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The h^*-Polynomial of the Cut Polytope of $K_{2,m}$ in the Lattice Spanned by its Vertices

Ryuichi Sakamoto
Department of Mathematical Sciences
Graduate School of Science and Technology
Kwansei Gakuin University
Sanda, Hyogo 669-1337
Japan
dpm86391@kwansei.ac.jp

Abstract

The cut polytope of a graph is an important object in several fields, such as functional analysis, combinatorial optimization, and probability. For example, Sturmfels and Sullivant showed that the toric ideals of cut polytopes are useful in algebraic statistics. In the theory of lattice polytopes, the h^*-polynomial is an important invariant. However, except for trees, there are no classes of graphs for which the h^*-polynomial of their cut polytope is explicitly specified. In the present paper, we determine the h^*-polynomial of the cut polytope of the complete bipartite graph $K_{2,m}$ using the theory of Gröbner bases of toric ideals.

1 Introduction

Given integer vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l \in \mathbb{Z}^d$, let

$$\operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l) = \left\{ \sum_{i=1}^{l} r_i \boldsymbol{v}_i \;\middle|\; 0 \le r_i \in \mathbb{R}, \sum_{i=1}^{l} r_i = 1 \right\}.$$

A set $\mathcal{P} \subset \mathbb{R}^d$ is called a *lattice polytope* if there exist $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l \in \mathbb{Z}^d$ such that $\mathcal{P} = \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l)$. Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d, where $\mathcal{P} \cap \mathbb{Z}^d =$

$\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$. Let $\mathcal{A}_\mathcal{P}$ be the integer matrix

$$\mathcal{A}_\mathcal{P} = \begin{pmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \\ 1 & \cdots & 1 \end{pmatrix}.$$

The *normalized Ehrhart polynomial* $i(\mathcal{P}, m)$ is given by the following equation:

$$i(\mathcal{P}, m) = |m\mathcal{P}' \cap \mathbb{Z}\mathcal{A}_\mathcal{P}|,$$

where $m \in \mathbb{N}$, $\mathcal{P}' = \mathrm{conv}(\binom{\boldsymbol{a}_1}{1}, \ldots, \binom{\boldsymbol{a}_n}{1})$, $m\mathcal{P}' = \{m\boldsymbol{a} \mid \boldsymbol{a} \in \mathcal{P}'\}$, and

$$\mathbb{Z}\mathcal{A}_\mathcal{P} = \mathbb{Z}\begin{pmatrix} \boldsymbol{a}_1 \\ 1 \end{pmatrix} + \cdots + \mathbb{Z}\begin{pmatrix} \boldsymbol{a}_n \\ 1 \end{pmatrix}.$$

In general, $i(\mathcal{P}, m)$ satisfies the following fundamental properties [4]:

- $i(\mathcal{P}, m)$ is a polynomial of degree d in m;
- $i(\mathcal{P}, 0) = 1$.

The *h^*-polynomial* $h^*(\mathcal{P}, x)$ of $\mathcal{P}$ in the lattice $\mathbb{Z}\mathcal{A}_\mathcal{P}$ is defined by

$$1 + \sum_{m=1}^{\infty} i(\mathcal{P}, m)x^m = \frac{h^*(\mathcal{P}, x)}{(1-x)^{d+1}}.$$

In general, $h^*(\mathcal{P}, x)$ satisfies the following properties:

- $h^*(\mathcal{P}, x) = \sum_{i=0}^{d} h_i^* x^i$, where each h_i^* is a nonnegative integer [18];
- $i(\mathcal{P}, m) = \sum_{i=0}^{d} h_i^* \binom{m+d-i}{d}$;
- If $h_d^* > 0$, then we have $h_i^* \geq h_1^*$ $(1 \leq i \leq d-1)$ [8, Theorem 1.1].

The third property is *Hibi's lower bound theorem*. Since the h^*-polynomial is defined in the lattice $\mathbb{Z}\mathcal{A}_\mathcal{P}$, $\mathcal{P}$ is a *spanning lattice polytope* [10], and a generalization of Hibi's lower bound theorem is known:

- $h_i^* \geq h_1^*$ $(1 \leq i \leq \deg(h^*(\mathcal{P}, x)) - 1)$ [9, Corollary 1.6].

