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On Certain Reciprocal Sums

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Abstract

We use tail sums of convergent series of positive real numbers to define a sequence of non-negative integers, and explicitly determine this sequence for classes of series defined by reciprocal sums of polynomials and rational functions. For this purpose we develop a new difference calculus method to approximate infinite series.

1 Introduction

Let $(x_m)_{m=1}^{\infty}$ be a sequence of positive real numbers satisfying $\sum_{m=1}^{\infty} x_m < \infty$. With $(x_m)_{m\geq 1}$ one associates a sequence of non-negative integers $(a_n)_{n=1}^{\infty}$ by defining

$$a_n = \left\lfloor \frac{1}{\sum_{m=n}^{\infty} x_m} \right\rfloor \quad (n \geq 1),$$

where $\lfloor . \rfloor$ is the floor function. We call $(a_n)_{n=1}^{\infty}$ the *reciprocal sequence* of $(x_m)_{m=1}^{\infty}$. The sequence $(a_n)_{n=1}^{\infty}$ is non-decreasing and divergent. Reciprocal sequences capture the rate of convergence of their defining series, and sometimes give rise to nice arithmetic and combinatorial structures. One can replace the floor function by the ceiling function or nearest integer function, and create variants of this notion.

Many authors have already studied reciprocal sequence of sequences defined by reciprocals of linear recurrence. Ohtsuka and Nakamura [4] derived a formula for reciprocal sequence of

$$x_m = \frac{1}{F_m} \quad (m \geq 1),$$

where F_m is the m^{th} Fibonacci number. Ramifications and generalizations of this result appear in several places, e.g., [3, 2]. Some papers have investigated the problem beyond reciprocals of linear recurrence relation. Reciprocal sequences of $(\frac{1}{m^4})_{m\geq 1}$ and $(\frac{1}{m^5})_{m\geq 1}$ are recorded in sequences A248230, A248234 respectively. Xin [6] and Xu [7] studied reciprocal sequences of

$$x_m = \frac{1}{m^k} \quad (m \geq 1)$$

for $k = 2, 3, 4, 5$. They showed that reciprocal sequences of these sequences are given by polynomials or polynomial like functions (for precise formulas see Section 4.3.2). In present article we extend results of Xin and Xu to general classes of sequences arising from reciprocals of polynomials and rational functions.

Let $\mathbb{K}$ be a subfield of $\mathbb{R}$. It is sufficient to consider $\mathbb{K} = \mathbb{R}, \mathbb{Q}$ for the purposes of this article. Suppose that $P(X) \in \mathbb{K}[X]$ is a polynomial of degree $k \geq 2$ with positive leading coefficient. Now $\mathbb{Q}$ is a subfield of $\mathbb{K}$ and $\mathbb{K}$ is dense in $\mathbb{R}$ with respect to Euclidean topology. Hence there exists $M_0 \in \mathbb{K}$ so that $P(x) > 0$ for all real $x \geq M_0 + 1$. For fixed choice of such M_0, define a sequence of positive real numbers by

$$x_m = \frac{1}{P(m + M_0)} \quad (m \geq 1). \tag{1}$$

Since $k \geq 2$ we have $\sum_{m\geq 1} x_m < \infty$. To calculate terms of reciprocal sequence of $(x_m)_{m\geq 1}$ one needs to estimate sums of the form

$$\sum_{m=n}^{\infty} \frac{1}{P(m + M_0)}. \tag{2}$$

The standard way to approximate sums of this form is to apply summation formulas from analysis (e.g., the Euler-Maclaurin summation formula [1, p. 806]). For reciprocal power sums, i.e., $P(X) = X^k$ and $M_0 = 0$, a precise estimate of (2) is readily available from the asymptotic expansion of the polygamma function [1, p. 260]. Though these summation formulas produce estimates up to higher order, they often lead to complicated computations and problems regarding convergence.

The goal of this paper is to present an improvised technique based on difference calculus, which bypasses analytic tools and provides a good upper bound as well as a lower bound for (2). The central idea behind our method is to find a suitable polynomial f so that the rational function $\frac{1}{P(X)}$ is *approximately* equal to $\frac{1}{f(X)} - \frac{1}{f(X+1)}$, i.e., $\frac{1}{f(X)}$ acts like the difference primitive of $-\frac{1}{P(X)}$. Once we have determined f, bounds on (2) simply follow by telescoping. This strategy bears resemblance to the methods employed by Xin [6] and Xu [7].

Though the technique looks naive and insufficient, it turns out to be powerful enough to calculate the first $k - 1$ terms in the asymptotic series of (2). Our main result is as follows:

Theorem 1. *Let $(x_m)_{m\geq 1}$ be the sequence defined by (1). There exists a polynomial $h(X) \in \mathbb{K}[X]$ of degree $k-1$ and a positive integer N_0 depending on h such that*

$$0 < h(n) \leq \frac{1}{\sum_{m=n}^{\infty} x_m} < h(n) + 1 \quad (n \geq N_0). \tag{3}$$

Moreover, h is algorithmically computable and uniquely determined by $P(X)$ and M_0 up to a constant term.

The leftmost inequality in (3) implies that leading coefficient of h is positive. Now (3) can be restated as

$$0 < \frac{1}{h(n)+1} < \sum_{m=n}^{\infty} \frac{1}{P(m+M_0)} \leq \frac{1}{h(n)} \quad (n \geq N_0). \tag{4}$$

Note that $\frac{1}{h(n)} - \frac{1}{h(n)+1} = \frac{1}{h(n)(h(n)+1)} = O(n^{-2k+2})$ and $\frac{1}{h(n)}, \frac{1}{h(n)+1}$ are of $O(n^{-k+1})$. Hence

$$\sum_{m=n}^{\infty} \frac{1}{P(m+M_0)} = \frac{1}{h(n)} + O(n^{-2k+2}) \quad (n \geq N_0).$$

Thus we obtain an estimate such that the main term is $O(n^{-k+1})$ and error term is $O(n^{-2k+2})$. This improvement is due to a formulation as a fractional expression that resembles the classical Padé approximants. Therefore, Theorem 1 turns out to be a more convenient approximation technique than the usual summation formulas, and one can directly obtain the first $k-1$ terms of the asymptotic expansion for suitably large n, as explained in Section 4.2.

We can use Theorem 1 to deduce estimates for classical reciprocal power sums.

Corollary 2. *Let k be an integer ≥ 2. There is a polynomial $h(X) \in \mathbb{Q}[X]$ of degree $k-1$ and a positive integer N_0 depending on h such that*

$$0 < h(n) \leq \frac{1}{\sum_{m=n}^{\infty} m^{-k}} < h(n) + 1 \quad (n \geq N_0).$$

Moreover, h is algorithmically computable and unique up to a constant term.

Theorem 1 allows us to study sequences defined by rational functions. Let $P(X), Q(X) \in \mathbb{K}[X]$ be two nonzero polynomials with positive leading coefficients such that $\deg_{\mathbb{K}} P - \deg_{\mathbb{K}} Q = k \geq 2$. Set $R(X) = \frac{P(X)}{Q(X)} \in \mathbb{K}(X)$. Here $R(X)$ determines k and it is independent of presentation. Since P and Q have positive leading coefficients, there is an $M_0 \in \mathbb{K}$ so that $P(x), Q(x) > 0$ for all real numbers $x \geq M_0 + 1$. For fixed choice of M_0, consider the sequence

$$x_m = \frac{1}{R(m+M_0)} \quad (m \geq 1). \tag{5}$$

As before $\sum_{m\geq 1} x_m < \infty$. We have

Theorem 3. *Let $(x_m)_{m\geq 1}$ be the sequence defined by (5). There exists a polynomial $h(X) \in \mathbb{K}[X]$ of degree $k-1$ and a positive integer N_0 depending on h such that*

$$0 < h(n) \leq \frac{1}{\sum_{m=n}^{\infty} x_m} < h(n) + 1 \quad (n \geq N_0). \tag{6}$$

Moreover, h is algorithmically computable and uniquely determined by $R(X)$ and M_0 up to a constant term.

Observe that if $Q(X) = 1$, then Theorem 3 reduces to Theorem 1. Though Theorem 3 is a generalization of Theorem 1, it follows easily from the earlier one. We prove both the theorems in Section 3. Theorem 3 also offers a convenient asymptotic expression, as described in paragraphs above. More discussion about this point is postponed to Section 4.2.

It is clear from Theorem 3 that the n-th term of the reciprocal sequence of (5) is either $\lfloor h(n) \rfloor$ or $\lfloor h(n) \rfloor + 1$. We pin down the exact expression of the reciprocal sequence if P and Q have coefficients in $\mathbb{Q}$, by carefully choosing constant term of h. This procedure and its arithmetic aspects are described in Section 4.1.

1.1 Notation and conventions

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ have their conventional meaning. In our convention $0 \notin \mathbb{N}$ and the set of nonnegative integers is $\mathbb{Z}_{\geq 0}$. We index sequences by $\mathbb{N}$.

Let $\mathbb{K}$ be a field. Then $\mathbb{K}^\times = \mathbb{K} - \{0\}$ and $\mathbb{K}[X]$ is the ring of polynomials with coefficients in $\mathbb{K}$. Also $\mathbb{K}(X)$ is the field of rational functions. If there is more than one variable, we use a boldface symbol to denote a tuple, e.g., $\mathbf{X} = (X_1, \ldots, X_n)$. For a fixed integer $d \geq 0$, there is a bijection between $\mathbb{K}^\times \times \mathbb{K}^d$ and $\mathbb{K}[X]_d$, subset polynomials of degree d, given by $\iota_d : (a_0, \ldots, a_d) \to a_0 X^d + \cdots + a_d$. Here, by convention, $\mathbb{K}^\times \times \mathbb{K}^0 = \mathbb{K}^\times$. The degree of the zero polynomial is $-\infty$. If P is a polynomial with coefficients in $\mathbb{R}$, then

$$\operatorname{sgn} P = \begin{cases} 0, & \text{if } P = 0; \\ 1, & \text{if the leading coefficient of } P \text{ is positive;} \\ -1, & \text{if the leading coefficient of } P \text{ is negative.} \end{cases}$$

We write $\lfloor . \rfloor$, $\lceil . \rceil$ for the floor and ceiling functions, respectively. A statement $S(n)$ concerning natural numbers holds for '$n \gg 1$' if there exists a real number C (depending on S) so that $S(n)$ is true for all $n \geq C$.

2 Approximate difference primitive

We begin by introducing some preliminary concepts necessary to define approximate difference primitives. The main result of this section is the existence of canonical approximants satisfying definite requirements. Our arguments are algebraic in nature, and most of the conclusions are valid even in a formal situation, which is briefly mentioned at the end of section.

Let $\mathbb{K}$ be a field of characteristic 0 and k be an integer ≥ 2. Suppose that $g(X) \in \mathbb{K}[X]_k$. Write

$$g(X) = a_0 X^k + \cdots + a_k. \tag{7}$$

Here $(a_0, \ldots, a_k)$ is the unique point in $\mathbb{K}^\times \times \mathbb{K}^k$ which corresponds to $g(X)$ under ι_k. Now $f(X) \in \mathbb{K}[X]$ be a nonzero polynomial such that $\frac{1}{f(X)}$ is a difference primitive of $-\frac{1}{g(X)}$, i.e., $\frac{1}{g(X)} = \frac{1}{f(X)} - \frac{1}{f(X+1)}$ holds in $\mathbb{K}(X)$. Then

$$f(X)f(X+1) = g(X)(f(X+1) - f(X)). \tag{8}$$

Suppose $\deg_{\mathbb{K}} f = d$. Since the left-hand side of (8) is not zero, we have $d \geq 1$. Comparing the degrees of both sides gives $d = k - 1$.

