n \geq 1$, we have*

$$p_{3,3}(n) \equiv \begin{cases} 1 \pmod 2, & \text{if } 3n+1 \text{ is a square;} \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

This characterization can be applied rather easily to prove several infinite families of parity results for $p_{3,3}(n)$. We share a number of these corollaries here.

Corollary 8. *For all $m \geq 0$ and all $n \geq 0$, we have*

$$p_{3,3}\left(4^{m+1}n + \frac{7\cdot 4^m - 1}{3}\right) \equiv 0 \pmod 2$$

and

$$p_{3,3}\left(4^{m+1}n + \frac{10\cdot 4^m - 1}{3}\right) \equiv 0 \pmod 2.$$

Proof. Note first that

$$3\left(4^{m+1}n+\frac{7\cdot 4^m-1}{3}\right)+1=4^m(12n+7)$$

after straightforward simplification. Note that 4^m is a square while $12n+7$ cannot be. (This is clear since $12n+7\equiv 3 \pmod 4$ and all squares are either 0 or 1 modulo 4). Thus

$$3\left(4^{m+1}n+\frac{7\cdot 4^m-1}{3}\right)+1$$

is not a square, and the result is proved thanks to Theorem 7.

Next, note that

$$3\left(4^{m+1}n+\frac{10\cdot 4^m-1}{3}\right)+1=4^m(12n+10)$$

after straightforward simplification. Note that 4^m is a square while $12n+10$ cannot be. (This is clear since $12n+10\equiv 2 \pmod 4$ and all squares are either 0 or 1 modulo 4). Thus

$$3\left(4^{m+1}n+\frac{10\cdot 4^m-1}{3}\right)+1$$

is not a square, and the result is proved thanks to Theorem 7. □

Corollary 9. *For all $m\geq 0$ and all $n\geq 0$, we have*

$$p_{3,3}\left(2\cdot 4^{m+1}n+\frac{13\cdot 4^m-1}{3}\right)\equiv 0 \pmod 2.$$

Proof. Note that

$$3\left(2\cdot 4^{m+1}n+\frac{13\cdot 4^m-1}{3}\right)+1=4^m(24n+13)$$

after straightforward simplification. Note that 4^m is a square while $24n+13$ cannot be. (This is clear since $24n+13\equiv 5 \pmod 8$ and all squares are either $0,1$ or 4 modulo 8). Thus

$$3\left(2\cdot 4^{m+1}n+\frac{13\cdot 4^m-1}{3}\right)+1$$

is not a square, and the result is proved thanks to Theorem 7. □

Corollary 10. *Let $p\geq 5$ be prime, and let r, $1\leq r\leq p-1$ be such that $3r+1$ is a quadratic non–residue modulo p. Then, for all $n\geq 0$, we have $p_{3,3}(pn+r)\equiv 0 \pmod 2$.*

Proof. Note that $3(pn+r)+1$ can never be square. This is because

$$3(pn+r)+1=3pn+3r+1\equiv 3r+1 \pmod p$$

and $3r+1$ is assumed to be a quadratic non–residue modulo p. So by Theorem 7, it follows that $p_{3,3}(pn+r)\equiv 0 \pmod 2$ for all $n\geq 0$. □

3 Additional congruences

We now consider parity results for $p_{t,t}(n)$ for $t \geq 5$. While characterizations modulo 2 for these functions do not appear to be readily available, we can still prove a significant set of Ramanujan–like congruences for several of these functions.

Theorem 11. *For all $n \geq 0$, we have*

$$\begin{aligned}
p_{5,5}(10n+j) &\equiv 0 \pmod 2, j \in \{2,6\},\\
p_{7,7}(14n+j) &\equiv 0 \pmod 2, j \in \{7,9,13\},\\
p_{11,11}(22n+j) &\equiv 0 \pmod 2, j \in \{2,8,12,14,16\},\\
p_{13,13}(26n+j) &\equiv 0 \pmod 2, j \in \{2,10,16,18,20,22\},\\
p_{17,17}(34n+j) &\equiv 0 \pmod 2, j \in \{11,15,17,19,25,27,29,33\},\\
p_{19,19}(38n+j) &\equiv 0 \pmod 2, j \in \{2,8,10,20,24,28,30,32,34\},\\
p_{23,23}(46n+j) &\equiv 0 \pmod 2, j \in \{11,15,21,23,29,31,35,39,41,43,45\}.
\end{aligned}$$