A polynomial $f(x)$ of degree s is said to be *palindromic* if $f(x) = x^s f(x^{-1})$. Let $K[\mathcal{A}_\mathcal{P}]$ be the toric ring of $\mathcal{P}$. (The toric ring will be defined in Section 1.) If $K[\mathcal{A}_\mathcal{P}]$ is normal (i.e., $\mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P} = \mathbb{Z}\mathcal{A}_\mathcal{P} \cap \mathbb{Q}_{\geq 0}\mathcal{A}_\mathcal{P}$, see [21, Proposition 13.5]) and Gorenstein, then it is known by [7, Lemma 4.22 (b)] and [3, P.235] that the h^*-polynomial of $\mathcal{P}$ is palindromic. In the theory of lattice polytopes, h^*-polynomials are important objects to study. For example, the h^*-polynomials of stable set polytopes of graphs, order polytopes, and chain polytopes of posets were studied in [1, 19]. Let G be a finite connected simple graph with vertex set

$V(G) = \{1, 2, \ldots, m\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_r\}$. For two subsets A and B of $V(G)$ such that $A \cap B = \emptyset, A \cup B = V(G)$, we define a vector $\delta_{A|B} = (d_1, d_2, \ldots, d_r) \in \{0, 1\}^r$ by

$$d_i = \begin{cases} 1, & |A \cap e_i| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\delta_{A|B} = \delta_{B|A}$. The *cut polytope* of G is the 0/1 polytope

$$\mathrm{Cut}(G) = \mathrm{conv}(\delta_{A|B} \, | A, B \subset V(G), A \cap B = \emptyset, A \cup B = V(G)).$$

Example 1. Let G be a cycle of length 4, where $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{e_1 = \{1, 2\}, e_2 = \{2, 3\}, e_3 = \{3, 4\}, e_4 = \{1, 4\}\}$. For subset $A = \{1, 2\} \subset V(G)$, we compute $\delta_{A|B} = (d_1, d_2, d_3, d_4)$, where $B = \{3, 4\}$. Since $|A \cap e_1| = 2$, we have $d_1 = 0$. Similarly, we obtain $d_2 = 1, d_3 = 0$, and $d_4 = 1$. Hence, $\delta_{A|B} = (0, 1, 0, 1)$. By computing $\delta_{A|B}$ for all subsets $A \subset V(G)$, we obtain the cut polytope

$$\mathrm{Cut}(G) = \\ \mathrm{conv}((0, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1), (1, 1, 1, 1)).$$

We define the graph theoretical terminology used in the present paper. A *bridge* of a graph G is an edge of G whose deletion increases the number of connected components, and a graph G is said to be *bridgeless* if G has no bridges. An *induced cycle* of G is a cycle of G that is an induced subgraph of G. A graph G is said to be *chordal* if G has no induced cycles of length ≥ 4. A graph H is called a *minor* of a graph G if H is obtained from G by a sequence of contractions and deletions of edges. On the other hand, if we cannot obtain H as a minor of G, then G is said to be *H-minor free*. A complete graph with n vertices is denoted by K_n, a complete bipartite graph with $m + n$ vertices is denoted by $K_{m,n}$, and a cycle of length n is denoted by C_n. There is only one class of cut polytopes for which the h^*-polynomial is explicitly known. Nagel and Petrović [12] showed that, if G is a tree with $n \geq 1$ edges, then the h^*-polynomial of the cut polytope in the lattice $\mathbb{Z}\mathcal{A}_{\mathrm{Cut}(G)}$ is the Eulerian polynomial

$$A_n(x) := \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)}$$

of degree $n - 1$. Here, $\mathfrak{S}_n$ is a symmetric group and $\mathrm{des}(w) = |\{i \mid w_i > w_{i+1}\}|$ for $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$. It is known that $A_n(x)$ is palindromic and unimodal. Ohsugi [14] showed that the toric ring of the cut polytope $\mathrm{Cut}(G)$ of a graph G is normal and Gorenstein if and only if G is K_5-minor free and satisfies one of the following:

1. G is a bipartite graph with no induced cycle of length ≥ 6.

2. G is a bridgeless chordal graph.

Thus, if G satisfies one of the above conditions, then the h^*-polynomial of the cut polytope of G is palindromic, since the toric ring is normal and Gorenstein.

In the present paper, we determine the h^*-polynomial of the cut polytope of a complete bipartite graph $K_{2,n-2}$ and show that the h^*-polynomial is $(x+1)(A_{n-2}(x))^2$ using the theory of Gröbner bases of toric ideals. See [7, 21] for the details of Gröbner bases and toric ideals.