Let $x_0, x_1, \ldots, x_{k-1}$ be k unknowns. Set

$$F(X, \mathbf{x}) = x_0 X^{k-1} + \cdots + x_{k-1} \in \mathbb{Z}[X, x_0, \ldots, x_{k-1}].$$

With each $\mathbf{c} = (c_0, \ldots, c_{k-1}) \in \mathbb{K}^\times \times \mathbb{K}^{k-1}$ one associates $F(X, \mathbf{c}) \in \mathbb{K}[X]_{k-1}$. Using (8) we see that $\frac{1}{F(X,\mathbf{c})}$ is a difference primitive of $-\frac{1}{g(X)}$ if and only if

$$F(X+1, \mathbf{c})F(X, \mathbf{c}) = g(X)(F(X+1, \mathbf{c}) - F(X, \mathbf{c})).$$

To solve the equation above, one defines a collection of polynomials

$$\big\{y_i(\mathbf{x}), u_j(\mathbf{x}), v_l(\mathbf{x}, g) \mid 1 \leq i \leq k-1, 0 \leq j, l \leq 2k-2\big\} \subseteq \mathbb{K}[x_0, \ldots, x_{k-1}]$$

by the relations

$$\begin{aligned} F(X+1, \mathbf{x}) &= x_0 X^{k-1} + (x_1 + y_1(\mathbf{x}))X^{k-2} + \cdots \\ &\quad \cdots + (x_{k-2} + y_{k-2}(\mathbf{x}))X + (x_{k-1} + y_{k-1}(\mathbf{x})), \qquad (9) \\ H(X, \mathbf{x}) &= F(X+1, \mathbf{x})F(X, \mathbf{x}) \\ &= u_0(\mathbf{x})X^{2k-2} + \cdots + u_{2k-3}(\mathbf{x})X + u_{2k-2}(\mathbf{x}), \qquad (10) \\ G(X, \mathbf{x}, g) &= g(X)\big(F(X+1, \mathbf{x}) - F(X, \mathbf{x})\big) \\ &= v_0(\mathbf{x}, g)X^{2k-2} + \cdots + v_{2k-3}(\mathbf{x}, g)X + v_{2k-2}(\mathbf{x}, g). \qquad (11) \end{aligned}$$

The coefficient of X^{k-1} in $F(X+1, \mathbf{x})$ is x_0 and the degrees of $H(X, \mathbf{x}), G(X, \mathbf{x}, g)$ in X are at most $2k-2$. Therefore, the equations above are justified. Note that the polynomials $\{y_i(\mathbf{x}), u_j(\mathbf{x}) \mid 0 \leq i \leq k-1, 0 \leq j \leq 2k-2\}$ are independent of g and defined over $\mathbb{Z}[x_0, \ldots, x_{k-1}]$.

As a consequence of the binomial theorem

$$y_i(\mathbf{x}) = \binom{k-i}{1} x_{i-1} + \binom{k-i+1}{2} x_{i-2} + \cdots + \binom{k-1}{i} x_0 \tag{12}$$

for all $1 \leq i \leq k-1$. For convenience put $y_0(\mathbf{x}) = y_k(\mathbf{x}) = 0$.

Equation (9), (10), and (11) together imply that

$$u_j(\mathbf{x}) = \sum_{r=0}^{j} x_r(x_{j-r} + y_{j-r}(\mathbf{x})) \quad (0 \leq j \leq k-1), \tag{13}$$

$$v_l(\mathbf{x}, g) = \sum_{r=0}^{l} a_r y_{l-r+1}(\mathbf{x}) \quad (0 \leq l \leq k-1). \tag{14}$$

Subtracting (10) from (11) gives

$$G(X, \mathbf{x}, g) - H(X, \mathbf{x}) = \sum_{i=0}^{2k-2} (v_i(\mathbf{x}, g) - u_i(\mathbf{x}))X^{2k-2-i}. \tag{15}$$

To construct an approximate difference primitive we need to find a point $(c_0, \ldots, c_{k-1}) \in \mathbb{K}^k$ so that the first few coefficients in (15) vanish at the point $(c_0, \ldots, c_{k-1})$. But for generic $(a_0, \ldots, a_k)$, one cannot expect to find a common zero of all the coefficients. Moreover, we would like to have $c_0 \neq 0$, so that $F(X, c_0, \ldots, c_{k-1})$ is actually of degree $k-1$. These considerations lead to a formal definition.

2.1 Good approximants

Let $f(X) \in \mathbb{K}[X]$ be a nonzero polynomial. Set

$$\delta(f, g, X) := \frac{1}{f(X)} - \frac{1}{f(X+1)} - \frac{1}{g(X)}, \tag{16}$$

$$N(f, g, X) := g(X)\big(f(X+1) - f(X)\big) - f(X+1)f(X). \tag{17}$$

A *good approximant* of g is a polynomial $f \in \mathbb{K}[X]_{k-1}$ so that $N(f, g, X)$ is a polynomial of degree $\leq k-1$.

Note that

$$\delta(f, g, X) = \frac{N(f, g, X)}{f(X)f(X+1)g(X)} \tag{18}$$

and $N(f, g, X) = 0$ if and only if $\frac{1}{f(X)}$ is a difference primitive of $-\frac{1}{g(X)}$.

The following lemma ensures existence of a canonical good approximant:

Lemma 4. *Let $k \geq 2$ and $g \in \mathbb{K}[X]_k$. Consider the system of k equations*

$$u_i(\boldsymbol{x}) = v_i(\boldsymbol{x}, g) \quad (0 \leq i \leq k-1) \tag{19}$$

in k unknowns $x_0, \ldots, x_{k-1}$. The system of equations (19) has a unique solution in $\mathbb{K}^\times \times \mathbb{K}^{k-1}$, i.e., there is a unique tuple $(c_0(g), \ldots, c_{k-1}(g)) \in \mathbb{K}^k$ with $c_0(g) \neq 0$, which is a solution to the system of equation (19).

Proof. To fix notation, assume that g is given in the form (7). This polynomial remains fixed throughout the discussion, and we omit it from the notation. Now u_i depends on $\{x_0, \ldots, x_i, y_0, \ldots, y_i\}$. Using (12), one concludes that the set of variables appearing in u_i is $\{x_0, \ldots, x_i\}$. Similarly, v_i depends on $\{y_1, \ldots, y_{i+1}\}$, i.e., on $\{x_0, \ldots, x_i\}$. These statements hold for all $0 \leq i \leq k-1$. Therefore one can use a recursive approach to solve the system of equations.

Let $0 \leq i \leq k-1$ and consider subsystems of $i+1$ equations

$$u_0 = v_0, \ldots, u_i = v_i \tag{20}$$

in $i+1$ variables $x_0, \ldots, x_i$. We would like to show that (20) has unique solution in $\mathbb{K}^\times \times \mathbb{K}^i$ for each $0 \leq i \leq k-1$.

For the base case, consider the equation $u_0(x_0) = v_0(x_0)$. We have $u_0 = x_0^2$, and $v_0 = y_1 = a_0\binom{k-1}{1}x_0 = a_0(k-1)x_0$. Now $a_0(k-1) \neq 0$ and it is the only nonzero solution to $u_0 = v_0$. Put $c_0 = a_0(k-1)$.

Assume that the statement holds for some $0 \leq i \leq k-2$, i.e., there is a unique solution to (20) in $\mathbb{K}^\times \times \mathbb{K}^i$. Note that the first coordinate of this solution is necessarily c_0. Let the unique solution be $(c_0, \ldots, c_i)$. We now construct $c_{i+1} \in \mathbb{K}$ so that $(c_0, \ldots, c_i, c_{i+1})$ is the unique solution of $u_{i+1} = v_{i+1}$ in $\mathbb{K}^\times \times \mathbb{K}^{i+1}$. In what follows, we consider u_{i+1} and v_{i+1} as polynomials of x_{i+1} with coefficients in $\mathbb{K}[x_0, \ldots, x_i]$. Equation (13) and (14) together imply that u_{i+1}, v_{i+1} are linear in the variable x_{i+1}.

There are two possibilities.

2.1.1 Case I: $i < k-2$

Here $i+1 \leq k-2$. From (13) and (14) we deduce that coefficient of x_{i+1} in u_{i+1} is $2x_0$ (one x_0 arises from term $x_0(x_{i+1}+y_{i+1})$ and other x_0 arises from the term $x_{i+1}(x_0+y_0)$). Similarly coefficient of x_{i+1} in v_{i+1} is a_0 times coefficient of x_{i+1} in y_{i+2}, i.e., $a_0\binom{k-i-2}{1} = a_0(k-i-2)$.

Hence $u_{i+1} = v_{i+1}$ can be rewritten as

$$\big(2x_0 - a_0(k-i-2)\big)x_{i+1} = \text{a polynomial in } x_0, \ldots, x_i \text{ over } \mathbb{K}. \tag{21}$$

But $\big(2c_0 - a_0(k-i-2)\big) = a_0(k+i) \neq 0$. Therefore we can substitute $x_0 = c_0, \ldots, x_i = c_i$ in (21) and solve for x_{i+1} to get a tuple $(c_0, \ldots, c_{i+1}) \in \mathbb{K}^{i+2}$ that is a solution to the system of equations

$$u_0 = v_0, \ldots, u_{i+1} = v_{i+1}.$$

If $(C_0, \ldots, C_{i+1})$ is another solution with $C_0 \neq 0$ then by recursion hypothesis $(c_0, \ldots, c_i) = (C_0, \ldots, C_i)$. Using (21) we have $C_{i+1} = c_{i+1}$. Hence the uniqueness. So for $i < k-2$ a solution to the first $i+1$ equations of (19) in $\mathbb{K}^\times \times \mathbb{K}^i$ extends to a unique solution to the first $i+2$ equations in $\mathbb{K}^\times \times \mathbb{K}^{i+1}$.

2.1.2 Case II: $i = k-2$

This case is essentially similar to Case I. Here coefficient of x_{k-1} in v_{k-1} is 0. Now $u_{k-1} = v_{k-1}$ can be rewritten as

$$2x_0x_{k-1} = \text{a polynomial in } x_0, \ldots, x_{k-2} \text{ over } \mathbb{K}.$$

Since $c_0 \neq 0$, the arguments of the previous case go through to yield a unique tuple $(c_0, \ldots, c_{k-1})$ that is a solution to (19).

In this way we can recursively construct $\big(c_0(g), \ldots, c_{k-1}(g)\big) \in \mathbb{K}^k$ so that $\big(c_0(g), \ldots, c_{k-1}(g)\big)$ is the unique solution to (19) in $\mathbb{K}^\times \times \mathbb{K}^{k-1}$. Hence the lemma is proved. □

Lemma 4 constructs a point $\mathbf{c}(g) = \big(c_0(g), \ldots, c_{k-1}(g)\big) \in \mathbb{K}^k$ such $c_0(g) \neq 0$ and $G\big(X, \mathbf{c}(g), g\big) - H\big(X, \mathbf{c}(g)\big)$ is of degree $\leq k-2$. Therefore $F\big(X, \mathbf{c}(g)\big)$ is a good approaximant of g. The condition that the first k coefficients of $G\big(X, \mathbf{c}(g), g\big) - H\big(X, \mathbf{c}(g)\big)$ vanish is better than expected, but control on one extra term turns out to be useful. For simplicity we frequently omit g from the notation for the solution, if it is understood from the context.

