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Crossings over Permutations Avoiding Some Pairs of Patterns of Length Three

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Abstract

In this paper, we compute the distributions of the statistic number of crossings over permutations avoiding one of the pairs $\{321, 231\}$, $\{123, 132\}$ and $\{123, 213\}$. The obtained results are new combinatorial interpretations of two known triangles in terms of restricted permutations statistic. For other pairs of patterns of length three, we find relationships between the polynomial distributions of the crossings over permutations that avoid the pairs containing the pattern 231 on the one hand, and the pattern 312 on the other hand.

1 Introduction and main results

The statistic number of crossings is among the complicated statistics on permutations. Its survey arises from the works of de Médicis and Viennot [6], Randrianarivony [12, 13], Corteel [4], Burrill et al. [2] to Corteel et al. [5]. Recently, the first author of this paper introduced the study of this statistic on permutations avoiding a single pattern of length three [11]. This

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one is devoted on the distribution of crossings on permutations avoiding a pair of patterns of length three. The technique we use in this paper differs from that of these known works who generally used a bijection between permutations and a family of paths. Here, we simply manipulate the structure of our combinatorial objects and use some trivial bijections that we will present in the next sections.

A permutation σ of $[n] := \{1, 2, \dots, n\}$ is a bijection from $[n]$ to itself that can be written linearly as $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$. We shall refer n as the length of σ (i.e., $n = |\sigma|$) and we let S_n denote the set of all permutations of length n. A *crossing* of a given permutation σ is a pair of indices (i, j) such that $i < j < \sigma(i) < \sigma(j)$ or $\sigma(i) < \sigma(j) \leq i < j$. We let $\mathrm{cr}(\sigma)$ denote the number of crossings of σ. For graphical understanding, we usually draw arc diagrams, i.e., draw an upper (resp., a lower) arc from i to $\sigma(i)$ if $\sigma(i) > i$ (resp., $\sigma(i) < i$).

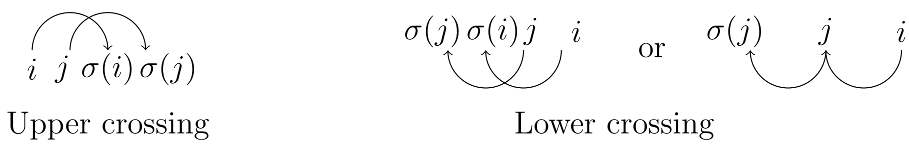

Figure 1: Arc diagrams of crossings.

Example: the crossings of the permutation $\pi = 4735126 \in S_7$ drawn in Figure 2 are $(1, 2)$, $(5, 6)$ and $(6, 7)$. So we have $\mathrm{cr}(\pi) = 3$.

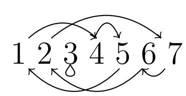

Figure 2: Arc diagrams of $\pi = 4735126 \in S_7$ with $\mathrm{cr}(\pi) = 3$.

Let $\sigma \in S_n$ and $\tau \in S_k$ with $1 \leq k \leq n$. For a given sequence of integers

$$i_1 < i_2 < \cdots < i_k,$$

we say that a subsequence $s = \sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)$ of σ is an occurrence of τ if s and τ are in order isomorphic, i.e., $\sigma(i_x) < \sigma(i_y)$ if and only if $\tau(x) < \tau(y)$. If there is no occurrence of the pattern τ in σ, we say that σ is τ-*avoiding*. Example: the permutation $\pi = 4162375 \in S_7$ is 321-avoiding and it has five occurrences of the pattern 312 namely 312, 423, 623, 625 and 635. We let $S_n(\tau)$ denote the set of all τ-avoiding permutations of $[n]$. For a subset of patterns $T = \{\tau_1, \tau_2, \dots\}$, we usually write $S_n(\tau_1, \tau_2, \dots)$ for $S_n(T)$ and $S(T) := \cup_{n\geq 0} S_n(T)$. There are three useful trivial involutions on S_n namely *reverse* r, *complement* c and *inverse* i defined as follows: for a permutation $\sigma \in S_n$,

- the reverse of σ is $\mathrm{r}(\sigma) = \sigma(n)\sigma(n-1)\cdots\sigma(1)$,
- the complement of σ is $\mathrm{c}(\sigma) = (n+1-\sigma(1))(n+1-\sigma(2))\cdots(n+1-\sigma(n))$,

- the inverse of σ is $\mathrm{i}(\sigma) = p(1)p(2)\cdots p(n)$ where $p(i)$ is the position of i in σ. We often write $\mathrm{i}(\sigma) = \sigma^{-1}$.

Example: for $\pi = 4135762 \in S_7$, we have $\mathrm{r}(\pi) = 2675314$, $\mathrm{c}(\pi) = 4753126$, $\pi^{-1} = 2731465$, $\mathrm{r} \circ \mathrm{c}(\pi) = 6213574$ and $\mathrm{r} \circ \mathrm{c} \circ \mathrm{i}(\pi) = 3247516$ where $\circ$ denotes the composition operation. Let $\mathrm{fg} := \mathrm{f} \circ \mathrm{g}$ for an involution f and g in $\{\mathrm{r}, \mathrm{c}, \mathrm{i}\}$. By composition $\circ$, these defined involutions generate the dihedral group $\mathcal{D} = \{\mathrm{id}, \mathrm{r}, \mathrm{c}, \mathrm{i}, \mathrm{rc}, \mathrm{ri}, \mathrm{ci}, \mathrm{rci}\}$ and they greatly simplify enumeration of pattern-avoiding permutations statistics through the fundamental property by Simion and Smith [15]

$$\varphi(S_n(T)) = S_n(\varphi(T)) \text{ for } \varphi \in \mathcal{D} \text{ and a subset of patterns } T. \tag{1}$$

For a given statistic st, we say that two subsets T_1 and T_2 are st-*Wilf-equivalent* if and only if the polynomial distributions of st over the sets $S_n(T_1)$ and $S_n(T_2)$ are the same for all integers n. In other word, for every integer n, we have

$$\sum_{\sigma \in S_n(T_1)} x^{\mathrm{st}(\sigma)} = \sum_{\sigma \in S_n(T_2)} x^{\mathrm{st}(\sigma)}.$$