Remark 2. We discuss the normalized Ehrhart polynomial

$$i(\mathcal{P}, m) = |m\mathcal{P}' \cap \mathbb{Z}\mathcal{A}_{\mathcal{P}}|$$

instead of the ordinary *Ehrhart polynomial*

$$|m\mathcal{P} \cap \mathbb{Z}^d|$$

because the lattice spanned by the $\delta_{A|B}$ is important in the study of the cut polytopes. In fact, the characterization of the graph G satisfying $\mathbb{Z}_{\geq 0}B_G = \mathbb{Z}B_G \cap \mathbb{Q}_{\geq 0}B_G$, where $B_G = \mathrm{Cut}(G) \cap \mathbb{Z}^d$, is an important open problem. See [6] and the references therein. In addition, it has been conjectured that $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathrm{Cut}(G)} = \mathbb{Z}\mathcal{A}_{\mathrm{Cut}(G)} \cap \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathrm{Cut}(G)}$ if and only if G is K_5-minor free [22]. See [13, 22].

2 Standard monomials of cut ideals of $K_{2,n-2}$

Let $K[\boldsymbol{x}] = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field K. Let $\mathcal{M}_n$ be the set of all monomials of $K[\boldsymbol{x}]$. A total order $<$ on $\mathcal{M}_n$ is called a *monomial order* if $<$ satisfies the following properties:

- For all $1 \neq u \in \mathcal{M}_n$, $1 < u$.
- If $u < v$ ($u, v \in \mathcal{M}_n$), then $w \cdot u < w \cdot v$ for all $w \in \mathcal{M}_n$.

We fix a monomial order $<$. For a nonzero polynomial f which belongs to $K[\boldsymbol{x}]$, the *initial monomial* $\mathrm{in}_<(f)$ of f is the greatest monomial in f with respect to $<$. The *initial ideal* of an ideal $I \subset K[\boldsymbol{x}]$ with respect to $<$ is defined by $\mathrm{in}_<(I) = \langle\ \mathrm{in}_<(f) \mid 0 \neq f \in I\rangle$. A finite subset $G = \{g_1, \dots, g_s\} \subset I$ is called a *Gröbner basis* of I with respect to $<$ when $\mathrm{in}_<(I) = \langle \mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_s)\rangle$. See [7, Chapter 1] for the basic theory of Gröbner bases.

A $d \times n$ integer matrix $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$ is called a *configuration* if there exists a vector $\boldsymbol{c} \in \mathbb{R}^d$ such that for all $1 \leq i \leq n$, the inner product $\boldsymbol{a}_i \cdot \boldsymbol{c}$ is equal to 1. Let $K[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_d^{\pm 1}]$ be a Laurent polynomial ring over K. For an integer vector $\boldsymbol{b} = (b_1, b_2, \dots, b_d) \in \mathbb{Z}^d$, we define the Laurent monomial $\boldsymbol{t}^{\boldsymbol{b}} = t_1^{b_1} t_2^{b_2} \cdots t_d^{b_d} \in K[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_d^{\pm 1}]$ and the *toric ring* $K[A] = K[\boldsymbol{t}^{\boldsymbol{a}_1}, \boldsymbol{t}^{\boldsymbol{a}_2}, \dots, \boldsymbol{t}^{\boldsymbol{a}_n}]$. Let π be a homomorphism $\pi : K[\boldsymbol{x}] \to K[A]$, where $\pi(x_i) = \boldsymbol{t}^{\boldsymbol{a}_i}$. The kernel of π is called the *toric ideal* of A and denoted by I_A. We often regard $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ as a set $A = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$. Suppose that a set Δ consists of simplices and that each vertex of $\sigma \in \Delta$ belongs to A. Then Δ is called a *covering* of $\mathrm{conv}(A)$ if

$$\mathrm{conv}(A) = \bigcup_{F \in \Delta} F.$$

We say that a covering Δ is a *triangulation* of $\mathrm{conv}(A)$ if Δ is a simplicial complex. A triangulation Δ of a polytope $\mathrm{conv}(A)$ is *unimodular* if the normalized volume of any maximal simplex is equal to 1. For a configuration A, the *initial complex* with respect to $<$ is defined by

$$\Delta(\mathrm{in}_<(I_A)) := \left\{ \mathrm{conv}(B) \;\middle|\; B \subset \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}, \prod_{\boldsymbol{a}_i \in B} x_i \notin \sqrt{\mathrm{in}_<(I_A)} \right\}.$$

It is known that $\Delta(\mathrm{in}_<(I_A))$ is a triangulation of $\mathrm{conv}(A)$. Moreover, $\Delta(\mathrm{in}_<(I_A))$ is unimodular if and only if $\mathrm{in}_<(I_A)$ is generated by squarefree monomials.