2.2 Consequences of Lemma 4

The technique used to prove Lemma 4 has several corollaries that are indispensable for later developments.

Corollary 5.

(i) Let $0 \leq i_0 \leq k-1$. Consider the subsystem of equations

$$u_i(\boldsymbol{x}) = v_i(\boldsymbol{x}, g) \quad (0 \leq i \leq i_0).$$

Observe that variables appearing in these equations are $x_0, \ldots, x_{i_0}$. This system has a unique solution in $\mathbb{K}^\times \times \mathbb{K}^{i_0}$ given by the first $i_0 + 1$ coordinates of $\boldsymbol{c}$, i.e., $(c_0, \ldots, c_{i_0})$.

(ii) Let $g_1, g_2 \in \mathbb{K}[X]_k$ with $g_1 - g_2 \in \mathbb{K}$. Then $\boldsymbol{c}(g_1) = \boldsymbol{c}(g_2)$.

Proof.

(i) Follows from the recursion argument in the proof of Lemma 4.

(ii) A consequence of the fact that for any $g \in \mathbb{K}[X]_k$, the polynomials

$$\{v_i(\mathbf{X}, g) \mid 0 \leq i \leq k-1\}$$

do not depend on the constant term of g.

□

The next corollary characterizes all good approximants and provides a necessary and sufficient condition for existence of difference primitives.

Corollary 6.

(i) For each $c \in \mathbb{K}$ the polynomial $F(X, c_0, \ldots, c_{k-2}, c)$ is a good approximant of g and every good approximant of g is in this form.

(ii) There exists a nonzero polynomial $f(X) \in \mathbb{K}[X]$ so that $\frac{1}{f(X)}$ is a difference primitive of $-\frac{1}{g(X)}$ if and only if the tuple $\boldsymbol{c} = (c_0, \ldots, c_{k-1})$ constructed in Lemma 4 satisfies

$$u_i(\boldsymbol{c}) = v_i(\boldsymbol{c}, g) \quad (k \le i \le 2k-2).$$

If this condition holds then $f(X)$ is uniquely determined and equals $F(X, \boldsymbol{c})$.

Proof.

(i) By definition, all good approximants of g have degree $k-1$. Let $\mathbf{C} = (C_0, \ldots, C_{k-1}) \in \mathbb{K}^\times \times \mathbb{K}^{k-1}$. Using (15) one sees that $F(X, \mathbf{C})$ is good approximant if and only if $u_i(\mathbf{C}) = v_i(\mathbf{C}, g)$ for each $0 \le i \le k-2$. Now the result follows from uniqueness part of Corollary 5.

(ii) We have already seen that f has to be of degree $k-1$. From (15) it follows that such f exists if and only if the system of $2k-1$ equations

$$u_i(\mathbf{x}) = v_i(\mathbf{x}, g) \quad (0 \le i \le 2k-2)$$

has a solution in $\mathbb{K}^\times \times \mathbb{K}^{k-1}$. Lemma 4 implies that if such solution exists it is unique and given by the tuple $\mathbf{c} = (c_0, \ldots, c_{k-1}) \in \mathbb{K}^\times \times \mathbb{K}^{k-1}$ constructed in lemma. Thus both parts of assertion are proved.

□

The following corollary investigates effect of scaling g.

Corollary 7. *Let $g_1(X) \in \mathbb{K}[X]$ be a nonzero scalar multiple of $g(X)$, i.e., $g_1(X) = \alpha g(X)$ for some $\alpha \in \mathbb{K}^\times$. Then $\big(\alpha c_0(g), \ldots, \alpha c_{k-1}(g)\big) \in \mathbb{K}^\times \times \mathbb{K}^{k-1}$ is the unique solution to system of equations*

$$u_i(\boldsymbol{x}) = v_i(\boldsymbol{x}, g_1) \quad (0 \le i \le k-1).$$

Proof. One writes coefficients as functions of polynomials. By assumption $a_r(g_1) = \alpha a_r(g)$ for all $0 \le r \le k-1$. From explicit expressions (12), (13) and (14) it follows that u_j is quadratic polynomial of $\{x_0, \ldots, x_{k-1}\}$ while v_l is linear in both $\{x_0, \ldots, x_{k-1}\}$ and $\{a_0, \ldots, a_{k-1}\}$. Therefore the system of equations

$$u_i(\mathbf{x}) = v_i(\mathbf{x}, g) \quad (0 \le i \le k-1)$$

is invariant under transformation $x_i \to \alpha x_i$, $0 \le i \le k-1$, and $a_r \to \alpha a_r$, $0 \le r \le k-1$. Hence $\big(\alpha c_0(g), \ldots, \alpha c_{k-1}(g)\big)$ is a solution to the system corresponding to g_1. But $\alpha c_0(g) \ne 0$. The result follows by uniqueness. □

Let $c \in \mathbb{K}$. Define $f_g(c, X) \in \mathbb{K}[X]$ by

$$f_g(c, X) := c_0(g)X^{k-1} + \ldots + c_{k-2}(g)X + c \tag{22}$$

By Corollary 6 $f_g(c)$ is a good approximant of g and all good approximants are in this form. The leading term of $N\big(f_g(c), g, X\big)$ is X^{k-1} and its coefficient is

$$\big(v_{k-1}(c_0, \ldots, c_{k-2}, c; g) - u_{k-1}(c_0, \ldots, c_{k-1}, c)\big).$$

By (14) x_{k-1} does not appear in $v_{k-1}(\mathbf{x})$ and from the proof of Lemma 4, we know that the term involving x_{k-1} in $u_{k-1}(\mathbf{x})$ is $2x_{k-1}x_0$. Now

$$\begin{aligned} &v_{k-1}(c_0, \ldots, c_{k-2}, c; g) - u_{k-1}(c_0, \ldots, c_{k-1}, c) \\ &= v_{k-1}(c_0, \ldots, c_{k-2}, c_{k-1}; g) - u_{k-1}(c_0, \ldots, c_{k-1}, c) \\ &= u_{k-1}(c_0, \ldots, c_{k-2}, c_{k-1}; g) - u_{k-1}(c_0, \ldots, c_{k-1}, c) \\ &= 2c_0(c_{k-1} - c). \end{aligned} \tag{23}$$

Here in third step one uses Lemma 4. Therefore coefficient of X^{k-1} in $N(f_g(c), g, X)$ is $2c_0\big(c_{k-1} - c\big)$.

2.3 Formal algebraic version

We conclude the section with a formal version of Lemma 4. Let $k \geq 2$ and $a_0, \ldots, a_k$ be formal variables. Suppose that $g(X, \mathbf{a})$ is an element of $\mathbb{Z}[a_0, \ldots, a_k]$ given by

$$g(X, \mathbf{a}) = a_0X^k + \cdots + a_k. \tag{24}$$

Define auxiliary polynomials

$$\big\{y_i(\mathbf{x}), u_j(\mathbf{x}), v_l(\mathbf{x}, \mathbf{a}) \mid 0 \leq i \leq k-1, 0 \leq j, l \leq 2k-2\big\} \subseteq \mathbb{Z}[\mathbf{x}, \mathbf{a}]$$

using (9), (10) and (11). It is easy to see that these polynomials satisfy (12), (13) and (14). Consider the system of equations

$$u_i(\mathbf{x}) = v_i(\mathbf{x}, \mathbf{a}) \quad (0 \leq i \leq k-1) \tag{25}$$

in variables $x_0, \ldots, x_{k-1}$.

Lemma 8. *Let $\mathbb{F} = \mathbb{Q}(a_0, \ldots, a_k)$, the field of rational functions in variables $a_0, \ldots, a_k$ with coefficients in $\mathbb{Q}$. Then there is a unique tuple of rational functions $(c_0(\boldsymbol{a}), \ldots, c_{k-1}(\boldsymbol{a})) \in \mathbb{F}^k$ with $c_0(\boldsymbol{a}) \neq 0$, which is a solution to the system of equations (25). Further, $c_i(\boldsymbol{a}) \in \mathbb{Q}[a_0, \ldots, a_i][a_0^{-1}]$ for all $0 \leq i \leq k-1$.*

Proof. Similar to the proof of Lemma 4. The base case holds since $c_0 = a_0(k-1) \neq 0$. Let $0 \leq i \leq k-2$. The recursion step goes through, since u_{i+1} is independent of $\{a_0, \ldots, a_k\}$ and v_{i+1} depends only on $\{a_0, \ldots, a_{i+1}\}$ and to determine c_{i+1} one needs to divide by an element of $\mathbb{Q}^\times a_0$. Recursively, one deduces

$$c_i(\mathbf{a}) \in \mathbb{Q}[a_0, \ldots, a_i][a_0^{-1}] \quad (0 \leq i \leq k-1).$$

□

In the formal version of the theory, it is enough to consider only one polynomial, namely, the universal polynomial $g(X, \mathbf{a})$. Statements analogous to Corollary 5(i) and Corollary 7 hold in this situation, i.e., one can restrict to suitable subsystems and scaling of variables results into scaling of solution. The notion of good approximant with coefficients in $\mathbb{F}$ can be introduced in exactly same manner. Lemma 8 constructs a canonical good approximant for $g(X, \mathbf{a})$ and classification of Corollary 6 continues to hold.

3 Proofs of the theorems

In this section we use the theory developed in Section 2 to prove Theorem 1. An appropriate application of the same ideas yields Theorem 3. Corollary 2 is an easy consequence of Theorem 1.

We initiate the discussion with a useful remark. In what follows, $\mathbb{K}$ is always a subfield of $\mathbb{R}$ unless otherwise specified.

Remark 9.

(i) Let $\phi(X) = a_0X^d + a_1X^{d-1} + \cdots + a_d \in \mathbb{K}[X]$ is a polynomial with $\operatorname{sgn}\phi = 1$. Then $\phi(x) > 0$ for all real x satisfying

$$x \geq \max\{1, \frac{(d+1)|a_j|}{|a_0|} \mid 1 \leq j \leq d\}. \tag{26}$$

If $d = 0$ then the right-hand side is interpreted as 1. Note that expression on the right-hand side is an element of $\mathbb{K}$.

(ii) The lower bound appearing in the first part is not best possible. To determine the best possible bound, we need to locate the real zeroes of ϕ.

(iii) Using (i) one can effectively determine the constant M_0 appearing in the statement of Theorems 1 and 3.

3.1 Proof of Theorem 1

Let $P(X) \in \mathbb{K}[X]$ be of degree $k \geq 2$ with $\operatorname{sgn} P = 1$. Suppose that M_0 is an element of $\mathbb{K}$ with property that $P(x) > 0$ for all real $x \geq M_0 + 1$.

With the notation of Section 2 we use Lemma 4 for $g(X) = P(X)$. Note that a_0, the leading coefficient of $P(X)$, is positive. Let $(c_0, \dots, c_{k-1}) \in \mathbb{K}^\times \times \mathbb{K}^{k-1}$ be the unique solution to the system of equations (19) corresponding to P.