Proof. Let $t \geq 5$ be an odd number. Taking (1) modulo 2, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{t,t}(n)q^n &\equiv \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{tn(n+1)/2} \pmod 2\\
&\equiv \frac{(q^t;q^t)_\infty^3}{(q;q)_\infty} \pmod 2 \qquad \text{(by (6))}. \qquad (7)
\end{aligned}$$

On the other hand, the generating function for t-core partitions [7, A175595] is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}.$$

Thus, from (7) we know

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{t,t}(n)q^n &\equiv \frac{(q^t;q^t)_\infty^3}{(q;q)_\infty} \pmod 2\\
&= \frac{1}{(q^t;q^t)_\infty^{t-3}}\sum_{n=0}^{\infty} a_t(n)q^n.
\end{aligned}$$

Since t is odd, we then know

$$\sum_{n=0}^{\infty} p_{t,t}(n)q^n \equiv \frac{1}{(q^{2t};q^{2t})_\infty^{(t-3)/2}}\sum_{n=0}^{\infty} a_t(n)q^n \pmod 2. \qquad (8)$$

The t-dissection of (8) yields, for each $r \in \{0, 1, \ldots, t-1\}$,

$$\sum_{n=0}^{\infty} p_{t,t}(tn+r)q^n \equiv \frac{1}{(q^2;q^2)_\infty^{(t-3)/2}} \sum_{n=0}^{\infty} a_t(tn+r)q^n \pmod 2.$$

Now, after 2-dissecting both sides of the last expression we are left with

$$\sum_{n=0}^{\infty} p_{t,t}(2tn+r)q^n \equiv \frac{1}{(q;q)_\infty^{(t-3)/2}} \sum_{n=0}^{\infty} a_t(2tn+r)q^n \pmod 2, \tag{9}$$

$$\sum_{n=0}^{\infty} p_{t,t}(2tn+t+r)q^n \equiv \frac{1}{(q;q)_\infty^{(t-3)/2}} \sum_{n=0}^{\infty} a_t(2tn+t+r)q^n \pmod 2. \tag{10}$$

Lastly, thanks to Radu and Sellers ([6, Theorem 1.4]) we know that, for all $n \geq 0$, we have

$$\begin{aligned}
a_5(10n+j) &\equiv 0 \pmod 2, j \in \{2, 6\},\\
a_7(14n+j) &\equiv 0 \pmod 2, j \in \{7, 9, 13\},\\
a_{11}(22n+j) &\equiv 0 \pmod 2, j \in \{2, 8, 12, 14, 16\},\\
a_{13}(26n+j) &\equiv 0 \pmod 2, j \in \{2, 10, 16, 18, 20, 22\},\\
a_{17}(34n+j) &\equiv 0 \pmod 2, j \in \{11, 15, 17, 19, 25, 27, 29, 33\},\\
a_{19}(38n+j) &\equiv 0 \pmod 2, j \in \{2, 8, 10, 20, 24, 28, 30, 32, 34\},\\
a_{23}(46n+j) &\equiv 0 \pmod 2, j \in \{11, 15, 21, 23, 29, 31, 35, 39, 41, 43, 45\}.
\end{aligned}$$

This, combined with (9) and (10), implies the results of this theorem. □

4 Closing thoughts

Several remarks are in order as we close. First, Andrews and Newman [2] introduce a second family of functions which they denote by $p_{2t,t}(n)$. The reader is referred to [2] for the details of this family of functions. Our hope was that this family would also satisfy various Ramanujan–like congruence properties. Unfortunately, based on extensive computations, this does not appear to be the case.

Secondly, we close with two potential paths for future work. Namely, it would be nice to have combinatorial proofs of these parity results. It would also be gratifying to have a fully elementary proof of Theorem 11 (since Radu and Sellers relied on modular forms to carry out their proof of their parity results for t–core partition functions which were mentioned in the proof of Theorem 11).

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