Example 3. Let A be a configuration

$$A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The toric ideal of A is given by $I_A = \langle x_1 x_5^2 - x_2 x_3 x_4 \rangle$. Let $<$ be the lexicographic order on $K[x_1, x_2, x_3, x_4, x_5]$ induced by the ordering $x_1 > x_2 > x_3 > x_4 > x_5$. Then

$$\begin{aligned} \mathrm{in}_<(x_1 x_5^2 - x_2 x_3 x_4) &= x_1 x_5^2 \\ \mathrm{in}_<(I_A) &= \langle x_1 x_5^2 \rangle \\ \sqrt{\mathrm{in}_<(I_A)} &= \langle x_1 x_5 \rangle. \end{aligned}$$

Thus, the maximal simplices of $\Delta(\mathrm{in}_<(I_A))$ are

$$\sigma_1 = \mathrm{conv}(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4) \text{ and } \sigma_2 = \mathrm{conv}(\boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5).$$

The triangulation $\Delta(\mathrm{in}_<(I_A))$ is not unimodular, since the normalized volume of σ_1 is 2.

A monomial is said to be a *standard monomial* of a toric ideal I_A with respect to a monomial order $<$ if the monomial does not belong to $\mathrm{in}_<(I_A)$. If $\Delta(\mathrm{in}_<(I_A))$ is unimodular, then the number of squarefree standard monomials of degree i corresponds to the number of $(i-1)$-dimensional faces of $\Delta(\mathrm{in}_<(I_A))$. See [7, 21] for details.

Let G be a graph with m vertices. We consider the configuration

$$\mathcal{A}_{\mathrm{Cut}(G)} = \begin{pmatrix} \delta_{A_1|B_1} & \delta_{A_2|B_2} & \cdots & \delta_{A_N|B_N} \\ & & & \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $A_i \cap B_i = \emptyset, A_i \cup B_i = V(G)$ for $1 \le i \le N$, and $N = 2^{m-1}$. The toric ideal of $\mathcal{A}_{\mathrm{Cut}(G)}$ is called the *cut ideal* of G and is denoted by I_G. The notion of cut ideals was introduced in [22]. The toric ring and ideal of $\mathcal{A}_{\mathrm{Cut}(G)}$ were investigated in, e.g., [5, 12, 11, 13, 16, 17]. The cut ideal I_G of a graph G is generated by quadratic binomials if and only if G is K_4-minor free [5]. The cut ideal I_G of a graph G has a quadratic Gröbner basis if G satisfies one of the following:

- G is (K_4, C_5)-minor free, where K_4 is a complete graph with 4 vertices, and C_5 is a cycle of length 5 [17, Corollary 2.4];
- G is a cycle of length ≤ 7 [16, Theorem 2.3].

An *unordered partition* $A|B$ of the vertex set $V(G)$ consists of subsets $A, B \subset V(G)$ such that $A \cap B = \emptyset, A \cup B = V(G)$. Given an unordered partition $A|B$, we associate a variable $q_{A|B}$. In particular, $q_{A|B} = q_{B|A}$. Let $K[q]$ be the polynomial ring in $N = 2^{m-1}$ variables over a field K defined by

$$K[q] = K[q_{A_1|B_1}, \dots, q_{A_N|B_N}],$$

where $\{A_1|B_1, \dots, A_N|B_N\}$ is the set of all unordered partitions of $V(G)$. Let $<$ be a reverse lexicographic order [7, Example 1.8 (b)] on $K[q]$ that satisfies $q_{A|B} < q_{C|D}$ with $\min\{|A|, |B|\} < \min\{|C|, |D|\}$. A quadratic Gröbner basis of the cut ideal of $K_{2,n-2}$ with respect to $<$ is given by the following proposition.

Proposition 4 ([17, Theorem 2.3])**.** *Let $K_{2,n-2}$ be the complete bipartite graph on the vertex set $\{1,2\} \cup \{3, \dots, n\}$ for $n \geq 4$. Then a Gröbner basis of $I_{K_{2,n-2}}$ with respect to $<$ consists of*

1. $q_{\{1\}\cup A|\{2\}\cup B} q_{\{1\}\cup B|\{2\}\cup A} - q_{\emptyset|[n]} q_{\{1,2\}|\{3,\dots,n\}}$,
2. $q_{A|B} q_{C|D} - q_{A\cap C|B\cup D} q_{A\cup C|B\cap D}$ $(1 \in A \cap C, 2 \in B \cap D, A \not\subset C, C \not\subset A)$,
3. $q_{A|B} q_{C|D} - q_{A\cap C|B\cup D} q_{A\cup C|B\cap D}$ $(1, 2 \in A \cap C, A \not\subset C, C \not\subset A)$.

The initial monomial of each binomial is the first monomial.