Now $(c_{k-1} - 1, c_{k-1}) \cap \mathbb{K}$ is nonempty since $\mathbb{K}$ is dense in $\mathbb{R}$. Let c be an element of $(c_{k-1} - 1, c_{k-1}) \cap \mathbb{K}$. Define $f_P(c, X) \in \mathbb{K}[X]$ by (22). Since $c_0 = a_0(k-1) > 0$ there exists a $M_{f_P(c)} \in \mathbb{R}$ so that $f_P(c, x) > 0$ for all real $x \geq M_{f_P(c)}$. We know that $f_P(c, X)$ is a good approximant of $P(X)$, and the coefficient of X^{k-1} in $N\big(f_P(c), P, X\big)$ is $2c_0(c_{k-1} - c)$. But $2c_0(c_{k-1} - c) > 0$. Hence there is a $M_1 \in \mathbb{R}$ such that $N(f_P(c), P, x) > 0$ for all real $x \geq M_1$.

Let $M_1' = \max\{M_1, M_{f_P(c)}\}$. Suppose that m is a positive integer $\geq M_1' - M_0$. Then by (16) and (17)

$$\delta\big(f_P(c), P, m + M_0\big) > 0$$

i.e.,

$$\frac{1}{f_P(c, m + M_0)} - \frac{1}{f_P(c, m + 1 + M_0)} > \frac{1}{P(m + M_0)}.$$

Using telescoping summation we have

$$\frac{1}{f_P(c, n + M_0)} > \sum_{m=n}^{\infty} \frac{1}{P(m + M_0)} \tag{27}$$

for all positive integers $n \geq M_1' - M_0$.

Now let $C = c + 1 \in \mathbb{K}$, and consider $f_P(C, X) \in \mathbb{K}[X]$. It is a good approximant of P and coefficient of X^{k-1} in $N\big(f_P(C), P, X\big)$ is $2c_0(c_{k-1} - C)$. But $c_{k-1} - c - 1 < 0$. Therefore $2c_0(c_{k-1} - C) < 0$ and there exists a $M_2 \in \mathbb{R}$ so that $N(f_P(C), P, x) < 0$ for all real $x \geq M_2$.

Let $M_2' = \max\{M_2, M_{f_P(c)}\}$ and m be a positive integer $\geq M_2' - M_0$. It follows that

$$\delta\big(f_P(C), P, m + M_0\big) < 0$$

i.e.,

$$\frac{1}{f_P(c, m + M_0) + 1} - \frac{1}{f_P(c, m + 1 + M_0) + 1} < \frac{1}{P(m + M_0)}.$$

By telescoping we have

$$\frac{1}{f_P(c, n + M_0) + 1} < \sum_{m=n}^{\infty} \frac{1}{P(m + M_0)} \tag{28}$$

for all positive integers $n \geq M_2' - M_0$.

Suppose that $M_3 = \max\{M_1', M_2'\}$. Using (27) and (28)

$$0 < \frac{1}{f_P(c, n + M_0) + 1} < \sum_{m=n}^{\infty} \frac{1}{P(m + M_0)} < \frac{1}{f_P(c, n + M_0)}$$

for all positive integers $n \geq M_3 - M_0$. The leftmost inequality is a consequence of $M_3 \geq M_{f_P(c)}$.

Let $h(X) = f_P(c, X + M_0)$. Note that it has degree $k-1$. Since $M_0 \in \mathbb{K}$, the polynomial $h(X) \in \mathbb{K}[X]$. Put $N_0 = \max\{\lceil M_3 - M_0 \rceil, 1\}$. Then

$$0 < \frac{1}{h(n)+1} < \sum_{m=n}^{\infty} \frac{1}{P(m+M_0)} < \frac{1}{h(n)}$$

for all integers $n \geq N_0$. It is clear that h and N_0 so defined have properties required by Theorem 1. This construction proves the first part of Theorem 1.

Further Lemma 4 algorithmically determines the tuple $(c_0, \ldots, c_{k-1})$ and c is any element of $(c_{k-1}-1, c_{k-1}) \cap \mathbb{K}$. Therefore we can determine $f_P(c)$ algorithmically. Since M_0 is part of hypothesis the polynomial h is also algorithmically computable. To finish off proof we need to show uniqueness. This part of assertion is a consequence of Lemma 10. Thus the proof of Theorem 1 is complete, modulo Lemma 10. □

Corollary 2 is a special case of Theorem 1.

3.1.1 Proof of Corollary 2

Follows from Theorem 1 with $\mathbb{K} = \mathbb{Q}$, $P(X) = X^k \in \mathbb{Q}[X]$, and $M_0 = 0$. □

3.2 Proof of Theorem 3

In this subsection we prove Theorem 3. Main idea behind the proof is to approximate the rational function by appropriate polynomial.

Let $P(X), Q(X)$ be two nonzero polynomials in $\mathbb{K}[X]$ so that $\deg_{\mathbb{K}} P - \deg_{\mathbb{K}} Q = k \geq 2$ and $\operatorname{sgn} P = \operatorname{sgn} Q = 1$. Suppose that $R(X) = \frac{P(X)}{Q(X)}$ and $M_0 \in \mathbb{K}$ with $P(x), Q(x) > 0$ for all real $x \geq M_0 + 1$.

Using the division algorithm, we construct polynomials $A(X), B(X) \in \mathbb{K}[X]$ such that

$$P(X) = A(X)Q(X) + B(X)$$

and $\deg_{\mathbb{K}} B < \deg_{\mathbb{K}} Q$. It is easy to see that $A(X)$ is of degree k and $\operatorname{sgn} A = 1$. Now

$$R(X) = A(X) + \frac{B(X)}{Q(X)}. \tag{29}$$

Note that $A(X)$ is determined by $R(X)$ and does not depend on individual polynomials $P(X)$ and $Q(X)$. It is the unique polynomial so that the difference $R(X) - A(X)$ is either 0 or is given by a rational function whose denominator has degree strictly larger than degree of numerator. Since $\deg_{\mathbb{K}} B < \deg_{\mathbb{K}} Q$, $\frac{B(X)}{Q(X)} \to 0$ as $x \to \infty$ on real line. If $B = 0$ then $R(X) = A(X)$ and the result is already true by Theorem 1.

Let $\epsilon \in \mathbb{K} \cap (\operatorname{sgn} B)\,\mathbb{R}_{>0}$. If $B = 0$ then $\epsilon = 0$. Note that $Q(x) > 0$ for $x \geq M_0 + 1$. Hence there is a real number $M_\epsilon \geq M_0 + 1 > 0$ such that for all real $x \geq M_\epsilon$ we have $(\operatorname{sgn} B)B(x) \geq 0$ and $0 \leq |\frac{B(x)}{Q(x)}| \leq |\epsilon|$.

Define $A_\epsilon(X) = A(X) + \epsilon \in \mathbb{K}[X]$. It is easy to see that $A_\epsilon(X)$ is a polynomial of degree k with $\operatorname{sgn} A_\epsilon = 1$. Moreover, if $\operatorname{sgn} B \geq 0$ then

$$A(x) \leq R(x) \leq A_\epsilon(x) \quad (x \geq M_\epsilon) \tag{30}$$

and if $\operatorname{sgn} B = -1$ then

$$A_\epsilon(x) \leq R(x) \leq A(x) \quad (x \geq M_\epsilon). \tag{31}$$

Consider the system of equation (19) in Lemma 4 for the polynomials $A(X)$ and $A_\epsilon(X)$. By Corollary 5(ii) the same tuple $(c_0, \ldots, c_{k-1}) \in \mathbb{K}^\times \times \mathbb{K}^{k-1}$ is a solution to (19) for both $A(X)$ and $A_\epsilon(X)$. Since $\operatorname{sgn} A = 1$ we have $c_0 > 0$. Let $c \in (c_{k-1} - 1, c_{k-1}) \cap \mathbb{K}$ and consider $f_A(c, X) \in \mathbb{K}[X]$ defined in Section 2. It is a good approximant of each of $A(X)$ and $A_\epsilon(X)$. The coefficient of X^{k-1} in both $N\big(f_A(c), A, X\big)$ and $N\big(f_A(c), A_\epsilon, X\big)$ is $2c_0(c_{k-1} - c) > 0$. Similarly, the coefficient of X^{k-1} in each of $N(f_A(c) + 1, A, X)$ and $N(f_A(c) + 1, A_\epsilon, X)$ is $2c_0(c_{k-1} - c - 1) < 0$.

Let $M' \in \mathbb{R}$ be such that $A(x), A_\epsilon(x) > 0$ for all real $x \geq M'$. Imitating the proof of Theorem 1, we can find M_3', $M_{3,\epsilon}' \geq M'$ so that

$$0 < \frac{1}{f_A(c, n + M_0) + 1} < \sum_{m=n}^{\infty} \frac{1}{A(m + M_0)} < \frac{1}{f_A(c, n + M_0)},$$
$$0 < \frac{1}{f_A(c, n + M_0) + 1} < \sum_{m=n}^{\infty} \frac{1}{A_\epsilon(m + M_0)} < \frac{1}{f_A(c, n + M_0)}$$

holds for all positive integers $n \geq M_3' - M_0$ and $n \geq M_{3,\epsilon}' - M_0$ respectively.

Let $M_4 = \max\{M_3', M_{3,\epsilon}', M_\epsilon\}$. From inequalities (30), (31) and choice of $M_3', M_{3,\epsilon}'$ it follows that

$$0 < \frac{1}{f_A(c, n + M_0) + 1} < \sum_{m=n}^{\infty} \frac{1}{R(m + M_0)} < \frac{1}{f_A(c, n + M_0)}$$

for all positive integers $n \geq M_4 - M_0$. Set $h(X) = f_A(c, X + M_0)$ and $N_0 = \max\{\lceil M_4 - M_0 \rceil, 1\}$. One easily sees that $h(X) \in \mathbb{K}[X]_{k-1}$ is a polynomial that satisfies the requirements of the statement of the theorem for the prescribed choice of N_0. Note that A, B are algorithmically computable. Therefore, by reasoning similar to that in the proof of Theorem 1, h is also algorithmically computable.

Proof of uniqueness is postponed to Lemma 10. □

3.2.1 Effectiveness of constants

Using Remark 9 we can determine effective choices for the constants $M_{f_P(c)}, M_1, M_2$ appearing in the proof of Theorem 1. Hence M_1', M_2', M_3, and in particular, N_0 are effective. These constants depend on choice of c. In the proof of Theorem 3, ϵ is any number in $\mathbb{K}\cap(\operatorname{sgn} B)\mathbb{R}_{>0}$ and the inequalities $(\operatorname{sgn} B)B(x) \geq 0$, $0 \leq |\frac{B(x)}{Q(x)}| \leq |\epsilon|$ effectively determine M_ϵ. Now one can use Remark 9 to show that the constants appearing in the proof of Theorem 3 are also effective. These numbers depend on choices of P, Q, c, and ϵ.

3.3 Admissible polynomials

This subsection studies all polynomials which approximate reciprocals of tail sums. First, we prove a lemma which implies uniqueness part of Theorem 1 and 3. Recall that to retrieve Theorem 1 from Theorem 3 one needs to substitute $Q = 1$. The notation is same as in the statement of Theorem 3.