Various statistic-Wilf-equivalence classes for subset of patterns of length three are known in [1, 7, 9, 11, 14]. In particular, Rakotomamonjy [11] provided the Wilf-equivalence classes modulo cr for single pattern of length three. He proved bijectively that the only non singleton class is $\{132, 213, 321\}$, i.e., we have

$$\sum_{\sigma \in S_n(321)} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma \in S_n(132)} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma \in S_n(213)} q^{\mathrm{cr}(\sigma)}. \tag{2}$$

To prove the first identity of (2), he exploited the bijection $\Theta : S_n(321) \to S_n(132)$ exhibited by Elizalde and Pak [8] and proved that Θ is cr-preserving [11, Thm. 3.10]. The second identity of (2) is simply obtained from the fact that the reverse-complement-inverse rci preserves the number of crossings [11, Lem. 4.2]). Using the q,p-Catalan numbers defined by Randrianarivony [13], Rakotomamonjy also proved the following result:

Theorem 1. [11] *Let $\tau \in \{321, 132, 213\}$. The polynomial $F_n(\tau; q) := \sum_{\sigma \in S_n(\tau)} q^{\mathrm{cr}(\sigma)}$ satisfies*

$$F_n(\tau; q) = F_{n-1}(\tau; q) + \sum_{k=0}^{n-2} q^k F_k(\tau; q) F_{n-1-k}(\tau; q).$$

Moreover, we have

$$\sum_{\sigma \in S(\tau)} q^{\mathrm{cr}(\sigma)} z^{|\sigma|} = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{z}{1 - \cfrac{qz}{1 - \cfrac{qz}{1 - \cfrac{q^2 z}{1 - \cfrac{q^2 z}{\ddots}}}}}}}.$$

Burstein and Elizalde found that this continued fraction expansion is the distribution of the statistic number of occurrences of the generalized pattern 31-2 in 231-avoiding permutations [3, Thm. 3.11]. For interested reader, knowing that Corteel [4] established the connection between occurrences of patterns, crossings and nestings on permutations, finding any correspondence between these results may be interesting. Notice also that finding the polynomial distributions of the number of crossings over the sets $S_n(\tau)$, for $\tau \in \{123, 231, 312\}$, remain open. The first result of this paper is the following.

Theorem 2. *We have the following identities:*

$$\sum_{\sigma \in S(231,321)} q^{\mathrm{cr}(\sigma)} z^{|\sigma|} = \frac{1-qz}{1-(1+q)z-(1-q)z^2}, \tag{3}$$

$$\sum_{\sigma \in S(123,\tau)} q^{\mathrm{cr}(\sigma)} z^{|\sigma|} = 1 + \frac{(1-qz)z}{(1-z)(1-(1+q)z)} \text{ for } \tau \in \{132, 213\}. \tag{4}$$

We observe throughout the paper of Bukata et al. [1] that identities (3) and (4) are, respectively, new combinatorial interpretations of the triangles A076791 and A299927 of the On-Line Encyclopedia of Integer Sequences (OEIS) [16]. Bukata et al. interpreted these triangles in terms of number of double descents (ddes) and number of double ascents(dasc) over permutations avoiding some pairs of patterns of length three [1, Prop. 7 and Prop. 11]. The statistics ddes and dasc are, respectively, defined by $\mathrm{ddes}(\sigma) := |\{i|\sigma(i) > \sigma(i+1) > \sigma(i+2)\}|$ and $\mathrm{dasc}(\sigma) := |\{i|\sigma(i) < \sigma(i+1) < \sigma(i+2)\}|$ for a permutation σ. Notice that the triangle A299927 is new in the OEIS and it was first discovered by Bukata et al. .

Let $\tau \in \{132, 213\}$. For an integer $n \geq 1$ and $k \geq 0$, as direct consequence of identity (4), we have

$$|\{\sigma \in S_n(123, \tau)|\mathrm{cr}(\sigma) = k\}| = \delta_{k,0} + \binom{n-1}{k+1}.$$

The next result of this paper concerns various relationships between the distributions of the number of crossing over permutations that avoid the pattern 231 on the first hand and permutations that avoid the pattern 312 on the second hand. For that, we let $F(T; q, z) := \sum_{\sigma \in S(T)} q^{\mathrm{cr}(\sigma)} z^{|\sigma|}$ for a subset of patterns T.

Theorem 3. *We have the following identities:*

$$F(312; q, z) = \frac{1}{1 - zF(231; q, z)},$$

$$F(312, 123; q, z) = 1 + \left(\frac{z}{1-z}\right)^2 + zF(231, 123; q, z),$$

$$\text{and } F(312, \tau; q, z) = 1 + \left(\frac{z}{1-z}\right) F(231, \tau'; q, z) \text{ for } (\tau, \tau') \in \{132, 213\}^2.$$

The aim of this paper is to find the polynomial distributions of the number of crossings over permutations avoiding a pair of patterns in S_3. The tool that we use is not sufficient to treat all cases. However, these relationships we found will obviously reduce the number of the remain cases to be processed.

We organize the rest of this paper in three sections. Section 2 is for notation and preliminary in which we will prove one fundamental proposition that will play a central role in the proof of our results. In Section 3, we will provide the proof of our main results. In Section 4, we will end this paper with two additional results. The first one is about the distributions of the number of crossings over the sets $S_n(321, 213)$ and $S_n(321, 132)$. The second one is about a cr-preserving bijection between S_n^k and S_n^{n+1-k}.

2 Notation and preliminary

Let n be a positive integer. For $k \in [n]$, we write

$$S_n^k := \{\sigma \in S_n | \sigma(k) = 1\}$$
$$S_{n,k} := \{\sigma \in S_n | \sigma(n) = k\}.$$

We let $F_n(T;q)$, $F_n^k(T;q)$ and $F_{n,k}(T;q)$ denote, respectively, the polynomial distributions of cr over the sets $S_n(T)$, $S_n^k(T)$ and $S_{n,k}(T)$, for any subset of patterns T and any integer $k \in [n]$. In particular, we let $F_n(q) := F_n(\emptyset;q)$, $F_n^k(q) := F_n^k(\emptyset;q)$ and $F_{n,k}(q) = F_{n,k}(\emptyset;q)$.

Let m and n be two integers such that $m > 1$. Let $T \subset S_m$ and $k \in [n]$. We also write $T^{-1} := \{\tau^{-1} | \tau \in T\}$ and $T(i) := \{\tau(i) | \tau \in T\}$ for $i \in [m]$. In this section, we will prove the following fundamental proposition that will help us to solve our problems in the next sections.