Example 5. Let $K_{2,3}$ be the complete bipartite graph on the vertex set $\{1,2\} \cup \{3,4,5\}$. Then the configuration $\mathcal{A}_{\mathrm{Cut}(K_{2,3})}$ is

$$\mathcal{A}_{\mathrm{Cut}(K_{2,3})} = \begin{pmatrix} 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&0&0&1&1&0&0&0&0&1&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1 \\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix},$$

and a Gröbner basis of $I_{K_{2,3}}$ with respect to the monomial order $<$ consists of the following binomials:

$$\begin{gathered} q_{\{1\}|\{2,3,4,5\}} q_{\{2\}|\{1,3,4,5\}} - q_{\emptyset|\{1,2,3,4,5\}} q_{\{1,2\}|\{3,4,5\}}, \\ q_{\{1,3\}|\{2,4,5\}} q_{\{2,3\}|\{1,4,5\}} - q_{\emptyset|\{1,2,3,4,5\}} q_{\{1,2\}|\{3,4,5\}}, \\ q_{\{1,4\}|\{2,3,5\}} q_{\{2,4\}|\{1,3,5\}} - q_{\emptyset|\{1,2,3,4,5\}} q_{\{1,2\}|\{3,4,5\}}, \\ q_{\{1,5\}|\{2,3,4\}} q_{\{2,5\}|\{1,3,4\}} - q_{\emptyset|\{1,2,3,4,5\}} q_{\{1,2\}|\{3,4,5\}}, \end{gathered}$$

$$
\begin{gathered}
q_{\{3\}|\{1,2,4,5\}}q_{\{4,5\}|\{1,2,3\}} - q_{\emptyset|\{1,2,3,4,5\}}q_{\{1,2\}|\{3,4,5\}}, \\
q_{\{5\}|\{1,2,3,4\}}q_{\{3,4\}|\{1,2,5\}} - q_{\emptyset|\{1,2,3,4,5\}}q_{\{1,2\}|\{3,4,5\}}, \\
q_{\{4\}|\{1,2,3,5\}}q_{\{3,5\}|\{1,2,4\}} - q_{\emptyset|\{1,2,3,4,5\}}q_{\{1,2\}|\{3,4,5\}}, \\
q_{\{4\}|\{1,2,3,5\}}q_{\{5\}|\{1,2,3,4\}} - q_{\emptyset|\{1,2,3,4,5\}}q_{\{4,5\}|\{1,2,3\}}, \\
q_{\{3\}|\{1,2,4,5\}}q_{\{5\}|\{1,2,3,4\}} - q_{\emptyset|\{1,2,3,4,5\}}q_{\{3,5\}|\{1,2,4\}}, \\
q_{\{3\}|\{1,2,4,5\}}q_{\{4\}|\{1,2,3,5\}} - q_{\emptyset|\{1,2,3,4,5\}}q_{\{3,4\}|\{1,2,5\}}, \\
q_{\{3,5\}|\{1,2,4\}}q_{\{4,5\}|\{1,2,3\}} - q_{\{5\}|\{1,2,3,4\}}q_{\{1,2\}|\{3,4,5\}}, \\
q_{\{3,4\}|\{1,2,5\}}q_{\{3,5\}|\{1,2,4\}} - q_{\{3\}|\{1,2,4,5\}}q_{\{1,2\}|\{3,4,5\}}, \\
q_{\{3,4\}|\{1,2,5\}}q_{\{4,5\}|\{1,2,3\}} - q_{\{4\}|\{1,2,3,5\}}q_{\{1,2\}|\{3,4,5\}} \\
q_{\{1,4\}|\{2,3,5\}}q_{\{1,5\}|\{2,3,4\}} - q_{\{1\}|\{2,3,4,5\}}q_{\{2,3\}|\{1,4,5\}}, \\
q_{\{1,3\}|\{2,4,5\}}q_{\{1,5\}|\{2,3,4\}} - q_{\{1\}|\{2,3,4,5\}}q_{\{2,4\}|\{1,3,5\}}, \\
q_{\{2,3\}|\{1,4,5\}}q_{\{2,4\}|\{1,3,5\}} - q_{\{2\}|\{1,3,4,5\}}q_{\{1,5\}|\{2,3,4\}}, \\
q_{\{1,3\}|\{2,4,5\}}q_{\{1,4\}|\{2,3,5\}} - q_{\{1\}|\{2,3,4,5\}}q_{\{2,5\}|\{1,3,4\}}, \\
q_{\{2,3\}|\{1,4,5\}}q_{\{2,5\}|\{1,3,4\}} - q_{\{2\}|\{1,3,4,5\}}q_{\{1,4\}|\{2,3,5\}}, \\
q_{\{2,4\}|\{1,3,5\}}q_{\{2,5\}|\{1,3,4\}} - q_{\{2\}|\{1,3,4,5\}}q_{\{1,3\}|\{2,4,5\}}.
\end{gathered}
$$