Lemma 10. *Suppose that $h_1(X), h_2(X) \in \mathbb{K}[X]$ are two nonzero polynomials which satisfy*

$$0 < h_j(n) \leq \frac{1}{\sum_{m=n}^{\infty} \frac{1}{R(m+M_0)}} < h_j(n) + 1 \quad (j = 1, 2)$$

for infinitely many $n \in \mathbb{N}$. Then

(i) $\deg_{\mathbb{K}} h_1 = \deg_{\mathbb{K}} h_2 = k - 1$,

(ii) $h_1(X) - h_2(X) \in \mathbb{K}$,

(iii) $|h_1 - h_2| < 2$,

(iv) if there is an infinite subset of $\mathbb{N}$ on which hypothesis of the lemma hold simultaneously for h_1 and h_2, then $|h_1 - h_2| < 1$.

Proof. In what follows the statements hold for both $j = 1, 2$. By assumption

$$0 < \frac{1}{h_j(n) + 1} < \sum_{m=n}^{\infty} \frac{1}{R(m + M_0)} \leq \frac{1}{h_j(n)} \tag{32}$$

for infinitely many $n \in \mathbb{N}$. These inequalities imply that h_j is non-constant and $\operatorname{sgn} h_j = 1$.

Let $h(X)$ be the polynomial constructed in the proof of Theorem 3. Comparing with (32)

$$0 < \frac{1}{h(n) + 1} < \frac{1}{h_j(n)},$$
$$0 < \frac{1}{h_j(n) + 1} < \frac{1}{h(n)}$$

for infinitely many $n \in \mathbb{N}$. Hence on an infinite subset of $\mathbb{N}$

$$h(n) - 1 < h_j(n) < h(n) + 1. \tag{33}$$

It follows that $|h(n) - h_j(n)| < 1$ on an infinite subset. But this inequality forces the difference to be constant.

The conclusion above proves both (i) and (ii). The proof of (iii) follows from (33).

Without loss of generality assume $h_1 - h_2 = C \geq 0$. If $C \geq 1$ then

$$0 < \frac{1}{h_1(n)} \leq \frac{1}{h_2(n)+1} \quad (n \gg 1).$$

This contradicts the hypothesis of (iv). Hence (iv) is proved by way of contradiction. □

3.3.1 Admissible polynomials

We introduce a terminology for convenience. A polynomial $h(X) \in \mathbb{K}[X]$ is *admissible* for (R, M_0) if

$$0 < h(n) \leq \frac{1}{\sum_{m=n}^{\infty} \frac{1}{R(m+M_0)}} < h(n) + 1 \quad (n \gg 1).$$

In rest of the section let $(c_0, \ldots, c_{k-1}) \in \mathbb{K}^{\times} \times \mathbb{K}^{k-1}$ denote the unique solution to system of equations (19) associated with $A(X)$. The polynomial A depends only on R and is determined by (29). Since $\operatorname{sgn} A = 1$ it follows that $c_0 > 0$. Moreover, the tuple does not depend on the constant term of A (Corollary 5).

Lemma 10 implies that every admissible polynomial for (R, M_0) is of degree $k-1$ and has positive leading coefficient. Further if h_1, h_2 are two admissible polynomials then $h_1 - h_2$ is constant and $|h_1 - h_2| < 1$. We have shown in Section 3.2 that for each $c \in (c_{k-1} - 1, c_{k-1}) \cap \mathbb{K}$ the polynomial $f_A(c, X + M_0)$ is admissible for (R, M_0). In fact it satisfies a stronger inequality, namely,

$$0 < f_A(c, n + M_0) < \frac{1}{\sum_{m=n}^{\infty} \frac{1}{R(m+M_0)}} < f_A(c, n + M_0) + 1 \quad (n \geq N_0).$$

Constant term of this polynomial is $f_A(c, M_0)$. Now for all $x, y \in \mathbb{K}$

$$f_A(x, X) - f_A(y, X) = x - y. \tag{34}$$

Since $c \in (c_{k-1} - 1, c_{k-1}) \cap \mathbb{K}$ is infinite, there are infinitely many distinct choices for c which give rise to infinitely many admissible polynomials.

3.3.2 Effect of scaling

Let $\alpha \in \mathbb{K}^{\times}$ be positive. Now $\alpha R(X) = \frac{\alpha P(X)}{Q(X)}$. Therefore same constant M_0 has desired positivity properties and the quotient polynomial is $A_\alpha(X) := \alpha A(X)$. By Corollary 7 $\mathbf{c}(A_\alpha) = \alpha \mathbf{c}(A)$. Therefore if $h(X)$ and $h_\alpha(X)$ are admissible polynomials with respect to (R, M_0) and $(\alpha R, M_0)$ resp., then $h_\alpha(X) - \alpha h(X) \in \mathbb{K}$.

3.3.3 Choice of constant term

Let $c \in (c_{k-1}-1, c_{k-1}) \cap \mathbb{K}$. We have seen that $f_A(c, M_0)$ is the constant term of a polynomial admissible for (R, M_0). Let $C \in \mathbb{K}$ be the constant term of some other admissible polynomial $H(X)$.

Lemma 11.

(i) There exists unique $c \in [c_{k-1} - 1, c_{k-1}] \cap \mathbb{K}$ *such that*

$$C = f_A(c, M_0),$$
$$H(X) = f_A(c, X + M_0).$$

(ii) Both $f_A(c_{k-1} - 1, X + M_0)$ *and* $f_A(c_{k-1}, X + M_0)$ *cannot be admissible polynomials for* (R, M_0).

Proof.

(i) For brevity write $\mathcal{I} = (c_{k-1} - 1, c_{k-1}) \cap \mathbb{K}$.

Let $c = C - f_A(0, M_0) \in \mathbb{K}$. Substituting $x = c$, $y = 0$ and $X = M_0$ in (34) we have $f_A(c, M_0) = C$. Moreover, if $x \in \mathbb{K}$ satisfies $f_A(x, M_0) = C$ then $x = c$. By Lemma 10 (iv), $|C - f_A(x, M_0)| < 1$ holds for each $x \in \mathcal{I}$. Hence $|c - x| < 1$ for all $x \in \mathcal{I}$. Therefore $c \in [c_{k-1} - 1, c_{k-1}]$. Now

$$H(X) - f_A(c, X + M_0) = H(X) - f_A(x, X + M_0) + f_A(x, X + M_0) - f_A(c, X + M_0)$$

for all $x \in \mathbb{K}$. Letting $x \in \mathcal{I}$ and using admissibility of H we see that the right-hand side is constant (Lemma 10). But the constant term of the left-hand side is 0. Therefore $H(X) = f_A(c, X + M_0)$.

(ii) Since $f_A(c_{k-1}, X+M_0) - f_A(c_{k-1}-1, X+M_0) = 1$ the assertion follows from Lemma 10 (iv).

□

Lemma 11 characterizes all admissible polynomials with possible exception at boundary of the interval. Analysis of extremal situation is necessary for later development. We need mild improvement over formalism of Section 2 to discuss admissibility of boundary points.

3.3.4 Approximate primitive for rational functions

Let $\mathbb{K}$ be a field of characteristic 0 and $P(X), Q(X), f(X) \in \mathbb{K}[X] - \{0\}$. Suppose that $\deg_{\mathbb{K}} P - \deg_{\mathbb{K}} Q = k \geq 2$. Define

$$\delta(f, R, X) := \frac{1}{f(X)} - \frac{1}{f(X+1)} - \frac{1}{R(X)},$$
$$N(f, R, X) := P(X)\big(f(X+1) - f(X)\big) - Q(X)f(X)f(X+1)$$

where $R = \frac{P}{Q}$. These expressions depend on P and Q. However if $N(f, R, X) = 0$ for one presentation then it is 0 for all presentations. It is clear that

$$\delta(f, R, X) = \frac{N(f, R, X)}{f(X)f(X+1)P(X)}. \tag{35}$$

Further $\delta(f, R, X) = 0$ implies $\deg_{\mathbb{K}} f = k - 1$. Let $A(X), B(X) \in \mathbb{K}[X]$ be the polynomials obtained by the division algorithm, i.e., $P(X) = A(X)Q(X) + B(X)$ with $\deg_{\mathbb{K}} A = k$ and $\deg_{\mathbb{K}} B < \deg_{\mathbb{K}} Q$. Then

$$\delta(f, R, X) = \delta(f, A, X) + \frac{B(X)}{P(X)A(X)},$$

i.e,

$$\delta(f, R, X) = \frac{N(f, A, X)}{f(X)f(X+1)A(X)} + \frac{B(X)}{P(X)A(X)}.$$

We have $\deg_{\mathbb{K}} P(X)A(X) - \deg_{\mathbb{K}} B(X) > 2k$. Suppose that $f \in \mathbb{K}[X]_{k-1}$. If $N(f, A, X) \in \mathbb{K}[X]_{k-1}$ then $\deg_{\mathbb{K}} f(X)f(X+1)A(X) - \deg_{\mathbb{K}} N(f, A, X) = 2k-1$. Here hypothesis amounts to saying that f is a good approximant of A with constant term $\neq c_{k-1}$. In such situation behavior of $\delta(f, R, X)$ is dominated by $\delta(f, A, X)$. Now let f be an arbitrary element of $\mathbb{K}[X]_{k-1}$. From expression above it follows that $\delta(f, R, X) = 0$ if and only if

$$N(f, A, X)P(X) = -f(X)f(X+1)B(X). \tag{36}$$

If (36) holds then $\deg_{\mathbb{K}} N(f, A, X) = k - 2 + \deg_{\mathbb{K}} B - \deg_{\mathbb{K}} Q < k - 2$. Moreover, we have $N(f, A, X) = 0$ if $B(X) = 0$.

Notation and assumptions are identical to Section 3.2. Since $\operatorname{sgn} P = \operatorname{sgn} Q = 1$ it is easy to see that $\operatorname{sgn} N(f, R, X)$ depends only on R and is independent of presentation.

Lemma 12.

(i) Let $N(f_A(c_{k-1}), R, X) = 0$. Then $f_A(c_{k-1}, X + M_0)$ is admissible for (R, M_0).

(ii) Now suppose $N(f_A(c_{k-1}), R, X) \neq 0$. If $\operatorname{sgn} N(f_A(c_{k-1}), R, X) = 1$ then $f_A(c_{k-1}, X + M_0)$ is admissible and if $\operatorname{sgn} N(f_A(c_{k-1}), R, X) = -1$ then $f_A(c_{k-1} - 1, X + M_0)$ is admissible.

Proof. (i) By assumption $N(f_A(c_{k-1}), R, X) = \delta(f_A(c_{k-1}), R, X) = 0$. Therefore

$$\frac{1}{R(X)} = \frac{1}{f_A(c_{k-1}, X)} - \frac{1}{f_A(c_{k-1}, X+1)}. \tag{37}$$

By (37) the maximum of real zeroes of $f_A(c_{k-1}, X)$ is $< M_0 + 1$. Hence $f_A(c_{k-1}, n + M_0) > 0$ for all $n \in \mathbb{N}$. Telescoping

$$\sum_{m=n}^{\infty} \frac{1}{R(m + M_0)} = \frac{1}{f_A(c_{k-1}, n + M_0)} \quad (n \geq 1). \tag{38}$$

The coefficient of X^{k-1} in both $N(f_A(c_{k-1})+1, A, X)$ and $N(f_A(c_{k-1})+1, A_\epsilon, X)$ is $2c_0(c_{k-1}-c_{k-1}-1) = -2c_0 < 0$. Using (30) and (31) and imitating the proof of Theorem 3, we can show

$$\sum_{m=n}^{\infty} \frac{1}{R(m+M_0)} > \frac{1}{f_A(c_{k-1}, n+M_0)+1} \quad (n \gg 1).$$

Therefore $f_A(c_{k-1}, X+M_0)$ is admissible.

