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Ellipse Chains and Associated Sequences

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Abstract

We define circle and ellipse chains tangent to the branches of a hyperbola and the terms of the chains are mutually tangent to each other. Our goal is to derive recurrence relations for the parameters of chains elements and to establish some connections between integer sequences and chains.

1 Introduction

Let us consider the hyperbola $\mathcal{H}$ with the canonical equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{1}$$

and foci $(\pm c, 0)$, where a and b are positive real numbers and $c^2 = a^2 + b^2$. Lucca [3] examined a tangential chain of circles inside the branch $x > 0$ of the hyperbola so that the n-th circle with center $(x_n, 0)$ and radius r_n are tangent to the hyperbola and mutually tangent to each other. He showed that for certain a and b, the sequences $(x_n/x_0)_{n=0}^{\infty}$ and $(r_n/r_0)_{n=0}^{\infty}$ are integers. Belbachir et al. [2] extended the circle chains to ellipse chains inside the branch of hyperbola when the ratio of the minor and major axes is fixed. They described the recurrence relations for the ellipses' parameters and determined a connection between the parameters of the ellipse chains and integer sequences.

In the present paper, we give an extension of the papers [2, 3]. We examine special chains of circles and ellipses between the branches of hyperbola $\mathcal{H}$ (or outside $\mathcal{H}$), such that the circles (ellipses) are tangent to the hyperbola $\mathcal{H}$ and mutually tangent to each other. Furthermore, we give the recurrence relations for the tangential points. We define a tangential chain of ellipses between the branches of $\mathcal{H}$ where the centers of the ellipses coincide with the centers of the circles. We give recurrence relations for the ellipses' parameters.

Our other main purpose is to give integer sequences that describe the parameters of our chains. We find more than fifty such integer sequences that appear in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [4], and in this way, our investigation gives them geometrical interpretations. In what follows, we define $t := a/b$, $s := b/a$, $\lambda := 2t^2 + 1$, and $\mu := t^2 + 1$.

2 Circle chains between the branches of hyperbola

Let us define a chain of circles with the following properties.

The canonical equation of the n-th circle centered at point $(0, y_n)$, $(n \geq 0,\ y_n \geq 0)$ is

$$x^2 + (y - y_n)^2 = r_n^2, \tag{2}$$

where the center of each circle lies on the y-axis, and $r_n > 0$ is the radius (Figure 1).

The circles (2) are tangent to the hyperbola (1) and are mutually tangent, i.e.,

$$y_n - y_{n-1} = r_n + r_{n-1} \quad (n \geq 1).$$

Let $(\hat{x}_n, \hat{y}_n)$ be the tangential point of the branch $x > 0$ of hyperbola $\mathcal{H}$ and the n-th circle of the chain, given by $\hat{y}_n = s^2/(1+s^2)\ y_n$ and $\hat{x}_n^2 = 1/(t^2+s^2+2)\ y_n^2 + a^2$. Moreover

$$\hat{x}_n^2 = t^2\, \hat{y}_n^2 + a^2. \tag{3}$$

The sequences $(y_n)_{n\geq 0}$, $(r_n)_{n\geq 0}$, $(\hat{x}_n)_{n\geq 0}$, and $(\hat{y}_n)_{n\geq 0}$ satisfy the following recurrence relations:

Theorem 1. *The sequences $(y_n)_{n\geq 0}$, $(r_n)_{n\geq 0}$, and $(\hat{y}_n)_{n\geq 0}$ are second-order linear homogeneous recurrence sequences*

$$\ell_n = 2\lambda\ell_{n-1} - \ell_{n-2} \qquad (n \geq 2), \tag{4}$$

and the sequence $(\hat{x}_n^2)_{n\geq 0}$ is a third-order linear homogeneous recurrence sequence

$$\hat{x}_n^2 = (4\lambda^2 - 1)\,\hat{x}_{n-1}^2 - (4\lambda^2 - 1)\,\hat{x}_{n-2}^2 + \hat{x}_{n-3}^2 \qquad (n \geq 3), \tag{5}$$

and the initial values are $y_0 = 0$, $r_0 = a$, $\hat{x}_0^2 = a^2$, $\hat{y}_0 = 0$, $y_1 = 2a\mu$, $r_1 = a\lambda$, $\hat{x}_1^2 = a^2(4t^2+1)$, $\hat{y}_1 = 2a$, and $\hat{x}_2^2 = a^2(16t^2\lambda^2 + 1)$.