Proposition 4. *For all integer $n \geq 1$, the following properties hold:*

$$\textit{If } \min T^{-1}(1) > 1, \ \textit{we have } F_n^1(T;q) = F_{n-1}(T;q). \tag{5}$$

$$\textit{If } \min T^{-1}(1) > 2, \ \textit{we have } F_n^2(T;q) = qF_{n-1}(T;q) + (1-q)F_{n-2}(T;q). \tag{6}$$

$$\textit{If } \max T^{-1}(1) < m-1, \ \textit{we have } F_n^{n-1}(T;q) = qF_{n-1}(T^{-1};q) + (1-q)F_{n-1,n-1}(T^{-1};q). \tag{7}$$

$$\textit{If } \max T^{-1}(1) < m, \ \textit{we have } F_n^n(T;q) = F_{n-1}(T^{-1};q). \tag{8}$$

For that, we need some notation to be defined and some lemmas to be proved. So, we let $\sigma \in S_n$. We say that an integer i is an upper transient (resp., lower transient) of σ if and only if $\sigma^{-1}(i) < i < \sigma(i)$ (resp., $\sigma(i) < i < \sigma^{-1}(i)$). The numbers of upper and lower transients of a given permutation σ are denoted, respectively, by $\mathrm{ut}(\sigma)$ and $\mathrm{lt}(\sigma)$. With this definition, we have the following remark.

Remark 5. For any permutation σ, an integer i is a lower transient of σ if and only if $(i, \sigma^{-1}(i))$ is a lower crossing of σ.

For any given integer k, we also let $\mathrm{Ut}_k(\sigma) := \{i < k / \sigma^{-1}(i) < i < \sigma(i)\}$ and $\mathrm{Lt}_k(\sigma) := \{i < k / \sigma(i) < i < \sigma^{-1}(i)\}$ denote, respectively, the sets of upper and lower transients of σ less than k. Also define

$$\begin{aligned} \mathrm{ut}_k^-(\sigma) &:= |\mathrm{Ut}_k(\sigma)| \text{ and } \mathrm{ut}_k^+(\sigma) := \mathrm{ut}(\sigma) - \mathrm{ut}_k^-(\sigma), \\ \mathrm{lt}_k^-(\sigma) &:= |\mathrm{Lt}_k(\sigma)| \text{ and } \mathrm{lt}_k^+(\sigma) := \mathrm{lt}(\sigma) - \mathrm{lt}_k^-(\sigma), \\ \alpha_k(\sigma) &:= |\{i \geq k / \sigma(i) < k\}|. \end{aligned}$$

Observe that in particular we have $\mathrm{ut}_n^-(\sigma) = \mathrm{ut}_{n+1}^-(\sigma) = \mathrm{ut}(\sigma)$ and $\mathrm{lt}_n^-(\sigma) = \mathrm{lt}_{n+1}^-(\sigma) = \mathrm{lt}(\sigma)$, $\alpha_n(\sigma) = 1 - \delta_{n,\sigma(n)}$ and $\alpha_{n+1}(\sigma) = 0$ where δ is the usual Kronecker symbol. Now, let us recall one needed notation introduced by Rakotomamonjy [11]. Given a permutation σ and two integers a and b, we let $\sigma^{(a,b)}$ denote the permutation obtained from σ in the following way:

- add 1 to each number in σ which is greater or equal to b,
- then insert b at the a-th position of the modified σ.

We can simply write $\sigma^{-(a,b)}$ for $(\sigma^{-1})^{(a,b)}$. Example: we have $3142^{(2,3)} = 43152$ and $3142^{-(2,3)} = 23514$. Next, we prove a fundamental lemma which is a particular case of [11, Lem. 3.6].

Lemma 6. *Let $\sigma \in S_n$ and $k \in [n+1]$. We have*

$$\mathrm{cr}(\sigma^{(k,1)}) = \mathrm{cr}(\sigma) + \mathrm{ut}_k^-(\sigma) - \mathrm{lt}_k^-(\sigma) + \alpha_k(\sigma).$$

Proof. Let $\sigma \in S_n$ and $k \in [n+1]$. Firstly, we let $A_k(\sigma)$ (resp., $B_k(\sigma)$, $C_k(\sigma)$) denote the set of all crossings (i, j) of σ such that $j < k$ (resp., $i < k \leq j$, $k \leq i$). We obviously have $\mathrm{cr}(\sigma) = |A_k(\sigma)| + |B_k(\sigma)| + |C_k(\sigma)|$. Let us assume that $\pi = \sigma^{(k,1)}$. By definition, we have

$$\pi(i) = \sigma(i) + 1 \text{ if } i < k, \pi(k) = 1 \text{ and } \pi(i+1) = \sigma(i) + 1 \text{ if } i \geq k.$$

Let (i, j) be a pair of integers such that $i < j$. Based on this definition of π, we will examine the following three cases:

Case 1: Suppose that $j < k$. So we have $\pi(i) = \sigma(i) + 1$ and $\pi(j) = \sigma(j) + 1$.

- Assume that $(i, j) \in A_k(\sigma)$.
  - If $i < j < \sigma(i) < \sigma(j)$, then $i < j < \pi(i) < \pi(j)$ and $(i, j) \in A_k(\pi)$,
  - If $\sigma(i) < \sigma(j) \leq i < j$, then $\begin{cases} \pi(i) < \pi(j) \leq i < j, & \text{if } \sigma(j) < i; \\ \pi(i) \leq i < \pi(j) = i + 1 \leq j, & \text{if } \sigma(j) = i. \end{cases}$

    Thus, we have $\begin{cases} (i, j) \in A_k(\pi), & \text{if } \sigma(j) < i; \\ (i, j) \notin A_k(\pi), & \text{if } \sigma(j) = i. \end{cases}$

- Inversely, if $(i,j) \in A_k(\pi)$, the following properties hold:
  - if $i < j < \pi(i) < \pi(j)$, then $i < j \leq \sigma(i) < \sigma(j)$. So, we have

$$\begin{cases} (i,j) \in A_k(\sigma), & \text{if } \pi(i) > j+1 (\text{i.e., } \sigma(i) > j); \\ (i,j) \notin A_k(\sigma), & \text{if } \pi(i) = j+1 (\text{i.e., } \sigma(i) = j). \end{cases}$$

  - if $\pi(i) < \pi(j) \leq i < j$ then $\sigma(i) < \sigma(j) < i < j$, i.e., $(i,j) \in A_k(\sigma)$.