(ii) Let $N(f_A(c_{k-1}), R, X) \neq 0$. Suppose that $\operatorname{sgn} N(f_A(c_{k-1}), R, X) = 1$. Hence there is $M_1 \in \mathbb{R}$ such that $N(f_A(c_{k-1}), R, x) > 0$ for all real $x \geq M_1$. One can use (35) and telescoping to deduce

$$\sum_{m=n}^{\infty} \frac{1}{R(m+M_0)} < \frac{1}{f_A(c_{k-1}, n+M_0)} \quad (n \gg 1).$$

The coefficient of X^{k-1} in each of $N\big(f_A(c_{k-1})+1, A, X\big)$ and $N\big(f_A(c_{k-1})+1, A_\epsilon, X\big)$ is $2c_0(c_{k-1}-c_{k-1}-1) = -2c_0 < 0$. We can repeat the arguments from the proof of Theorem 3 to conclude (see part (i))

$$\sum_{m=n}^{\infty} \frac{1}{R(m+M_0)} > \frac{1}{f_A(c_{k-1}, n+M_0)+1} \quad (n \gg 1).$$

Hence $f_A(c_{k-1}, X+M_0)$ is an admissible polynomial.

Now assume that $\operatorname{sgn} N(f_A(c_{k-1}), R, X) = -1$. As before, we can use (35) and telescoping to conclude

$$\sum_{m=n}^{\infty} \frac{1}{R(m+M_0)} > \frac{1}{f_A(c_{k-1}, n+M_0)} \quad (n \gg 1).$$

Note that $f_A(c_{k-1})-1$ is a good approximant of A and A_ϵ. Further coefficient of X^{k-1} in both $N\big(f_A(c_{k-1})-1, A, X\big)$ and $N\big(f_A(c_{k-1})-1, A_\epsilon, X\big)$ is $2c_0(c_{k-1}-c_{k-1}+1) = 2c_0 > 0$. Therefore

$$\sum_{m=n}^{\infty} \frac{1}{R(m+M_0)} < \frac{1}{f_A(c_{k-1}, n+M_0)-1} \quad (n \gg 1).$$

Hence $f_A(c_{k-1}-1, X+M_0)$ is an admissible polynomial in this case.

□

3.3.5 Summation by telescoping

If there is $f(X) \in \mathbb{K}[X]-\{0\}$ so that $\frac{1}{R(X)} = \frac{1}{f(X)} - \frac{1}{f(X+1)}$ then $N(f,R,X)=0$. By (35) and (36) we have $\deg_{\mathbb{K}} f = k-1$, $\deg_{\mathbb{K}} N(f,A,X) < k-2$. Now (15) and Lemma 4 implies that

$f_A(c_{k-1})$ is the only polynomial which can possibly satisfy these two conditions. Therefore $-\frac{1}{R(X)}$ has an exact difference primitive if and only if $N(f_A(c_{k-1}), R, X) = 0$. Whenever this criterion holds, we can compute the exact value of the sum by telescoping. (Cf. Corollary 6.)

Remark 13. One can use Remark 9 to calculate effective lower bounds on n for inequalities of Lemma 12 to hold. Computations are analogous to Section 3.2 and we omit the details.

4 Explicit calculations

4.1 Reciprocal sequence

This subsection is devoted to explicit calculation of reciprocal sequence. We begin with an elementary observation. The situation is same as Theorem 3.

Remark 14. Let h be a polynomial given by Theorem 3 and N_0 be the corresponding integer. Suppose that $(a_n)_{n\geq 1}$ is reciprocal sequence of the sequence $(x_m)_{m\geq 1}$ given by $x_m = \frac{1}{R(m+M_0)}$. Then from (6) it follows that, for all $n \geq N_0$,

(i) a_n is either $\lfloor h(n) \rfloor$ or $\lfloor h(n) \rfloor + 1$;

(ii) if $h(n)$ is an integer for some n, then $a_n = h(n)$.

Let $P, Q \in \mathbb{Q}[X]$ be polynomials so that $\deg_{\mathbb{Q}} P - \deg_{\mathbb{Q}} Q = k \geq 2$ and $\operatorname{sgn} P = \operatorname{sgn} Q = 1$. Suppose that $P(x), Q(x) > 0$ for all real $x \geq 1$, i.e., 0 is a legitimate choice for M_0. One can always ensure this property by shifting the polynomials in hypothesis. Set $R(X) = \frac{P(X)}{Q(X)} \in \mathbb{Q}(X)$ and consider the sequence $(x_m)_{m\geq 1}$ given by

$$x_m = \frac{1}{R(m)} \quad (m \geq 1). \tag{39}$$

Subsequent paragraphs contain an algorithmic determination of $(a_n)_{n\geq 1}$, the reciprocal sequence of $(x_m)_{m\geq 1}$.

4.1.1 Algorithm for reciprocal sequence

Let $A(X)$ and $B(X)$ be the polynomials in $\mathbb{Q}[X]$ obtained by the division algorithm, i.e, $P(X) = A(X)Q(X) + B(X)$ and $\deg_{\mathbb{Q}} B < \deg_{\mathbb{Q}} Q$. Suppose that $(c_0, \ldots, c_{k-1}) \in \mathbb{Q}^\times \times \mathbb{Q}^{k-1}$ is the unique tuple which is solution to the system of equations (19) corresponding to $A(X)$. Write $c_i = \frac{p_i}{q_i}$ where p_i, q_i are integers with $q_i > 0$ and $\gcd(p_i, q_i) = 1$ for each $0 \leq i \leq k-1$. Note that if $p_i = 0$ then $q_i = 1$. Let $L = \operatorname{lcm}(q_0, \ldots, q_{k-2})$. Put

$$\begin{aligned} H(X) &= c_0 X^{k-1} + \cdots + c_{k-2} X, \\ H_L(X) &= L H(X). \end{aligned} \tag{40}$$

It is clear that $H_L(X) \in \mathbb{Z}[X]$. We determine a family of polynomial in $\mathbb{Q}[X]$ parameterized by residue classes modulo L such that for sufficiently large n value of a_n is obtained by evaluating one of these polynomials at n, and this polynomial depends only on residue class of n modulo L.

4.1.2 Case I: $q_{k-1} \nmid L$

Under this assumption $q_{k-1} \neq 1$. Write $c_{k-1} = \lfloor c_{k-1} \rfloor + \frac{r_{k-1}}{q_{k-1}}$ where r_{k-1} is a positive integer $\leq q_{k-1} - 1$. Since $\gcd(p_{k-1}, q_{k-1}) = 1$ it follows that $\gcd(r_{k-1}, q_{k-1}) = 1$. Therefore $\frac{r_{k-1}}{q_{k-1}} \notin \{\frac{n}{L} \mid n \in \mathbb{Z}\}$.

Suppose that $r \in \{1, \ldots, L\}$. By the argument above, there is a unique integer $l(r)$ that satisfies $l(r) - \frac{r}{L} < c_{k-1} < l(r) + 1 - \frac{r}{L}$. Now let $c(r) = l(r) - \frac{r}{L}$ and $h_r(X) = H(X) + c(r)$. Since $c(r) \in (c_{k-1} - 1, c_{k-1})$ there is an integer $N(r)$ so that

$$h_r(n) \leq \frac{1}{\sum_{m=n}^{\infty} \frac{1}{R(m)}} < h_r(n) + 1 \quad (n \geq N(r)).$$

Fix choice of $N(r)$ for each $r \in \{1, \ldots, L\}$. Put $N = \max\{N(1), \ldots, N(L)\}$. Let $n \geq N$. Now $r(n)$ be the unique element in $\{1, \ldots, L\}$ with $H_L(n) \equiv r(n) \pmod{L}$. Evidently $r(n)$ depends only on residue class $n \bmod L$. Note that

$$h_{r(n)}(n) = \frac{H_L(n)}{L} + l\big(r(n)\big) - \frac{r(n)}{L} \in \mathbb{Z} \quad (n \geq N).$$

Remark 14 (ii) implies $a_n = h_{r(n)}(n)$.

Therefore in this case we have a closed form formula for a_n depending on equivalence class of n modulo L whenever $n \geq N$.

4.1.3 Case II: $q_{k-1} \mid L$

Let $r \in \{1, \ldots, L\}$. If $\frac{r}{L} \neq 1 + \lfloor c_{k-1} \rfloor - c_{k-1}$ then there is a unique integer $l(r)$ with $l(r) - \frac{r}{L} < c_{k-1} < l(r) + 1 - \frac{r}{L}$. For these residue classes define $c(r) = l(r) - \frac{r}{L} \in (c_{k-1} - 1, c_{k-1})$.

Now let $r \in \{1, \ldots, L\}$ with $\frac{r}{L} = 1 + \lfloor c_{k-1} \rfloor - c_{k-1}$. Such residue class exists and is unique. Set $l(r) = 1 + \lfloor c_{k-1} \rfloor$. Note that $c_{k-1} = l(r) - \frac{r}{L}$. For this residue class define

$$c(r) = \begin{cases} c_{k-1}, & \text{if } \operatorname{sgn} N(f_A(c_{k-1}), P, X) \geq 0; \\ c_{k-1} - 1, & \text{if } \operatorname{sgn} N(f_A(c_{k-1}), P, X) = -1. \end{cases}$$

Let $h_r(X) = H(X) + c(r)$. By the proof of Theorem 3 and Lemma 12 it follows that for each r there is an integer $N(r)$ so that

$$h_r(n) \leq \frac{1}{\sum_{m=n}^{\infty} \frac{1}{R(m)}} < h_r(n) + 1 \quad (n \geq N(r)).$$

Suppose that $N = \max\{N(1), \dots, N(L)\}$. Let $n \geq N$ and $r(n)$ be the unique element in $\{1, \dots, L\}$ such that $H_L(n) \equiv r(n) \pmod{L}$. It follows that there is a positive integer N so that $a_n = h_{r(n)}(n)$ whenever $n \geq N$.

The discussion above can be summarized as follows:

Theorem 15. *Let $(x_m)_{m\geq 1}$ be the sequence defined by (39). There exist a positive integer L, algorithmically computable polynomials $h_{r,L}(X) \in \mathbb{Q}[X]$ parameterized by residue classes modulo L, a polynomial $H_L(X) \in \mathbb{Z}[X]$ of degree $k-1$ with constant term 0, and natural number N so that*

(i) $H_L(X) - Lh_{r,L}(X) \in \mathbb{Z}$ for all residue classes r,

(ii) $a_n = h_{r,L}(n)$ if $H_L(n) \equiv r \pmod{L}$ for all $n \geq N$.

Proof. The statement holds with $L = \operatorname{lcm}(q_0, \dots, q_{k-2})$, H_L given by (40), polynomials $(h_r(X))_{r \bmod L}$ and the number N constructed above. The tuple $(c_0, \dots, c_{k-1})$ is algorithmically constructible and once we have this datum, construction of $(h_r(X))_{r \bmod L}$ is already given in two possible cases. □

Remark 16. The choice of L in the proof of the theorem is algorithmically computable. Moreover, one can choose N effectively since by Section 3.2 and Remark 13 each of $N(r)$ is effective.

Theorem 15 is an analogue of Theorem 3 in context of reciprocal sequence. The modulus in the statement of the theorem adds an arithmetic aspect to the theory.