Proof. The system composed of the equations (1) and (2) gives

$$\begin{cases} y_n = \lambda\, y_{n-1} + 2\mu\, r_{n-1}, \\ r_n = 2t^2\, y_{n-1} + \lambda\, r_{n-1}. \end{cases}$$

Then $(y_n)_{n\geq 0}$ and $(r_n)_{n\geq 0}$ satisfy (4) and this is also the case for $(\hat{y}_n)_{n\geq 0}$, as $y_n = \mu\,\hat{y}_n$. Now for $\hat{x}_n^2$, let $C = 1/(t^2 + s^2 + 2)$. Then $\hat{x}_n^2 = Cy_n^2 + a^2$, $y_n = 2\lambda y_{n-1} - y_{n-2}$, substituting y_n in $\hat{x}_n^2$ we get $\hat{x}_n^2 = C(2\lambda y_{n-1} - y_{n-2})^2 + a^2 = 4\lambda^2\hat{x}_{n-1}^2 + \hat{x}_{n-2}^2 - 4C\lambda y_{n-1}y_{n-2} - 4\lambda^2 a^2$ and from the sum $\hat{x}_n^2 + \hat{x}_{n-1}^2$, we obtain the equation (5). □

Remark 2. Because of the equation (3), the recurrence relation for the sequence $(\hat{y}_n^2)_{n\geq 0}$ is the same as for the recurrence of $(\hat{x}_n^2)_{n\geq 0}$. Thus, the sequence of squared distances of tangential points and the origin $(d_n^2 = \hat{x}_n^2 + \hat{y}_n^2)_{n\geq 0}$ satisfies the recurrence (5) as well. In the last section, we give a second-order recurrence solving (5).

3 Ellipse chains between the branches of hyperbola

In this section, we define a tangential chain of ellipses between the branches of $\mathcal{H}$, where the ellipses' centers coincide with the circles' centers. Let us define a chain of ellipses with the following properties.

The canonical equation of the n-th ellipse centered at point $(0, y_n)$ $(y_n > 0,\ n \geq 0)$ is

$$\frac{x^2}{\alpha_n^2} + \frac{(y - y_n)^2}{\beta_n^2} = 1, \tag{6}$$

where $2\alpha_n > 0$ is the width and $2\beta_n > 0$ is the height of the ellipse (Figure 1).

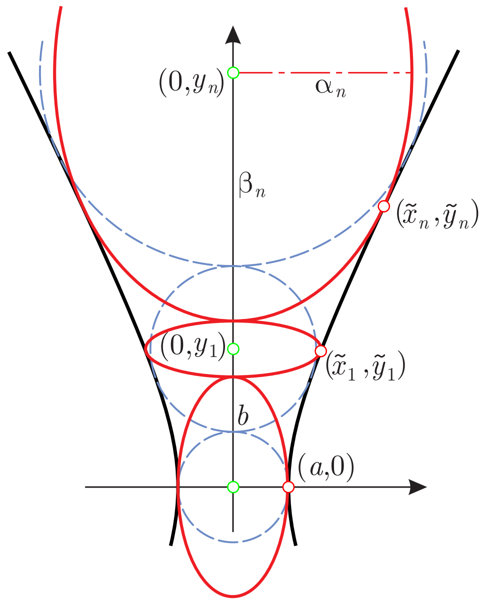

Figure 1: Ellipse and circle chains between the branches of hyperbola.

Let the equation of the 0-th ellipse be $x^2/a^2 + y^2/\beta_0^2 = 1$, and thus $y_0 = 0$, $\alpha_0 = a$. This implies that $0\beta_0 < r_0 + r_1 = 2a\mu$.

The ellipses (6) are tangent to the hyperbola (1) and are mutually tangent, i.e.,

$$y_n - y_{n-1} = \beta_n + \beta_{n-1} \quad (n \geq 1). \tag{7}$$

The sequence y_n is the same as the sequence of circle chains. However, we can give recurrence formulas for the parameter β_n of ellipses (10). Let us determine α_n^2. From the system composed by the equations (1) and (6) and from the tangency condition between the ellipses, we have

$$\left(\frac{a^2}{b^2} + \frac{\alpha_n^2}{\beta_n^2}\right) y^2 - \frac{2\alpha_n^2}{\beta_n^2} y_n y + \frac{\alpha_n^2}{\beta_n^2} y_n^2 + a^2 - \alpha_n^2 = 0. \tag{8}$$