Consequently, we obtain the following identity

$$|A_k(\sigma)| - |\{i|\sigma(i) < i < \sigma^{-1}(i) < k\}| = |A_k(\pi)| - |\{(i,j) \in A_k(\pi)|i < j < \pi(i) = j+1\}|. \quad (9)$$

Case 2: Suppose that $i < k \leq j$. We have $\pi(i) = \sigma(i) + 1$ and $\pi(j+1) = \sigma(j) + 1$.

- Assume that $(i,j) \in B_k(\sigma)$.
  - If $i < j < \sigma(i) < \sigma(j)$ then $(i, j+1) \in B_k(\pi)$,
  - If $\sigma(i) < \sigma(j) \leq i < j$ then $\begin{cases} (i,j+1) \in B_k(\pi), & \text{if } \sigma(j) < i; \\ (i,j+1) \notin B_k(\pi), & \text{if } \sigma(j) = i. \end{cases}$
- Inversely, if $(i,j) \in B_k(\pi)$,
  - if $i < j < \pi(i) < \pi(j)$, then $j > k$ since $\pi(k) = 1$. Thus, we have $i < j-1 < \sigma(i) < \sigma(j-1)$, i.e., $(i, j-1) \in B_k(\sigma)$,
  - if $\pi(i) < \pi(j) \leq i < j$, then $\sigma(i) < \sigma(j-1) < i < j-1$, i.e., $(i, j-1) \in B_k(\sigma)$.

Consequently, we obtain the following identity

$$|B_k(\sigma)| - |\{i|\sigma(i) < i < k \leq \sigma^{-1}(i)\}| = |B_k(\pi)|. \quad (10)$$

Case 3: Suppose now that $k \leq i < j$. We have $\pi(i+1) = \sigma(i) + 1$ and $\pi(j+1) = \sigma(j) + 1$.

- If $(i,j) \in C_k(\sigma)$, then we have
  - if $i < j < \sigma(i) < \sigma(j)$, then $(i+1, j+1) \in C_k(\pi)$,
  - if $\sigma(i) < \sigma(j) \leq i < j$ then $(i+1, j+1) \in C_k(\pi)$.
- Inversely, if $(i,j) \in C_k(\pi)$, we have
  - if $i < j < \pi(i) < \pi(j)$, then $k > i$ since $\pi(k) = 1$. Thus, we have $k \leq i-1 < j-1 < \sigma(i-1) < \sigma(j-1)$, i.e., $(i-1, j-1) \in C_k(\sigma)$,
  - if $\pi(i) < \pi(j) \leq i < j$, then $\sigma(i-1) < \sigma(j-1) \leq i-1 < j-1$. So, we get $\begin{cases} (i-1,j-1) \in C_k(\sigma), & \text{if } i > k; \\ (i-1,j-1) \notin C_k(\sigma), & \text{if } i = k. \end{cases}$

Similarly to the previous cases, we obtain

$$|C_k(\sigma)| = |C_k(\pi)| - |\{j > k|\pi(j) \leq k\}|. \quad (11)$$

By summing equations (9), (10) and (11), using the facts that $|\{i|\sigma(i) < i < \sigma^{-1}(i) < k\}| + |\{i|\sigma(i) < i < k \le \sigma^{-1}(i)\}| = \mathrm{lt}_k^-(\sigma)$, $|\{(i,j) \in A_k(\pi)|i < j < \pi(i) = j+1\}| = |\{j < k|\sigma^{-1}(j) < j < \sigma(j)\}| = \mathrm{ut}_k^-(\sigma)$ and $|\{j > k|\pi(j) \le k\}| = |\{j \ge k|\sigma(j) < k\}| = \alpha_k(\sigma)$, we get

$$\mathrm{cr}(\sigma) - \mathrm{lt}_k^-(\sigma) = \mathrm{cr}(\pi) - \mathrm{ut}_k^-(\sigma) - \alpha_k(\sigma). \tag{12}$$

We deduce from (12) the desired identity of our lemma. □

Lemma 7. *Let σ be a given permutation. If $\pi = \sigma^{-1}$ or $\mathrm{rc}(\sigma)$ then we have*

$$\mathrm{cr}(\pi) = \mathrm{cr}(\sigma) + \mathrm{ut}(\sigma) - \mathrm{lt}(\sigma).$$

Proof. Let $\sigma \in S_n$ and $\pi = \sigma^{-1}$ or $\mathrm{rc}(\sigma)$. Noticing that i or rc are simple symmetries on the arc diagram, they exchange lower and upper arcs including of course transients. Thus, we have $\mathrm{ut}(\pi) = \mathrm{lt}(\sigma)$ and $\mathrm{lt}(\pi) = \mathrm{ut}(\sigma)$. By this fact, Remark 5 explains how we get $\mathrm{cr}(\pi) = \mathrm{cr}(\sigma) + \mathrm{ut}(\sigma) - \mathrm{lt}(\sigma)$ and we complete the proof of our lemma. □

Let n be an integer and $k \in [n]$. Let us now define a bijection $\Phi_{n,k}$ as follows

$$\begin{aligned}\Phi_{n,k} : S_{n-1} &\longrightarrow S_n^k \\ \sigma &\longmapsto \sigma^{-(k,1)}.\end{aligned}$$

The properties of this bijection allow us to get some relations between F_n^{n-1}, F_n^n and F_n in Proposition 9 and we use its restricted version to prove Proposition 4.

Proposition 8. *The bijection $\Phi_{n,n}$ preserves the number of crossings and, for any $\sigma \in S_{n-1}$, we have*

$$\mathrm{cr}(\Phi_{n,n-1}(\sigma)) = \begin{cases} \mathrm{cr}(\sigma), & \text{if } \sigma(n-1) = n-1; \\ \mathrm{cr}(\sigma) + 1, & \text{if } \sigma(n-1) < n-1. \end{cases}.$$

Proof. Combining Lemma 6 and Lemma 7, it is not difficult to see that, for any $\sigma \in S_{n-1}$, we have

$$\mathrm{cr}(\sigma^{-(n,1)}) = \mathrm{cr}(\sigma) \text{ and } \mathrm{cr}(\sigma^{-(n-1,1)}) = \mathrm{cr}(\sigma) + 1 - \delta_{n-1,\sigma(n-1)}. \tag{13}$$

The proposition comes from (13). □

Let $\alpha \oplus \beta$ denote the *direct sum* of the two given permutations α and β defined as follows

$$\alpha \oplus \beta(i) = \begin{cases} \alpha(i), & \text{if } i \le |\alpha|; \\ |\alpha| + \beta(i - |\alpha|), & \text{if } i > |\alpha|. \end{cases}$$

Example: $1432 \oplus 4231 = 14328675$. An obvious property of the direct sum that we need is $\mathrm{cr}(\alpha \oplus \beta) = \mathrm{cr}(\alpha) + \mathrm{cr}(\beta)$ for any permutations α and β.