The h^*-polynomial $h^*(\mathrm{Cut}(K_{2,3}), x)$ is

$$h^*(\mathrm{Cut}(K_{2,3}), x) = x^5 + 9x^4 + 26x^3 + 26x^2 + 9x + 1 = (x+1)(x^2+4x+1)^2.$$

Since the dimension of the cut polytopes of $K_{2,n-2}$ is $2n-4$, the maximum degree of squarefree standard monomials is $2n-3$. Note that the initial monomial $q_{A|B}q_{C|D}$ of the binomials in the Gröbner basis in Proposition 4 satisfies one of the following conditions:

- $1 \in A \cap C$ and $2 \in B \cap D$,
- $1, 2 \in A \cap C$.

Hence, a squarefree monomial

$$q_{A_1|\{1,2\}\cup B_1} \cdots q_{A_k|\{1,2\}\cup B_k}\; q_{\{1\}\cup A'_1|\{2\}\cup B'_1} \cdots q_{\{1\}\cup A'_l|\{2\}\cup B'_l} \in K[q]$$

is standard if and only if both $q_{A_1|\{1,2\}\cup B_1} \cdots q_{A_k|\{1,2\}\cup B_k}$ and $q_{\{1\}\cup A'_1|\{2\}\cup B'_1} \cdots q_{\{1\}\cup A'_l|\{2\}\cup B'_l}$ are standard.

Proposition 6. *Each of the squarefree monomials*

(1) $q_{A_1|\{1,2\}\cup B_1} \cdots q_{A_k|\{1,2\}\cup B_k}$

(2) $q_{\{1\}\cup A'_1|\{2\}\cup B'_1} \cdots q_{\{1\}\cup A'_k|\{2\}\cup B'_k}$,

of degree $k \leq 2n-3$ is not divisible by the initial monomials if and only if, by changing indices if necessary,

(1) $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_k$,

(2) $A'_1 \subsetneq A'_2 \subsetneq \cdots \subsetneq A'_k$ and $(A'_1, A'_k) \neq (\emptyset, \{3, \dots, n\})$,

respectively. Moreover, the numbers of squarefree standard monomials of types (1) *and* (2) *above are*

$$(k-1)!\left\{ {n-2 \atop k-1} \right\} + 2k!\left\{ {n-2 \atop k} \right\} + (k+1)!\left\{ {n-2 \atop k+1} \right\},$$
$$2k!\left\{ {n-2 \atop k} \right\} + (k+1)!\left\{ {n-2 \atop k+1} \right\},$$

respectively, where $\left\{ {n \atop k} \right\}$ *is the Stirling number of the second kind.*

Proof. Let $m_1 = q_{A_1|\{1,2\}\cup B_1} \cdots q_{A_k|\{1,2\}\cup B_k}$ and $m_2 = q_{\{1\}\cup A'_1|\{2\}\cup B'_1} \cdots q_{\{1\}\cup A'_k|\{2\}\cup B'_k}$ be squarefree monomials.

First, suppose that m_1 and m_2 are standard. Since m_1 is squarefree and is not divisible by the initial monomials $q_{A|B}q_{C|D}$ $(1, 2 \in A \cap C, A \not\subset C, C \not\subset A)$, we can obtain $B_1 \supsetneq \cdots \supsetneq B_{k-1} \supsetneq B_k$ by changing indices if necessary. Then $A_1 \subsetneq \cdots \subsetneq A_{k-1} \subsetneq A_k$. Similarly, since m_2 is squarefree and is not divisible by the initial monomials $q_{A|B}q_{C|D}$ $(1 \in A \cap C, 2 \in B \cap D, A \not\subset C, C \not\subset A)$, we can obtain $A'_1 \subsetneq \cdots \subsetneq A'_{k-1} \subsetneq A'_k$ by changing indices if necessary. Moreover, since m_2 is not divisible by the initial monomial $q_{\{1\}|\{2,3,\dots,n\}}q_{\{1,3,\dots,n\}|\{2\}}$ of the binomial $q_{\{1\}|\{2,3,\dots,n\}}q_{\{1,3,\dots,n\}|\{2\}} - q_{\emptyset|[n]}q_{\{1,2\}|\{3,\dots,n\}}$, we have $(A'_1, A'_k) \neq (\emptyset, \{3, \dots, n\})$.