Since the discriminant of equation (8) is equal to zero and after simplification, we get

$$s^2\alpha_n^4 - \left(b^2 + y_n^2 - \beta_n^2\right)\alpha_n^2 - a^2\beta_n^2 = 0. \tag{9}$$

We put $\delta_n = \alpha_n^2$ and $\omega_n = b^2 + y_n^2 - \beta_n^2$. Then the solutions of the equation (9) are $\delta_{n,1} = (\omega_n + \sqrt{\omega_n^2 + 4b^2\beta_n^2})/2s^2$ and $\delta_{n,2} = (\omega_n - \sqrt{\omega_n^2 + 4b^2\beta_n^2})/2s^2$. Since $\delta_{n,2} < 0$, we get $\alpha_n^2 = t^2/2(\omega_n + \sqrt{\omega_n^2 + 4b^2\beta_n^2})$. Let $(\tilde{x}_n, \tilde{y}_n)$ be the tangential point of the branch $x > 0$ of $\mathcal{H}$ and the n-th ellipse of the chain. Then recurrence relations for $\tilde{x}_n$ and $\tilde{y}_n$ are as follows: $\tilde{y}_n = (\alpha_n^2 b^2/a^2\beta_n^2 + \alpha_n^2 b^2)y_n$ and $\tilde{x}_n^2 = (\alpha_n^4 a^2 b^2/(a^2\beta_n^2 + \alpha_n^2 b^2)^2)y_n^2 + a^2$, where $y_n^2 = (\alpha_n^2 - a^2)(a^2\beta_n^2 + \alpha_n^2 b^2)/a^2\alpha_n^2$.

The following theorem gives the recurrence relations for $(\beta_n)_{n\geq 0}$, $(\tilde{y}_n)_{n\geq 0}$, and $(\tilde{x}_n^2)_{n\geq 0}$.

Theorem 3. *The sequence* $(\beta_n)_{n\geq 0}$ *is a third-order linear homogeneous recurrence sequence*

$$\beta_n = (2\lambda - 1)\,\beta_{n-1} + (2\lambda - 1)\,\beta_{n-2} - \beta_{n-3} \quad (n \geq 3). \tag{10}$$

The sequence $(\tilde{y}_n)_{n\geq 0}$ *satisfies the second-order linear homogeneous recurrence sequence* (4) *and the sequence* $(\tilde{x}_n^2)_{n\geq 0}$ *is a third-order linear homogeneous recurrence sequence*

$$\tilde{x}_n^2 = (4\lambda^2 - 1)\,\tilde{x}_{n-1}^2 - (4\lambda^2 - 1)\,\tilde{x}_{n-2}^2 + \tilde{x}_{n-3}^2 \qquad (n \geq 3). \tag{11}$$

The initial values are β_0, $\beta_1 = 2a\mu - \beta_0$, $\beta_2 = 8at^2\mu + \beta_0$, $\tilde{y}_0 = 0$, $\tilde{x}_0^2 = a^2$,

$$\begin{aligned} \tilde{y}_1 &= \frac{\alpha_1(2 + 2a^2)}{a\beta_1^2 + \alpha_1^2 b^2}, \\ \tilde{x}_1^2 &= \frac{\alpha_1^8 a^2 b^2 (2a^2 + 2)^2}{(a^2\beta_1^2 + \alpha_1^2 b^2)^2 (a\beta_1^2 + \alpha_1^2 b^2)^2} + a^2, \textit{ and} \\ \tilde{x}_2^2 &= \frac{\alpha_2^2 b^2 (a^2 + \alpha_2^2)(-a^2 b^2 + \alpha_2^2 + (b^2 - \beta_2^2)x_2^2)}{(a^2\beta_2^2 + \alpha_2^2 b^2)^2} + a. \end{aligned}$$

Proof. We have y_n is defined as follows: $y_n = \lambda\, y_{n-1} + 2\mu\, r_{n-1}$ where r_n is the radius of the circles. Using the tangency condition (7) we get

$$\beta_n = 2t^2\, y_{n-1} + 2\mu\, r_{n-1} - \beta_{n-1}. \tag{12}$$

After calculations, we obtain $\beta_n = 4t^2\, y_{n-1} + \beta_{n-2}$. Subtracting β_{n-1} from β_n we obtain $\beta_n - \beta_{n-1} = 4t^2(y_{n-1} - y_{n-2}) + \beta_{n-2} - \beta_{n-3}$. Hence from Eq. (7), we find (10). The initial values come from (12). The proofs of the second-order and third-order linear homogeneous recurrence sequences of $(\tilde{y}_n)_{n\geq 0}$ and $(\tilde{x}_n^2)_{n\geq 0}$ are similar to the proof of Theorem 1. Now $C = \alpha_n^4 a^2 b^2 / (a^2\beta_n^2 + \alpha_n^2 b^2)^2$. □