Proposition 9. *Let n be a non-negative integer. The following recurrences hold*

$$F_n^n(q) = F_{n-1}(q) \text{ for } n \geq 1,$$
$$\text{and } F_n^{n-1}(q) = qF_{n-1}(q) + (1-q)F_{n-2}(q) \text{ for } n \geq 2.$$

Proof. Since the bijection $\Phi_{n,n}$ is cr-preserving, we have $F_n^n(q) = F_{n-1}(q)$. Now, using the property of the bijection $\Phi_{n,n-1}$, we get

$$\begin{aligned} F_n^{n-1}(q) &= q \times \sum_{\substack{\sigma \in S_{n-1}, \\ \sigma(n-1) \neq n-1}} q^{\mathrm{cr}(\sigma)} + \sum_{\substack{\sigma \in S_{n-1}, \\ \sigma(n-1) = n-1}} q^{\mathrm{cr}(\sigma)} \\ &= q\left(F_{n-1}(q) - F_{n-1,n-1}(q)\right) + F_{n-1,n-1}(q) \end{aligned}$$

Since $F_{n,n}(q) = \sum_{\sigma \oplus 1 \in S_n} q^{\mathrm{cr}(\sigma \oplus 1)} = \sum_{\sigma \in S_{n-1}} q^{\mathrm{cr}(\sigma)} = F_{n-1}(q)$ for all $n \geq 1$, we consequently obtain

$$F_n^{n-1}(q) = qF_{n-1}(q) + (1-q)F_{n-2}(q) \text{ for all } n \geq 1.$$

This ends the proof of the proposition. □

We may observe that Proposition 4 is none other than a restricted version of Proposition 9. In fact, the effect of restriction totally changes the obtained relations. For example, we have $F_n^n(321; q) = 1 \neq F_{n-1}(321; q)$. Now, we can give the poof of Proposition 4.

Proof. Our proof is simply based on the following obvious fact. Let T be a subset of S_m for any integer $m > 1$. For any integer $n \geq m$, we have

(i) If $k < \min T^{-1}(1)$, we have $\sigma^{(k,1)} \in S_n^k(T)$ if and only if $\sigma \in S_{n-1}(T)$.
(ii) If $n - m + \max T^{-1}(1) < k \leq n$, we have $\sigma^{-(k,1)} \in S_n^k(T)$ if and only if $\sigma \in S_{n-1}(T^{-1})$.

The two first relations (5) and (6) of Proposition 4 use the (i) of the fact. If $\min T^{-1}(1) \neq 1$, then we have $1 \oplus \sigma \in S_n^1(T)$ if and only if $\sigma \in S_{n-1}(T)$ for any $n \geq 1$. Thus we get relation (5) as follows

$$F_n^1(T; q) = \sum_{1 \oplus \sigma \in S_n^1(T)} q^{\mathrm{cr}(1 \oplus \sigma)} = \sum_{\sigma \in S_{n-1}(T)} q^{\mathrm{cr}(\sigma)} = F_{n-1}(T; q).$$

By the same way, if $\min T^{-1}(1) > 2$, we have $\sigma^{(2,1)} \in S_n^2(T)$ if and only if $\sigma \in S_{n-1}(T)$ for any $n \geq 1$. Moreover, we have $\mathrm{cr}(\sigma^{(2,1)}) = \mathrm{cr}(\sigma) + 1 - \delta_{1,\sigma(1)}$ for any permutation σ (see Lemma 6). By applying (5), we also get (6) as follows

$$\begin{aligned} F_n^2(T; q) &= q \times \sum_{\substack{\sigma \in S_{n-1}(T), \\ \sigma(1) \neq 1}} q^{\mathrm{cr}(\sigma)} + \sum_{\substack{\sigma \in S_{n-1}(T), \\ \sigma(1) = 1}} q^{\mathrm{cr}(\sigma)} \\ &= q\left(F_{n-1}(T; q) - F_{n-1}^1(T; q)\right) + F_{n-1}^1(T; q) \\ &= qF_{n-1}(T; q) + (1-q)F_{n-2}(T; q). \end{aligned}$$

For the two last relations (7) and (8) of the proposition, we obviously use the (ii) of the fact and we also exploit the bijections $\Phi_{n,n}$ and $\Phi_{n,n-1}$. If $\max T^{-1}(1) < m-1$ (i.e., $n - m + \max T^{-1}(1) < n-1$), we have $\sigma^{-(n-1,1)} \in S_n^{n-1}(T)$ if and only if $\sigma \in S_{n-1}(T^{-1})$. This implies that we have $\Phi_{n,n-1}(S_{n-1}(T^{-1})) = S_n^{n-1}(T)$. Using the property of the bijection $\Phi_{n,n-1}$ described in Theorem 8, we get (7) as follows

$$\begin{aligned} F_n^{n-1}(T;q) &= q \times \sum_{\substack{\sigma \in S_{n-1}(T^{-1}),\\ \sigma(n-1)\neq n-1}} q^{\mathrm{cr}(\sigma)} + \sum_{\substack{\sigma \in S_{n-1}(T^{-1}),\\ \sigma(n-1)= n-1}} q^{\mathrm{cr}(\sigma)} \\ &= q\left(F_{n-1}(T^{-1};q) - F_{n-1,n-1}(T^{-1};q)\right) + F_{n-1,n-1}(T^{-1};q) \\ &= qF_{n-1}(T^{-1};q) + (1-q)F_{n-1,n-1}(T^{-1};q). \end{aligned}$$

Notice that we generally have $F_{n,n}(T;q) \neq F_{n-1}(T;q)$ since the set $S_{n,n}(T)$ depends on T. By the same way we obtain the last relation (8) using the cr-preserving of the bijection $\Phi_{n,n}$. This ends the proof of Proposition 4. □

Let us end this preliminary section with some illustrations of Proposition 4. Since $\max\{321\}^{-1}(1) = 3 > 2$, by applying (6) we get

$$F_n^2(321;q) = qF_{n-1}(321;q) + (1-q)F_{n-2}(321;q) \text{ for } n \geq 2.$$

Since $123^{-1} = 123$ and $\max\{123\}^{-1}(1) = 1 < 2$, we can also apply (7) and get

$$F_n^{n-1}(123;q) = qF_{n-1}(123;q) + (1-q)F_{n-1,n-1}(123;q).$$

Since $S_{n,n}(123) = \{(n-1)\cdots 21n\}$, then we have $F_{n,n}(123;q) = 1$ and we consequently obtain

$$F_n^{n-1}(123;q) = qF_{n-1}(123;q) + 1 - q.$$

3 Proof of the main results

In this section, we will establish the proof of our results presented in Section 1. As fundamental tools, we use the Proposition 4 proved in the preceding section and the cr-preserving of the involution rci proved by Rakotomamonjy [11]. For that, we let $F(T;q,z) := \sum_{\sigma\in S(T)} q^{\mathrm{cr}(\sigma)} z^{|\sigma|}$ for any set of patterns T.