Contrarily, suppose that m_1 and m_2 satisfy $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_k$, $A'_1 \subsetneq A'_2 \subsetneq \cdots \subsetneq A'_k$ and $(A'_1, A'_k) \neq (\emptyset, \{3, \dots, n\})$. Then m_1 and m_2 are not divisible by the initial monomials $q_{A|B}q_{C|D}$ $(1, 2 \in A \cap C, A \not\subset C, C \not\subset A)$ and $q_{A|B}q_{C|D}$ $(1 \in A \cap C, 2 \in B \cap D, A \not\subset C, C \not\subset A)$. Hence, m_1 is standard. Suppose that m_2 is not standard. Then m_2 is divisible by the initial monomial $q_{\{1\}\cup A|\{2\}\cup B}q_{\{1\}\cup B|\{2\}\cup A}$. Since $A'_1 \subsetneq A'_2 \subsetneq \cdots \subsetneq A'_k$. Thus, we may assume that $A \subsetneq B$. Since $\{1\} \cup A|\{2\} \cup B$ is a partition, $A \cup B = \{3, \dots, n\}$. Therefore, $(A, B) = (\emptyset, \{3, \dots, n\})$. This contradicts the hypothesis $(A'_1, A'_k) \neq (\emptyset, \{3, \dots, n\})$. Hence, m_2 is standard.

The Stirling number of the second kind $\left\{ {a \atop b} \right\}$ represents the number of ways to partition a set of a objects into b nonempty subsets. We obtain Table 2 by considering four cases for the number of squarefree standard monomials of type (1). There is a restriction that

	$A_1 = \emptyset$	$A_1 \neq \emptyset$
$A_k = \{3, \dots, n\}$	$(k-1)!\left\{ {n-2 \atop k-1} \right\}$	$k!\left\{ {n-2 \atop k} \right\}$
$A_k \neq \{3, \dots, n\}$	$k!\left\{ {n-2 \atop k} \right\}$	$(k+1)!\left\{ {n-2 \atop k+1} \right\}$

Table 1: Number of squarefree standard monomials

$(A'_1, A'_k) \neq (\emptyset, \{3, \dots, n\})$ for type (2). Therefore, we obtain the desired number of squarefree standard monomials in each condition. $\square$

3 h^*-polynomial of the cut polytope of $K_{2,n-2}$

Let Δ be a triangulation of a lattice polytope $\mathcal{P}$. The *f-polynomial* $f_\Delta(x) = \sum_{i=0}^{d+1} f_{i-1}x^i$ of Δ encodes the number f_i of i-faces for $i = 0, 1 \ldots, d$ and $f_{-1} = 1$. The *h-polynomial* $h_\Delta(x) = \sum_{i=0}^{d+1} h_i x^i$ is given by the following relation [2, P.185]:

$$h_\Delta(x) = \sum_{i=0}^{d+1} f_{i-1}x^i(1-x)^{d+1-i}.$$

The following is known for h^*-polynomials and h-polynomials [2, Theorem 10.3].

Proposition 7. *If $\mathcal{P}$ is a d-dimensional lattice polytope that admits a unimodular triangulation Δ, then $h^*(\mathcal{P}, x) = h_\Delta(x)$.*

We have the following theorem for the h^*-polynomial of the cut polytope of $K_{2,n-2}$.

Theorem 8. *Let $\mathcal{P} = \mathrm{Cut}(K_{2,n-2})$ be the cut polytope of $K_{2,n-2}$, and let Δ be the unimodular triangulation $\Delta(\mathrm{in}_<(I_{\mathcal{A}_\mathcal{P}}))$ of $\mathcal{P}$ with respect to the monomial order $<$ in Proposition 4. Then the h^*-polynomial of $\mathcal{P}$ and the h-polynomial of Δ are*

$$h^*(\mathcal{P}, x) = h_\Delta(x) = (x+1)(A_{n-2}(x))^2,$$

where $A_n(x)$ is the Eulerian polynomial of degree $n-1$. In particular, the normalized volume of $\mathcal{P}$ is $h^*(\mathcal{P}, 1) = 2((n-2)!)^2$.

Proof. The Eulerian polynomial $A_n(x)$ satisfies the following condition [15, Theorem 9.1]:

$$A_n(x) = \sum_{k=1}^{n} k! \left\{ {n \atop k} \right\} (x-1)^{n-k}.$$

In addition, $\left\{ {n \atop k} \right\}$ is given by the following equation [20]:

$$\left\{ {n \atop k} \right\} = \sum_{m=1}^{k} (-1)^{k-m} m^n \binom{k}{m}.$$

The f-polynomial of Δ is given by the following equation:

$$f_\Delta(x) = \sum_{k=0}^{2n-3} f_{k-1}x^k.$$

Then f_{k-1} is equal to the number of squarefree standard monomials of degree k. Recall that a squarefree monomial