4.1.4 Choice of modulus

Let $\mathfrak{m}$ be a natural number and $\phi(X) \in \mathbb{Z}[X]$. A residue class r modulo $\mathfrak{m}$ is *nontrivial* with respect to ϕ if there exists one (and hence infinitely many) integer n such that $\phi(n) \equiv r \pmod{\mathfrak{m}}$. Nontrivial residue classes are exactly residue classes of $\{\phi(j) \mid 1 \leq j \leq \mathfrak{m}\}$. Observe that we need $h_{r,L}(X)$ only for the residue classes modulo L which are nontrivial with respect to H_L. For other residue classes, the constant term of $h_{r,L}(X)$ can be chosen arbitrarily. Let L_1 be another positive integer so that there exists $H_{L_1}(X) \in \mathbb{Z}[X]$ and polynomials $\{h_{r,L_1}(X) \mid r \bmod L_1\} \subseteq \mathbb{Q}[X]$ satisfying conditions (i) and (ii) in the statement of theorem. Using condition (i) and Lemma 10

$$H(X) = \frac{H_{L_1}(X)}{L_1} = \frac{H_L(X)}{L}.$$

Since $L = \operatorname{lcm}(q_0, \dots, q_{k-2})$ we have $L|L_1$ and $H_{L_1}(X) = \frac{L_1}{L} H_L(X)$. Non-constant part of the polynomials $\{h_{r,L_1}(X) \mid r \bmod L_1\}$ are same and equals $H(X)$. Now $H_L(n_1) \equiv H_L(n_2) \pmod{L}$ if and only if $H_{L_1}(n_1) \equiv H_{L_1}(n_2) \pmod{L_1}$ for all $n_1, n_2 \in \mathbb{Z}$. Hence $H_L(n) \to H_{L_1}(n)$ is a bijection between residue classes modulo L nontrivial with respect to H_L and residue classes modulo L_1 nontrivial with respect to H_{L_1}. Let r, r_1 be two nontrivial residue classes modulo

L, L_1 respectively. By condition (ii) of the theorem, $h_{r,L}(X) = h_{r_1,L_1}(X)$ if r corresponds to r_1 under the bijection mentioned above.

This phenomenon shows that the modulus in Theorem 15 is essentially unique.

4.2 Asymptotic of summation

In this subsection we write down the asymptotic form of Theorem 3. For this purpose one requires a familiar technique from complex analysis.

Lemma 17. *Let $\phi(z) = a_0 z^d + a_1 z^{d-1} + \cdots + a_d \in \mathbb{C}[z]$ be a polynomial of degree $d \geq 1$. There exists $M(\phi) > 0$ (depending on ϕ) so that on an open set containing the region $|z| \geq M(\phi)$*

$$\frac{1}{\phi(z)} = \sum_{j \geq d} \frac{A_j}{z^j}$$

where the right-hand side converges absolutely on the open set. Moreover, there are computable polynomials $F_j \in \mathbb{Z}[x_1, \ldots, x_d]$, $j \geq d$, which depend only on the integer d so that

(i) $A_j = a_0^{-1} F_j(\frac{a_1}{a_0}, \ldots, \frac{a_d}{a_0})$,

(ii) *F_d is identically 1 and for $d+1 \leq j \leq 2d-1$ the set of variables appearing in F_j is $\{x_1, \ldots, x_{j-d}\}$.*

Proof. We have

$$\phi(z)^{-1} = a_0^{-1} z^{-d} (1 + \frac{a_1}{a_0 z} + \cdots + \frac{a_d}{a_0 z^d})^{-1}.$$

Write $\Phi(z) = -(\frac{a_1}{a_0} z + \cdots + \frac{a_d}{a_0} z^d)$. It is easy to see that $|\Phi(\frac{1}{z})| \leq \frac{d}{d+1}$ whenever $|z| \geq M = \max\{1, \frac{(d+1)|a_j|}{|a_0|} \mid 1 \leq j \leq d\}$. Let $M(\phi) = 1 + M$ and the open set be $\{z \in \mathbb{C} \mid |z| > M\}$. Observe that $\phi(z)$ has no zeroes in region $|z| \geq M$. One obtains the first part of assertion by expanding $(1 - \Phi(\frac{1}{z}))^{-1}$ in a geometric series for $|z| > M$.

For any $p \geq 0$, coefficient of z^{-p} in $(1 - \Phi(\frac{1}{z}))^{-1}$ is the coefficient of z^{-p} in the finite sum $\sum_{r=0}^{p} \Phi(\frac{1}{z})^r$. Hence, the second part of the assertion is a consequence of the multinomial theorem. □

4.2.1 Asymptotic form of Theorem 3

With the notation of Theorem 3 we have $\deg_{\mathbb{K}} h(X) = k - 1 \geq 1$ and leading coefficient of $h(X)$ of is $c_0 > 0$. Further $h(n) > 0$ for $n \geq N_0$. The inequality (6) is equivalent to

$$0 < \frac{1}{h(n)+1} < \sum_{m=n}^{\infty} \frac{1}{R(m+M_0)} \leq \frac{1}{h(n)} \quad (n \geq N_0). \tag{41}$$

Now $\frac{1}{h(X)} - \frac{1}{h(X)+1} = \frac{1}{h(X)(h(X)+1)}$, and the first term in Taylor series of $\frac{1}{h(z)(h(z)+1)}$ is $c_0^{-2} z^{-2(k-1)}$. Therefore $\frac{1}{h(n)(h(n)+1)} = O(n^{-2k+2})$ and

$$\sum_{m=n}^{\infty} \frac{1}{R(m+M_0)} = \frac{1}{h(n)} + O(n^{-2k+2}) \quad (n \geq N_0). \tag{42}$$

Lemma 17 implies that for each $n \geq \max\{M(h), M(h+1), N_0\}$

$$0 < \frac{1}{h(n)} = \sum_{j \geq k-1} \frac{A_j^{(1)}}{n^j},$$
$$0 < \frac{1}{h(n)+1} = \sum_{j \geq k-1} \frac{A_j^{(2)}}{n^j}.$$

By the second part of the lemma, we have $A_j^{(1)}, A_j^{(2)} \in \mathbb{K}$ for all $j \geq k-1$. Since leading term in expansion of $\frac{1}{h(z)(h(z)+1)}$ is $c_0^{-2} z^{-2k+2}$ we have

$$A_j^{(1)} = A_j^{(2)} (= \text{say, } A_j) \qquad (k-1 \leq j \leq 2k-3),$$
$$A_{2k-2}^{(1)} = A_{2k-2}^{(2)} + c_0^{-2} > A_{2k-2}^{(2)}.$$

These statements also follow from lemma above. Moreover, from absolute convergence ensured by the lemma, it is easy to see that

$$\frac{1}{h(n)} = \sum_{j=k-1}^{2k-3} \frac{A_j}{n^j} + O_h(n^{-2k+2}),$$
$$\frac{1}{h(n)+1} = \sum_{j=k-1}^{2k-3} \frac{A_j}{n^j} + O_{h+1}(n^{-2k+2}) \quad (n \gg 1).$$

From (41) one deduces

$$\Big| \sum_{m=n}^{\infty} \frac{1}{R(m+M_0)} - \sum_{j=k-1}^{2k-3} \frac{A_j}{n^j} \Big|$$
$$\leq \max \Big\{ \Big| \frac{1}{h(n)} - \sum_{j=k-1}^{2k-3} \frac{A_j}{n^j} \Big|, \Big| \frac{1}{h(n)+1} - \sum_{j=k-1}^{2k-3} \frac{A_j}{n^j} \Big| \Big\} \quad (n \geq N_0).$$

Hence

$$\sum_{m=n}^{\infty} \frac{1}{R(m+M_0)} = \sum_{j=k-1}^{2k-3} \frac{A_j}{n^j} + O(n^{-2k+2}) \quad (n \gg 1). \tag{43}$$

Thus we have calculated the first $k-1$ terms in the asymptotic expansion of the sum in the left-hand side in the traditional sense. (Compare (42) and (43).)

The constants M and $M(\phi)$ appearing in Lemma 17 are effective. Let $|z| \geq M$. Then $|\Phi(\frac{1}{z})| \leq \frac{d}{d+1}$ and

$$\frac{1}{\phi(z)} = a_0^{-1} z^{-d} \Big(\sum_{r=0}^{p} \Phi(\frac{1}{z})^r + \sum_{r \geq p+1} \Phi(\frac{1}{z})^r \Big) \quad (p \geq 0).$$

Now suppose $j \geq d$ and $p = j - d$. It is clear that the terms $\sum_{r=d}^{j} \frac{A_r}{z^r}$ in expansion of the lemma appear from the first summation, namely, $a_0^{-1} z^{-d} \sum_{r=0}^{j-d} \Phi(\frac{1}{z})^r$. This summation contributes only finitely many terms in orders higher than $(\frac{1}{z})^j$. To estimate the second summation, note that in the region under consideration $|\Phi(\frac{1}{z})| \leq \frac{\alpha}{|z|}$ where $\alpha = |\frac{a_1}{a_0}| + \frac{d-1}{d+1}$. Thus, if $|z| \geq \alpha+1$ then $|\sum_{r \geq p+1} \Phi(\frac{1}{z})^r| \leq \frac{\alpha^{p+1}(\alpha+1)}{|z|^{p+1}}$. Hence for fixed $\epsilon > 0$ one can effectively determine $M_{\phi,j,\epsilon} > 0$ such that

$$\left| \sum_{r \geq j+1} \frac{A_r}{z^r} \right| \leq \epsilon |z|^{-j} \quad (|z| \geq M_{\phi,j,\epsilon}).$$

From this argument, we conclude that the constant and range appearing in O_h, O_{h+1}, and (43) are effective.

4.3 The example: $P(X) = X^k$

Let $k \geq 2$ and $P(X) = X^k$, $Q(X) = 1 \in \mathbb{Q}[X]$. With the notation of Section 3, $R(X) = P(X) = A(X) = X^k$, $B(X) = 0$ and we are in the situtation of Theorem 1. The objective of this section is to write down the first few coefficients of the admissible polynomial $f_P(X)$ as a function of k.

We introduce formal binomial coefficients for convenience. Let X be a formal variable and $r \in \mathbb{Z}_{\geq 0}$. Define $\binom{X}{r}$ to be an element of $\mathbb{Q}[X]$ given by the expression

$$\binom{X}{r} = \begin{cases} \frac{X(X-1)\cdots(X-r+1)}{r!}, & \text{if}\, r \geq 1; \\ 1, & \text{if}\, r = 0. \end{cases}$$

With the notation of Section 2 suppose that $g(X) = X^k$. For this polynomial

$$a_r = \begin{cases} 1, & \text{if } r = 0; \\ 0, & \text{if } 1 \leq r \leq k. \end{cases}$$

By (14)

$$\begin{aligned} v_l(\mathbf{x}, X^k) &= y_{l+1}(\mathbf{x}) \\ &= \binom{k-l-1}{1} x_l + \binom{k-l}{2} x_{l-1} + \cdots + \binom{k-1}{l+1} x_0 \end{aligned} \tag{44}$$

for all $0 \le l \le k-1$. If $l = k-1$ then one observes that (44) holds with formal binomial coefficients.