3.1 Proof of Theorem 2

Proof. It is obvious to see that we have $S_n(321,231) = S_n^1(321,231) \cup S_n^2(321,231)$ for all n. So we get

$$F_n(321,231;q) = F_n^1(321,231;q) + F_n^2(321,231;q).$$

Since $\min\{321, 231\}^{-1}(1) = 3 > 2$, we can apply the relations (5) and (6) of proposition 4 and we get

$$F_n(321, 231; q) = (1 + q)F_{n-1}(321, 231; q) + (1 - q)F_{n-2}(321, 231; q), \text{ for } n \geq 2. \quad (14)$$

Recurrence (14) is associated with the following functional equation

$$F(321, 231; q, z) = 1 + z + (1 + q)z(F(321, 231; q, z) - 1) + (1 - q)z^2 F(321, 231; q, z).$$

Solving it by $F(321, 231; q, z)$, we obtain the following identity equivalent to identity (3) of Theorem 2:

$$F(321, 231; q, z) = \frac{1 - qz}{1 - (1 + q)z - (1 - q)z^2}.$$

As structure, we also have $S_n(123, 132) = S_n^{n-1}(123, 132) \cup S_n^n(123, 132)$. Since we have $\max\{123, 132\}^{-1}(1) = 1 < 2$, we can also apply the two relations (7) and (8) of Proposition 4. Thus, since $\{123, 132\}^{-1} = \{123, 132\}$, we get from (7)

$$F_n^n(123, 132; q) = F_{n-1}(123, 132; q). \quad (15)$$

Moreover, since $F_{n,n}(123, 132; q) = 1$, we get from (8)

$$F_n^{n-1}(123, 132; q) = qF_{n-1}(123, 132; q) + 1 - q. \quad (16)$$

Summing (15) and (16), we obtain the following recurrence:

$$F_n(123, 132; q) = (1 + q)F_{n-1}(123, 132; q) + 1 - q \text{ for } n \geq 2. \quad (17)$$

Recurrence (17) corresponds to the following functional equation:

$$F(123, 132; q, z) = 1 + z + (1 + q)z(F(123, 132; q) - 1) + z\left(\frac{1}{1 - z} - 1 - z\right).$$

When solving this last equation by $F(123, 132; q, z)$, we obtain

$$F(123, 132; q, z) = 1 + \frac{z(1 - qz)}{(1 - z)(1 - (1 + q)z)}.$$

Finally, since $\{123, 213\} = \mathrm{rci}(\{123, 132\})$, we also have $F(123, 132; q, z) = F(123, 213; q, z)$. This completes the proof of identity (4) of Theorem 2 and Theorem 2 itself. □

Notice that when we solve (17) with the initial condition $F_1(123, 132; q) = 1$, we obtain the following closed form:

$$\sum_{\sigma \in S_n(123,\tau)} q^{\mathrm{cr}(\sigma)} = \frac{(1 + q)^{n-1} - 1 + q}{q} \text{ for } n \geq 1 \text{ and } \tau \in \{132, 213\}.$$

Furthermore, when we substitute $F_{n-1}(123,132;q)$ by $\frac{(1+q)^{n-2}-1+q}{q}$ for $n \geq 2$, we also get from (16)

$$\sum_{\sigma \in S_n^{n-1}(123,132)} q^{\mathrm{cr}(\sigma)} = (1+q)^{n-2} \text{ for } n \geq 2.$$

Since $\mathrm{rci}(\mathrm{S_n^{n-1}}(123,132)) = \mathrm{S_{n,2}}(123,213)$, we also get

$$\sum_{\sigma \in S_{n,2}(123,213)} q^{\mathrm{cr}(\sigma)} = (1+q)^{n-2} \text{ for } n \geq 2.$$

Corollary 10. *For $n \geq 2$ and $k \geq 0$, we have*

$$|\{\sigma \in S_n^{n-1}(123,132)|\mathrm{cr}(\sigma) = k\}| = |\{\sigma \in S_{n,2}(123,213)|\mathrm{cr}(\sigma) = k\}| = \binom{n-2}{k}.$$

We observe that Corollary 10 is a new combinatorial interpretation of the Pascal triangle A007318 in terms of crossings over restricted permutations. Finding a bijection with subsets of a given size to get a direct proof of Corollary 10 may be interesting and staying open.

3.2 Proof of Theorem 3

In this subsection, we will establish the proof of the result concerning some relationships between the distributions of crossings over the sets $S_n(312,T)$ and $S_n(231,T)$, where T is empty or a singleton of $\{123,132,213\}$. As we did in the preceding subsection, we will first find recurrences and we then compute the corresponding generating functions to get the desired relations.

Proposition 11. *For all integer $n \geq 1$, we have*

$$F_n(312;q) = \sum_{j=0}^{n-1} F_j(231;q)F_{n-1-j}(312;q). \tag{18}$$

Proof. We have $S_n^j(312) = \{\sigma_1 \oplus \sigma_2 | \sigma_1 \in S_j^j(312), \sigma_2 \in S_{n-j}(312)\}$ for all $j \geq 1$. So, we get using (8) the following identities:

$$F_n^j(312;q) = F_j^j(312;q)F_{n-j}(312;q) = F_{j-1}(231;q)F_{n-j}(312;q), \text{ for } 1 \leq j \leq n.$$

By summing $F_n^j(312;q)$ over $j \in [n]$, we obtain the desired relationship for $F_n(312;q)$. □

