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A (p, q)-Deformed Recurrence for the Bell Numbers

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Abstract

We obtain a (p, q)-deformation of the recurrence formula for the Bell numbers, using algebraic techniques. Specializing to the case $p = 1$ and the case $p = q = 1$, respectively, we recover the generalized recurrence formula for Bell numbers as obtained by Katriel and Spivey, in related papers.

1 Introduction

The Bell numbers, denoted by B_n, are given by the following formula:

$$B_n = \sum_{k=0}^{n} S(n, k), \tag{1}$$

where $S(n, k)$ are the Stirling numbers of the second kind, which appear as coefficients in the expansion of

$$x^n = \sum_{k=0}^{n} S(n, k) \prod_{i=0}^{k-1} (x - i).$$

The Stirling numbers of the second kind $S(n, k)$ count the number of ways to partition a set of size n into k nonempty subsets.

The Bell numbers (1) satisfy the following recursive formula:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{2}$$

In 2008, Spivey [11] obtained the following generalization of the recurrence formula (2):

$$B_{n+m} = \sum_{j=0}^{n} \sum_{k=0}^{m} k^{n-j} S(m,k) \binom{n}{j} B_j, \tag{3}$$

using techniques from combinatorics. This formula is known in the literature as "Spivey's Bell number formula". Subsequently, Katriel [6] proved a q-deformed version of Spivey's Bell number formula using algebraic techniques. The resulting formula can be written as follows:

$$B_{n+m}(q) = \sum_{k=0}^{m} \sum_{j=0}^{n} \binom{n}{j} S_q(m,k) [k]_q^{n-j} q^{jk} B_j(q), \tag{4}$$

where

$$S_q(m,k) = q^{k-1} S_q(m-1,k-1) + [k]_q S_q(m-1,k), \tag{5}$$

are the q-Stirling numbers of the second kind with the initial value $S_q(0,0) = 1$, and $[k]_q = \frac{1-q^k}{1-q}$. Here,

$$B_n(q) = \sum_{k=0}^{n} S_q(n,k) \tag{6}$$

are the q-Bell numbers.

An alternative derivation of (4) was obtained by Mangontarum [7], using techniques based on the analysis of the creation, annihilation and number operators in the q-Boson-Fock space [1]. It is worthwhile to mention that, Eq. (1) has also been extended in several ways by several authors, including various generalizations of Stirling and Bell numbers. We refer the reader to [3, 5, 9, 12] for the details.

In the present paper, we follow the techniques by Katriel [6], and obtain a (p,q)-deformed version of Spivey's Bell number formula (3), using algebraic methods.

2 A (p,q)-deformation of Spivey's Bell number formula

For $0 < q < p \leq 1$, define the following operators:

1. The operator X of multiplication by the variable x:

$$Xf(x) = xf(x).$$

2. The (p,q)-derivative operator:

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{x(p-q)}.$$

3. The Fibonacci operator [8]:

$$N_p f(x) = f(px).$$

These operators satisfy the (p,q)-commutation relation:

$$D_{p,q}X - qXD_{p,q} = N_p. \tag{7}$$

We can check easily that

$$N_p X = pXN_p, \tag{8}$$

and

$$D_{p,q}N_p = pN_pD_{p,q}. \tag{9}$$

Proposition 1. *For any positive integer n, we have*

$$D_{p,q}X^n = q^n X^n D_{p,q} + [n]_{p,q}X^{n-1}N_p, \tag{10}$$

where $[n]_{p,q} = \frac{p^n - q^n}{p-q}$.

Proof. The proof follows from an easy computation involving an induction on n, and using Eqs. (7) and (8). □

Corollary 2. *One can rewrite* (10) *as follows:*

$$(XD_{p,q})X^n = X^n\big([n]_{p,q}N_p + q^n(XD_{p,q})\big). \tag{11}$$

Proposition 3. *Let n be a nonnegative integer, then the following relation holds:*

$$(XD_{p,q})^n = \sum_{k=0}^{n} S_{p,q}(n,k)X^k N_p^{n-k} D_{p,q}^k, \tag{12}$$

where

$$S_{p,q}(n,k) = p^{n-k}q^{k-1}S_{p,q}(n-1,k-1) + [k]_{p,q}S_{p,q}(n-1,k), \tag{13}$$

are the (p,q)-Stirling numbers of the second kind with the initial value $S_{p,q}(0,0) = 1$, and consequently the numbers

$$B_n(p,q) = \sum_{k=0}^{n} S_{p,q}(n,k), \tag{14}$$

can be considered as the (p,q)-Bell numbers.