Substituting (12) into (13) we obtain

$$u_j(\mathbf{x}) = \sum_{r=0}^{j} x_{j-r}x_r + \sum_{r=1}^{j}\sum_{s=1}^{r} \binom{k-s}{r-s+1} x_{j-r}x_{s-1} \quad (0 \le j \le k-1). \tag{45}$$

Now one can proceed to solve the system of equations

$$u_i(\mathbf{x}) = v_i(\mathbf{x}, X^k) \quad (0 \le i \le k-1)$$

following the algorithm of Lemma 4. The first few solutions are as follows:

$$\begin{aligned}
c_0(k) &= k-1, \\
c_1(k) &= -\frac{(k-1)^2}{2}, \\
c_2(k) &= \frac{(k-1)^2(2k-3)}{12} \quad (k \ge 3), \\
c_3(k) &= -\frac{(k-1)^3(k-3)}{24} \quad (k \ge 4), \\
c_4(k) &= \frac{(k-1)^2}{720}(6k^3 - 47k^2 + 92k - 45) \quad (k \ge 5).
\end{aligned} \tag{46}$$

In general one can use (44), (45), and recursion argument of Lemma 7 to conclude that $c_r(k)$ is a rational function of k whenever $k \ge r+1$.

4.3.1 Asymptotic expansion of polygamma [1, p. 260]

Let $k \ge 1$. The polygamma function of order k is a holomorphic function on $\mathbb{C} - \{0, -1, -2, \ldots\}$ defined by the series

$$\psi^{(k)}(z) = (-1)^{k+1} k! \sum_{n=0}^{\infty} (z+n)^{-k-1}.$$

Let $0 < \theta < \frac{\pi}{2}$ and $\mathcal{S}_\theta = \{z \in \mathbb{C} \mid z \ne 0, |\arg z| \le \pi - \theta\}$. Finite order asymptotic expansion of polygamma function on $\mathcal{S}_\theta$ is

$$\psi^{(k)}(z) = (-1)^{k+1}\Big(\frac{(k-1)!}{z^k} + \frac{k!}{2z^{k+1}} + \sum_{n=1}^{p} B_{2n}\frac{(2n+k-1)!}{(2n)!z^{2n+k}} + O_{p,\theta}(\frac{1}{z^{k+2p+1}})\Big)$$

where p is any positive integer. Hence on $\mathcal{S}_\theta$

$$\begin{aligned}
&\frac{1}{\sum_{n=0}^{\infty}(z+n)^{-k-1}} \\
&= kz^k\Big(1 + \frac{k}{2z} + k\sum_{n=1}^{p}\frac{B_{2n}}{2n}\binom{k+2n-1}{2n-1}z^{-2n} + O_{p,\theta}(\frac{1}{z^{2p+1}})\Big)^{-1}.
\end{aligned} \tag{47}$$

As $z \to \infty$ on the sector one can expand the right-hand side to get asymptotic expansion up to arbitrary high order by choosing a large p. One can put $p = \lceil \frac{k}{2} \rceil$ to retrieve the first k terms in the asymptotic expansion of $\frac{1}{\sum_{n=0}^{\infty}(z+n)^{-k-1}}$. These formulas give an alternate gateway to results of this subsection as mentioned in introduction.

4.3.2 Numerical formula

Let $k \geq 2$. Consider the sequence defined by $x_m = m^{-k}$, $m \geq 1$. We explicitly determine the reciprocal sequences associated with $(x_m)_{m\geq 1}$ for small values of k. Set up is same as beginning of this section, i.e., $R(X) = P(X) = A(X) = X^k$, $Q(X) = 1$ and $B(X) = 0$. Notation and arguments from the proof of Theorem 1 and Section 4.1 are frequently used in the discussion below. Here $M_0 = 0$, which is compatible with theory in Section 4.1.

4.3.3 $k = 2$

(Xin [6]) Using (46) we have $(c_0, c_1) = (1, -\frac{1}{2})$. Hence $L = 1$ and one needs to use Case I. It is clear that $c(1) = -1$. From explicit expression of $f_P(-1, X)$ and numerator polynomials it follows that 2 is a legitimate choice for N. Hence

$$a_n = n - 1 \quad (n \geq 2).$$

4.3.4 $k = 3$

(Xin [6]) By (46) one concludes $(c_0, c_1, c_2) = (2, -2, 1)$. Here $L = 1$ and we are in the situation of Case II. Therefore, the constant term has to be c_{k-1} or $c_{k-1} - 1$. Note that $N(f_P(c_{k-1}), P, X)$ is a degree 0 polynomial with negative leading coefficient. Hence $c(1) = c_{k-1} - 1 = 0$. It is easy to see that $N = 2$ is a legitimate choice. Thus

$$a_n = 2n(n-1) \quad (n \geq 2).$$

4.3.5 $k = 4$

(Xu [7]) From (46) it follows that $(c_0, c_1, c_2, c_3) = (3, -\frac{9}{2}, \frac{15}{4}, -\frac{9}{8})$. Here $L = 4$ and Case I holds. All residue classes are nontrivial and $c(1) = -\frac{5}{4}$, $c(2) = -\frac{3}{2}$, $c(3) = -\frac{7}{4}$, $c(4) = -2$. The choice of constants $M_{f_P(c)} = 1$, $M_1 = M_2 = 1$ works for $c = c(2), c(3), c(4)$. If the constant term is $c(1)$, then a legitimate choice is $M_{f_P(c)} = M_2 = 0$, $M_1 = 5$. Therefore $N(2), N(3), N(4) = 1$ and $N(1) = 5$ is a legitimate choice of constants. Hence $N = 5$ and for all $n \geq 5$

$$a_n = \begin{cases} 3n^3 - \frac{9}{2}n^2 + \frac{15}{4}n - \frac{5}{4}, & \text{if } n \equiv 1 \pmod 4; \\ 3n^3 - \frac{9}{2}n^2 + \frac{15}{4}n - \frac{3}{2}, & \text{if } n \equiv 2 \pmod 4; \\ 3n^3 - \frac{9}{2}n^2 + \frac{15}{4}n - \frac{7}{4}, & \text{if } n \equiv 3 \pmod 4; \\ 3n^3 - \frac{9}{2}n^2 + \frac{15}{4}n - 2, & \text{if } n \equiv 4 \pmod 4. \end{cases}$$

4.4 $k = 5$

(Xu [7]) Using (46) we see that $(c_0, c_1, c_2, c_3, c_4) = (4, -8, \frac{28}{3}, -\frac{16}{3}, -\frac{2}{9})$. Hence $L = 3$ and Case I holds. Nontrivial residue classes modulo L with respect to $H_L(X)$ are $\{2, 3\}$ and $c(2) = -\frac{2}{3}, c(3) = -1$. The choice of constants $M_{f_P(c)} = 2$, $M_1 = 1$, $M_2 = 5$ works for $c \in \{c(2), c(3)\}$. Therefore $N = 5$ and for all $n \geq 5$

$$a_n = \begin{cases} 4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n - 1, & \text{if } n \equiv 1 \pmod 3; \\ 4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n - \frac{2}{3}, & \text{if } n \equiv 2 \pmod 3; \\ 4n^4 - 8n^3 + \frac{28}{3}n^2 - \frac{16}{3}n - 1, & \text{if } n \equiv 3 \pmod 3). \end{cases}$$

Remark 18. The results for $k = 4, 5$ answer questions of Kotesovec [5]. Theorem 15 provides an algorithmically computable answer to the problem of determining reciprocal sequence for a large class.

5 Complements: Sequence of polynomials

The discussion in Section 4.3 leads to new kind of formalism. Let $\mathbb{K}$ be a field of characteristic 0 and $(P_k(X))_{k\geq 1}$ be sequence of polynomials in $\mathbb{K}[X]$ such that $d_k = \deg_{\mathbb{K}} P_k \geq 2$ for all $k \geq 1$ and $d_k \to \infty$ as $k \to \infty$. For $r \in \mathbb{Z}_{\geq 0}$ define

$$m_r = \min\{k_0 \in \mathbb{N} \mid k \geq k_0 \implies d_k \geq r + 1\}.$$

Write $\mathcal{I}_r = \{k \in \mathbb{N} \mid k \geq m_r\}$. By Lemma 4 one has a well defined function $c_r : \mathcal{I}_r \to \mathbb{K}$ given by $c_r(k) = c_r(P_k)$. Since the system equations

$$u_i(\mathbf{X}) = v_i(\mathbf{X}, P_k) \quad \mathbf{X} = (X_0, \ldots, X_r), (0 \leq i \leq r \leq d_k - 1)$$

determines $c_r(k)$, it depends only on coefficients of $X^{d_k}, \ldots, X^{d_k - r}$ in $P_k(X)$. These coefficients are denoted by $a_0(k), \ldots, a_r(k)$ respectively.

We can study behavior of this function for different sequences. The subsequent remark summarizes some examples of interest.

Example 19.

(i) Let $P_k(X) = X^{k+1} \in \mathbb{Q}[X]$. Note that

$$a_r(k) = \begin{cases} 1, & r = 0; \\ 0, & r \geq 1. \end{cases}$$

One uses recursion argument of Lemma 4 to conclude that there is a rational function $C_r(X) \in \mathbb{Q}(X)$ so that $c_r(k) = C_r(X)$ for all $k \in \mathcal{I}_r$ (cf. Section 4.3). For $r \geq 1$ these rational functions are divisible by X^2. Moreover, the asymptotic expansion in (47) implies that these rational functions are actually **polynomials**.

(ii) Let $a \in \mathbb{K}$, and $P(X) \in \mathbb{K}[X]$ be a non-constant polynomial. Suppose that $P_k(X) = P(X)(X+a)^k \in \mathbb{K}[X]$. For this sequence $a_0(k) = a_0(P)$. Suppose that $k \geq r$. Then $a_r(k)$ is $\mathbb{K}$-linear combination of elements of the form

$$\left\{\binom{k}{j} a^j \mid 0 \leq j \leq r\right\}$$

with coefficients independent of k. By recursion argument there exists a rational function $C_r(X) \in \mathbb{K}(X)$ such that $c_r(k) = C_r(k)$ for $k \gg 1$.

(iii) Let $P(X), Q(X) \in \mathbb{K}[X]$ be so that $\deg_{\mathbb{K}} P, \deg_{\mathbb{K}} Q \geq 1$. Construct a sequence by $P_k(X) = P(X)Q(X)^k$. Assume leading coefficient of $P(X)$ is $a_0(P)$ and

$$Q(X) = X^d + A_1 X^{d-1} + \cdots + A_d \in \mathbb{K}[X]$$

with $d \geq 1$. Therefore $a_0(k) = a_0(P)$. Using multinomial theorem we deduce that if $k \geq dr$ then $a_r(k)$ is a $\mathbb{K}$-linear combination of elements

$$\left\{\binom{k}{\mathbf{j}} \mathbf{A}^{\mathbf{j}} \mid \mathbf{j} = (j_0, \ldots, j_d) \in \mathbb{Z}_{\geq 0}^{d+1},\ \sum_{s=0}^{d} j_s = k,\ 0 \leq \sum_{s=0}^{d} s j_s \leq r\right\}$$

whose coefficients are independent of k. Observe that constraints on $\mathbf{j}$ imply that $j_0 \geq k - rd$ and $j_s \leq r$ for all $1 \leq s \leq r$. For $k \geq dr$ all possible choices for $\mathbf{j}$ appear and these depend only on d, r. Then each of these multinomial terms is a polynomial in k of degree $\leq dr$. Using recursion, we can construct $C_r(X) \in \mathbb{K}(X)$ such that $c_r(k) = C_r(X)$ for $k \gg 1$.

In special cases one can use recursion to find finer properties of these functions which indicate possibility of richer structure.

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