Proposition 12. *For all integer $n \geq 2$, we have*

$$F_n(123,312;q) = n - 1 + F_{n-1}(123,231;q). \tag{19}$$

Proof. We have $S_n(123,312) = \{\pi_1, \pi_2, \ldots, \pi_{n-1}\} \cup S_n^n(123,312)$ with $\pi_j = j \cdots 21n(n-1)\cdots(j+1)$ for all $j \in [n]$. So we get

$$F_n(123,312;q) = \sum_{j=1}^{n-1} q^{\mathrm{cr}(\pi_j)} + F_n^n(123,312;q).$$

It is not difficult to see that we have $\mathrm{cr}(\pi_j) = 0$ for all $j \in [n]$. Thus, we immediately obtain the proposition using again (8). □

Proposition 13. *For any τ_1, τ_2 and $\tau_3 \in \{132, 213\}$ and for all $n \geq 2$, we have*

$$F_n(312,\tau_1;q) = F_{n-1}(312,\tau_2;q) + F_{n-1}(231,\tau_3;q). \tag{20}$$

Proof. Since $S_n(312,213) = S_n^1(312,213) \cup S_n^n(312,213)$, we get

$$F_n(312,213;q) = F_n^1(312,213;q) + F_n^n(231,213;q).$$

So for all $n \geq 2$, we get from (5) and (6) the following identity:

$$F_n(312,213;q) = F_{n-1}(312,213;q) + F_{n-1}(231,213;q).$$

To complete the proof of the proposition, we just use the facts that $\mathrm{rci}(\{312,132\}) = \{312,213\}$ and $\mathrm{rci}(\{231,132\}) = \{231,213\}$. □

Now, to prove Theorem 3, we just compute the corresponding generating functions of the three recurrences (18), (19) and (20) and deduce the desired relations.

Proof. From (18), we obtain the functional equation

$$F(312;q,z) = 1 + zF(312;q,z).F(231;q,z)$$

which leads to

$$F(312;q,z) = \frac{1}{1 - zF(231;q,z)}.$$

The associated generating function with (19) is

$$F(123,312;q,z) = 1 + z + \left(\frac{z}{1-z}\right)^2 + z(F(123,231;q,z) - 1).$$

This functional equation is equivalent to the following one:

$$F(312,123;q,z)1 + \left(\frac{z}{1-z}\right)^2 + zF(231,123;q,z).$$

From (20), when we set $\tau = \tau_1 = \tau_2$ and $\tau' = \tau_3$, we get the functional equation

$$F(312,\tau;q,z) = 1 + z + z\left(F(312,\tau;q,z) + F(231,\tau';q,z) - 2\right).$$

Solving it for $F(312,\tau;q,z)$, we obtain

$$F(312,\tau;q,z) = 1 + \left(\frac{z}{1-z}\right) F(231,\tau';q,z) \text{ for any } (\tau,\tau') \in \{132,213\}^2.$$

This completes the proof of Theorem 3. □

4 Additional results

We end this paper with two additional results. The first one is about $F_n(321, \tau; q)$, with $\tau \in \{213, 132\}$. The second one is inspired from the first section and is about a cr-preserving bijection between S_n^k and S_n^{n+1-k}.

For the first result, we remark that the distribution of cr over the set of permutations avoiding one of the pairs $\{321, 213\}$ and $\{321, 132\}$ can be computed. One of the tools that we may use is an interesting relationship proved by Randrianarivony [13]. He showed how the statistic cr is related to other usual statistics through the following identity:

$$\mathrm{cr}(\sigma) = \mathrm{inv}(\sigma) - \mathrm{exc}(\sigma) - 2\mathrm{nes}(\sigma), \tag{21}$$

where, for any permutation σ, $\mathrm{inv}(\sigma) := |\{(i,j)|i < j \text{ and } \sigma(i) > \sigma(j)\}|$, $\mathrm{exc}(\sigma) := |\{i|\sigma(i) > i\}|$ and $\mathrm{nes}(\sigma) := |\{(i,j)|i < j < \sigma(j) < \sigma(i) \text{ or } \sigma(j) < \sigma(i) \leq i < j\}|$ are respectively the numbers of inversions, excedances and nestings of σ. Below is an unexpected result in which we try to use identity (21) to get the proof.

Theorem 14. *Let $[n]_q = 1 + q + \cdots + q^{n-1}$ for any integer $n \geq 1$. For any $\tau \in \{132, 213\}$, we have*

$$\sum_{\sigma \in S_n(321,\tau)} q^{\mathrm{cr}(\sigma)} = 1 + \sum_{k=1}^{n-1} [n-k]_{q^k}.$$

Proof. It is easy to see that we have $S_n(321, 213) = S_n^1(321, 213) \cup \{\alpha_2, \alpha_3, \ldots, \alpha_n\}$ where $\alpha_j = (n-j+2)\cdots(n-1)n12\cdots(n+1-j)$ for all $j \in [n]$. From this structure, we get

$$F_n(321, 213; q) = F_n^1(321, 213; q) + \sum_{j=2}^{n} q^{\mathrm{cr}(\alpha_j)}.$$

Since every 321-avoiding permutations are nonesting [11, Lem. 5.1], we have

$$\mathrm{cr}(\alpha_j) = \mathrm{inv}(\alpha_j) - \mathrm{exc}(\alpha_j) = (j-1)(n-j) \text{ for all } j.$$

Using the fact that $F_n^1(321, 213; q) = F_{n-1}(321, 213; q)$, we obtain

$$F_n(321, 213; q) = F_{n-1}(321, 213; q) + \sum_{j=2}^{n} q^{(j-1)(n-j)}.$$

When we solve this recurrence with the initial condition $F_1(321, 213; q) = 1$, we obtain

$$F_n(321, 213; q) = 1 + \sum_{k=1}^{n-1} \sum_{j=1}^{k} q^{j(k-j)} = 1 + \sum_{k=1}^{n-1} [n-k]_{q^k}.$$

From the fact that $F_n(321, 213; q) = F_n(321, 132; q)$ since $\{321, 132\} = \mathrm{rci}(\{321, 213\})$, we complete the proof of the theorem. □

For the second additional result, we notice first that we have $S_n^k = \{\sigma^{(k,1)} | \sigma \in S_{n-1}\}$. We will show that the following well-defined and bijective map preserves the number of crossings:

$$\begin{aligned} \Psi_{n,k} : S_n^k &\longrightarrow S_n^{n+1-k} \\ \sigma^{(k,1)} &\longmapsto \mathrm{rc}(\sigma)^{(n+1-k,1)}. \end{aligned}$$