Proof. The proof follows by induction with respect to n, using Eqs. (7), (9), and (11). □

Remark 4. For $p=1$, Eqs. (13) and (14) reduce to the q-Stirling numbers of the second kind (5) and q-Bell numbers (6), respectively.

Definition 5. The (p,q)-exponential function is defined by the formula:

$$e_{p,q}(x) := \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!},$$

where $[n]_{p,q}! = \prod_{i=1}^{n} [i]_{p,q}$, and $[0]_{p,q}! = 1$.

Sadjang [10] showed that the (p,q)-exponential function $e_{p,q}(x)$ satisfies the following differential equation:

$$D^n_{p,q} e_{p,q}(x) = p^{\binom{n}{2}} e_{p,q}(p^n x), \tag{15}$$

where $D^n_{p,q}$ is the nth (p,q)-derivative.

Applying the (p,q)-exponential function $e_{p,q}(x)$ to both sides of Eq. (12), and dividing by $e_{p,q}(p^n x)$, we arrive at the identity:

$$\frac{1}{e_{p,q}(p^n x)} (XD_{p,q})^n e_{p,q}(x) = \sum_{k=0}^{n} \tilde{S}_{p,q}(n,k) x^k = \tilde{B}_n(p,q;x), \tag{16}$$

where $\tilde{S}_{p,q}(n,k) = p^{\binom{k}{2}} S_{p,q}(n,k)$ can be considered as new versions of the (p,q)-Stirling numbers of the second kind, with $\tilde{B}_n(p,q;x)$ being the associated (p,q)-Bell polynomials, which give for $x=1$ the new version of the (p,q)-Bell numbers $\tilde{B}_n(p,q;1) = \sum_{k=0}^{n} \tilde{S}_{p,q}(n,k)$. Moreover, putting $x=-1$ in the Equation (16), we obtain (p,q)-Rényi numbers $R_n(p,q) = \sum_{k=0}^{n} (-1)^k \tilde{S}_{p,q}(n,k)$ which reduce to the usual q-Rényi number [4] for $p=1$.

3 Main result

We are now in a position to prove the main result of this paper.

Theorem 6. *Let m and n be nonnegative integers. Then we have the following (p,q)-deformation of Spivey's Bell number formula:*

$$\tilde{B}_{n+m}(p,q;1) = \sum_{k=0}^{m} \sum_{j=0}^{n} \binom{n}{j} \tilde{S}_{p,q}(m,k) [k]^{n-j}_{p,q} q^{jk} \tilde{B}_j(p,q;p^{n+m-j}). \tag{17}$$

Proof. Using Eqs. (12) and (11), we obtain

$$\begin{aligned}
(XD_{p,q})^{n+m} &= \sum_{k=0}^{m} S_{p,q}(m,k)(XD_{p,q})^n X^k N_p^{m-k} D_{p,q}^k \\
&= \sum_{k=0}^{m} S_{p,q}(m,k) X^k \left([k]_{p,q} N_p + q^k (XD_{p,q})\right)^n N_p^{m-k} D_{p,q}^k \\
&= \sum_{k=0}^{m}\sum_{j=0}^{n} \binom{n}{j} S_{p,q}(m,k)[k]_{p,q}^{n-j} q^{jk} X^k N_p^{n-j} (XD_{p,q})^j N_p^{m-k} D_{p,q}^k,
\end{aligned}$$

where in the last equality, we have used the binomial expansion. Applying the above identity to the (p,q)-exponential function $e_{p,q}(x)$, and using Eqs. (15) and (16), we get the following:

$$\begin{aligned}
(XD_{p,q})^{n+m} e_{p,q}(x) &= \sum_{k=0}^{m}\sum_{j=0}^{n} \binom{n}{j} S_{p,q}(m,k)[k]_{p,q}^{n-j} q^{jk} X^k N_p^{n-j} (XD_{p,q})^j N_p^{m-k} D_{p,q}^k e_{p,q}(x) \\
&= \sum_{k=0}^{m}\sum_{j=0}^{n} \binom{n}{j} \tilde{S}_{p,q}(m,k)[k]_{p,q}^{n-j} q^{jk} X^k N_p^{n-j} (XD_{p,q})^j N_p^{m-k} e_{p,q}(p^k x) \\
&= \sum_{k=0}^{m}\sum_{j=0}^{n} \binom{n}{j} \tilde{S}_{p,q}(m,k)[k]_{p,q}^{n-j} q^{jk} X^k N_p^{n-j} (XD_{p,q})^j e_{p,q}(p^m x) \\
&= \sum_{k=0}^{m}\sum_{j=0}^{n} \binom{n}{j} \tilde{S}_{p,q}(m,k)[k]_{p,q}^{n-j} q^{jk} X^k N_p^{n-j} \tilde{B}_j(p,q,p^m x) e_{p,q}(p^{m+j} x) \\
&= \sum_{k=0}^{m}\sum_{j=0}^{n} \binom{n}{j} \tilde{S}_{p,q}(m,k)[k]_{p,q}^{n-j} q^{jk} x^k \tilde{B}_j(p,q;p^{n+m-j} x) e_{p,q}(p^{n+m} x).
\end{aligned}$$

Dividing both sides of the above identity by $e_{p,q}(p^{n+m}x)$ and setting $x=1$ gives

$$\tilde{B}_{n+m}(p,q;1) = \sum_{k=0}^{m}\sum_{j=0}^{n} \binom{n}{j} \tilde{S}_{p,q}(m,k)[k]_{p,q}^{n-j} q^{jk} \tilde{B}_j(p,q;p^{n+m-j}).$$

□

Remark 7. For $p=1$, Eq. (17) reduces to the q-deformation of Spivey's Bell number formula (4), and for $p=q=1$, it reduces to Spivey's Bell number formula (3).

Proposition 8 (Dobinski formula)**.** *The (p,q)-Bell polynomials $\tilde{B}_n(p,q;x)$, satisfy the following identity:*

$$\tilde{B}_n(p,q;x) = \frac{1}{e_{p,q}(p^n x)} \sum_{k=0}^{\infty} p^{\binom{k}{2}} \frac{[k]_{p,q}^n}{[k]_{p,q}!} x^k. \tag{18}$$

Consequently, the (p,q)-Bell numbers $\tilde{B}_n(p,q;1)$, are given by:

$$\tilde{B}_n(p,q;1) = \frac{1}{e_{p,q}(p^n)} \sum_{k=0}^{\infty} p^{\binom{k}{2}} \frac{[k]_{p,q}^n}{[k]_{p,q}!}.$$

Proof. Since $(XD_{p,q})x^m = [m]_{p,q}x^m$, it follows that

$$(XD_{p,q})^n x^m = [m]_{p,q}^n x^m,$$

and

$$(XD_{p,q})^n e_{p,q}(x) = \sum_{k=0}^{\infty} p^{\binom{k}{2}} \frac{[k]_{p,q}^n}{[k]_{p,q}!} x^k. \tag{19}$$

Multiplying both sides of Eq. (19) with $\frac{1}{e_{p,q}(p^n x)}$ and taking into account Eq. (16) proves the desired result. □

Remark 9. When $p = 1$, Eq. (18) reduces to the Dobinski formula for q-Bell polynomials given by Katriel [6]. Furthermore, when $p = q = 1$, we obtain the Dobinski formula for the ordinary Bell polynomials:

$$B_n(x) = \frac{1}{e} \sum_{k \geq 0} \frac{k}{k!} x^k.$$

Therefore, we can think of Eq. (18) as the Dobinski formula for (p,q)-Bell polynomials $\tilde{B}_n(p,q;x)$.

4 Final remarks

It seems likely that following the techniques due to Mangontarum [7], one can obtain an alternative proof of (17), using the creation, annihilation and number operators in the (p,q)-Fock space [2]. Our speculation stems from the following observation:

Let $l^2(\mathbb{N})$ be a Hilbert space with the standard orthonormal basis $(\delta_n)_{n=0}^{\infty}$. We define the creation, annihilation and number operators as follows:

1. The creation operator a^+ defined by

$$a^+ \delta_n = \sqrt{[n+1]_{p,q}} \cdot \delta_{n+1}, \quad n \geq 0.$$

2. The annihilation operator (adjoint of a^+) a^- defined by

$$a^- \delta_0 = 0, \quad a^- \delta_n = \sqrt{[n]_{p,q}} \cdot \delta_{n-1}, \quad n \geq 1.$$

3. The number operator a° defined by

$$a^\circ \delta_n = p^n \delta_n, \quad n \geq 0.$$

It follows by an easy computation that these operators satisfy the (p,q)-commutation relation

$$a^- a^+ - q a^+ a^- = a^\circ.$$

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