$$q_{A_1|\{1,2\}\cup B_1} \cdots q_{A_\alpha|\{1,2\}\cup B_\alpha} \; q_{\{1\}\cup A'_{\alpha+1}|\{2\}\cup B'_{\alpha+1}} \cdots q_{\{1\}\cup A'_k|\{2\}\cup B'_k} \in K[q]$$

is standard if and only if $q_{A_1|\{1,2\}\cup B_1} \dots q_{A_\alpha|\{1,2\}\cup B_\alpha}$ and $q_{\{1\}\cup A'_{\alpha+1}|\{2\}\cup B'_{\alpha+1}} \dots q_{\{1\}\cup A'_k|\{2\}\cup B'_k}$ are standard. From Proposition 6, we have $f_{k-1} = \sum_{\alpha=0}^{k} B_\alpha C_{k-\alpha}$, where

$$\begin{aligned} B_k &= (k-1)!\left\{ {n-2 \atop k-1} \right\} + 2k!\left\{ {n-2 \atop k} \right\} + (k+1)!\left\{ {n-2 \atop k+1} \right\}, \\ C_k &= 2k!\left\{ {n-2 \atop k} \right\} + (k+1)!\left\{ {n-2 \atop k+1} \right\}. \end{aligned}$$

Since $B_k = 0$ for any $k \geq n$, and $C_k = 0$ for any $k \geq n-1$,

$$f_\Delta(x) = \sum_{k=0}^{2n-3} \sum_{\alpha=0}^{k} B_\alpha C_{k-\alpha} x^k = \left(\sum_{k=0}^{n-1} B_k x^k \right) \left(\sum_{k=0}^{n-2} C_k x^k \right).$$

Let $X = x-1$. From [14, Remark 2.4, Theorem 3.4], $h_\Delta(x)$ is a palindromic polynomial of degree $2n-5$. Hence,

$$\begin{aligned} h_\Delta(x) = x^{2n-5} h_\Delta(x^{-1}) &= x^{2n-5} \sum_{k=0}^{2n-3} f_{k-1} x^{-k} (1-x^{-1})^{2n-3-k} = x^{-2} \sum_{k=0}^{2n-3} f_{k-1} X^{2n-3-k} &(1) \\ &= x^{-2} \left(\sum_{k=0}^{n-1} B_k X^{n-k-1} \right) \left(\sum_{k=0}^{n-2} C_k X^{n-k-2} \right). &(2) \end{aligned}$$

It then follows that

$$\begin{aligned} \sum_{k=0}^{n-1} B_k X^{n-k-1} &= \sum_{k=1}^{n} B_{k-1} X^{(n-1)-(k-1)} \\ &= \sum_{k=3}^{n} (k-2)! \left\{ {n-2 \atop k-2} \right\} X^{(n-2)-(k-2)} \\ &\qquad + 2X \sum_{k=2}^{n-1} (k-1)! \left\{ {n-2 \atop k-1} \right\} X^{(n-2)-(k-1)} + X^2 \sum_{k=1}^{n-2} k! \left\{ {n-2 \atop k} \right\} X^{(n-2)-k} \\ &= \sum_{k'=1}^{n-2} k'! \left\{ {n-2 \atop k'} \right\} X^{(n-2)-k'} \\ &\qquad + 2X \sum_{k''=1}^{n-2} k''! \left\{ {n-2 \atop k''} \right\} X^{(n-2)-k''} + X^2 \sum_{k=1}^{n-2} k! \left\{ {n-2 \atop k} \right\} X^{(n-2)-k} \\ &= A_{n-2}(x) + 2X A_{n-2}(x) + X^2 A_{n-2}(x) \\ &= x^2 A_{n-2}(x), \end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n-2} C_k X^{n-k-2} &= \sum_{k=0}^{n-2}\left(2k!\left\{ {n-2 \atop k} \right\} X^{n-k-2} + (k+1)!\left\{ {n-2 \atop k+1} \right\} X^{n-k-2}\right)\\
&= 2\sum_{k=1}^{n-2} k!\left\{ {n-2 \atop k} \right\} X^{(n-2)-k} + X\sum_{k=0}^{n-3}(k+1)!\left\{ {n-2 \atop k+1} \right\} X^{(n-3)-k}\\
&= 2A_{n-2}(x) + X\sum_{k'=1}^{n-2} k'!\left\{ {n-2 \atop k'} \right\} X^{(n-2)-k'}\\
&= 2A_{n-2}(x) + XA_{n-2}(x)\\
&= (1+x)A_{n-2}(x).
\end{aligned}$$

Therefore, $h^*(\mathcal{P}, x) = h_\Delta(x) = (x+1)(A_{n-2}(x))^2$. □

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