Theorem 15. *The bijection $\Psi_{n,k}$ preserves the number of crossings for $1 \le k \le n$.*

Proof. Let $\sigma^{(k,1)} \in S_n^k$ and $\pi^{(n+1-k,1)} = \Psi_{n,k}(\sigma^{(k,1)})$ for $\sigma \in S_{n-1}$. Knowing that rc exchanges lower and upper arcs, it is not difficult to see that we have

$$\mathrm{ut}_{n+1-k}^-(\pi) = \mathrm{lt}_k^+(\sigma) \text{ and } \mathrm{lt}_{n+1-k}^-(\pi) = \mathrm{ut}_k^+(\sigma). \tag{22}$$

Moreover, since $|\{i < k/\sigma(i) \ge k\}| = |\{i \ge k/\sigma(i) < k\}|$, we get

$$\alpha_{n+1-k}(\pi) = \alpha_k(\sigma). \tag{23}$$

Indeed, we have

$$\begin{aligned} \alpha_{n+1-k}(\pi) &= |\{n-i \ge n+1-k/\pi(n-i) < n+1-k\}|, \\ &= |\{i \le k-1/n-\sigma(i) < n+1-k\}|, \\ &= |\{i < k/\sigma(i) > k-1\}|, \\ &= |\{i < k/\sigma(i) \ge k\}|, \\ &= \alpha_k(\sigma). \end{aligned}$$

Consequently, combining (22) and (23) with Lemma 6 and Lemma 7, we get

$$\begin{aligned} \mathrm{cr}(\pi^{(n+1-k,1)}) &= \mathrm{cr}(\pi) + \mathrm{ut}_{n+1-k}^-(\pi) - \mathrm{lt}_{n+1-k}^-(\pi) + \alpha_{n+1-k}(\pi), \\ &= \mathrm{cr}(\sigma) + \mathrm{ut}(\sigma) - \mathrm{lt}(\sigma) + \mathrm{lt}_k^+(\sigma) - \mathrm{ut}_k^+(\sigma) + \alpha_k(\sigma), \\ &= \mathrm{cr}(\sigma) + \big(\mathrm{ut}(\sigma) - \mathrm{ut}_k^+(\sigma)\big) - \big(\mathrm{lt}(\sigma) - \mathrm{lt}_k^+(\sigma)\big) + \alpha_k(\sigma), \\ &= \mathrm{cr}(\sigma) + \mathrm{ut}_k^-(\sigma) - \mathrm{lt}_k^-(\sigma) + \alpha_k(\sigma), \\ &= \mathrm{cr}(\sigma^{(k,1)}). \end{aligned}$$

This proves the cr-preserving of the bijection $\Psi_{n,k}$ and also ends the proof of Theorem 15. □

Corollary 16. *For any integers n and $k \in [n]$, we have the following equidistributions:*

$$\sum_{\sigma \in S_{n,k}} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma \in S_n^{n+1-k}} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma \in S_n^k} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma \in S_{n,n+1-k}} q^{\mathrm{cr}(\sigma)}.$$

Proof. We have $S_n^{n+1-k} = \Psi_{n,k}(S_n^k)$ and $S_{n,n+1-k} = \mathrm{rci}(S_n^k)$ for any $k \in [n]$. So we get these identities from the facts that the bijections $\Psi_{n,k}$ and rci are cr-preserving. □

Corollary 17. *The number of permutations of $[2n]$ having r crossings is always even for all integers $n \ge 1$ and $r \ge 0$.*

Proof. The number of permutations of $[2n]$ having r crossings is $[q^r]F_{2n}(q)$ (i.e., the coefficient of the polynomial $F_{2n}(q)$), where $F_{2n}(q) = \sum_{k=1}^{2n} F_{2n}^k(q) = 2\sum_{k=1}^{n} F_{2n}^k(q)$. □

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References

[1] M. Bukata, R. Kulwicki, N. Lewandowski, L. Pudwell, J. Roth, and T. Wheeland, Distributions of statistics over pattern-avoiding permutations, *J. Integer Sequences* **22** (2019) Article 19.2.6.

[2] S. Burrill, M. Mishna, and J. Post, On k-crossing and k-nesting of permutations, *Proc. 22nd International Conf. on Formal Power Series and Algebraic Combinatorics (FPSAC 2010)*, *DMTCS Proc.*, Vol. AN (2010), 593–600. Available at `https://dmtcs.episciences.org/2873/`.

[3] A. Burstein and S. Elizalde, Total occurrence statistics on restricted permutations, *Pure Math. Appl.* **24** (2013), 103–123.

[4] S. Corteel, Crossing and alignments of permutations, *Adv. Appl. Math.* **38** (2007) 149–163.

[5] S. Corteel, M. Josuat-Vergès, and J. S. Kim, Crossings of signed permutations and q-Eulerian numbers of type B, *J. Comb.* **4** (2013) 191–228.

[6] A. de Médicis and X. G. Viennot, Moments des q-polynômes de Laguerre et la bijection de Foata-Zeilberger, *Adv. Appl. Math.* **15** (1994) 262–304.

[7] T. Dokos, T. Dwyer, B. P. Johnson, Bruce E. Sagan, and K. Selsor. Permutation patterns and statistics. *Discrete Math.* **312** (2012) 2760–2775.

[8] S. Elizalde and I. Pak, Bijections for refined restricted permutations, *J. Combin. Theory Ser. A* **105** (2004) 207–219.

[9] S. Elizalde, Multiple pattern-avoidance with respect to fixed points and excedances, *Electron. J. Combin.* **11** (2004) #R51.

[10] D. Knuth, *The Art of Computer Programming*, Vol. 3, Addison-Wesley, 1973.

[11] P. M. Rakotomamonjy, Restricted permutations refined by number of crossings and nestings, *Discrete Math.* **343** (2020) 111950.

[12] A. Randrianarivony, Fractions continues, q-nombres de Catalan et q-polynômes de Genocchi, *European J. Combin.* **18** (1997), 75–92.

[13] A. Randrianarivony, q, p-analogues des nombres de Catalan, *Discrete Math.* **178** (1998), 199–211.

[14] A. Robertson, D. Saracino, and D. Zeilberger, Refined restricted permutations, *Ann. Comb.* **6** (2003), 427–444.

[15] R. Simion and F. Schmidt, Restricted permutations, *European J. Combin.* **6** (1985), 383–406.

[16] N. J. A. Sloane et al., *The On-line Encyclopedia of Integer Sequences*, published electronically at https://oeis.org, 2019.

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