Equations (5) and (11) are of the form $V_n = (\theta - 1)V_{n-1} - (\theta - 1)V_{n-2} + V_{n-3} \quad (n \geq 3)$, and thus $W_n = V_n - V_{n-1}$ is a second-order linear recurrence $W_n = (\theta - 2)W_{n-1} - W_{n-2}$ with $\theta = 4\lambda^2$. We deduce an explicit form for $(V_n)_{n\geq 0}$ using $(W_n)_{n\geq 0}$ in the following theorem.

Theorem 4. *For all* $n \geq 2$, $V_n = V_0 + \sum_{k=1}^{n} W_k$.

Proof. The proof is left to the reader; it can be shown with some calculations. The explicit terms of the W_k are well known. □

4 Associated integer sequences of chains

In this section, we determine conditions to relate the circle and ellipse chains with integer sequences. In what follow, let $\beta_0 = b$.

4.1 Rectangular hyperbolas

Corollary 5. *If $\mathcal{H}$ is a rectangular hyperbola ($a = b$), then $r_n = \alpha_n = \beta_n$, $\hat{y}_n = \tilde{y}_n$, and $\hat{x}_n = \tilde{x}_n$ for all non-negative integers n. Moreover, the recurrence sequence of $(r_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$, $(\hat{y}_n)_{n\geq 0}$, and $(\hat{x}_n^2)_{n\geq 0}$ are, respectively, $\ell_n = 6\ell_{n-1} - \ell_{n-2} \quad (n \geq 2)$, with initial values $r_0 = a$, $r_1 = 3a$, $y_0 = 0$, $y_1 = 4a$, $\hat{y}_0 = 0$, and $\hat{y}_1 = 2a$ and $\hat{x}_n^2 = 35\hat{x}_{n-1}^2 - 35\hat{x}_{n-2} + \hat{x}_{n-3}^2 \quad (n \geq 3)$, with initial values $\hat{x}_0^2 = a^2$, $\hat{x}_1^2 = 5a^2$, and $\hat{x}_2^2 = 145a^2$.*

For rectangular hyperbolas $\mathcal{H}$, from Corollary 5, we deduce the following result:

Theorem 6. *If $\mathcal{H}$ is a rectangular hyperbola, then $(r_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$, $(\hat{y}_n)_{n\geq 0}$, and $(\hat{x}_n^2)_{n\geq 0}$ are integer sequences, respectively, if and only if a is a positive integer, $a = k/4$, $a = k/2$, and $a^2 = k$, respectively, where k is any positive integer.*

Now we give some examples of integer sequences. Some of them appear in the OEIS [4] for $a = 1$: $(r_n)_{n\geq 0} =$ A001541, $(y_n)_{n\geq 0} =$ A005319, $(\hat{y}_n)_{n\geq 0} =$ A001542, and $(\hat{x}_n^2)_{n\geq 0} = \{0,$ A076218$\}$ and for general a with the conditions of Theorem 6, $(r_n)_{n\geq 0} = a \cdot$ A001541, $(y_n)_{n\geq 0} = a\cdot$A005319 $= 2a\cdot$A001542, $(\hat{y}_n)_{n\geq 0} = a\cdot$A001542, and $(\hat{x}_n^2)_{n\geq 0} = a^2\cdot\{0,$ A076218$\}$. For more sequences, see Table 1.

We mention that among the sequences in Table 1 and Table 5, there are some sequences, e.g., A098706, which are defined without any combinatorial or geometrical interpretation. We provide a geometric interpretation. We also note that $k \cdot (y_n) = 2k \cdot (\hat{y}_n)$.

Remark 7. The sequence (A001109), e.g., appears, as $(\hat{y}_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ when $a = 1/2$ and $a = 1/4$, dealing with balancing numbers [1].

$a = b$	$(r_n)_{n\geq 0}$	$(y_n)_{n\geq 0}$	$(\hat{y}_n)_{n\geq 0}$	$(\hat{x}_n^2)_{n\geq 0}$
1	A001541	A005319	A001542	A076218
a	$a\cdot$A001541	$a\cdot$A005319	$a\cdot$A001542	$a^2\cdot$A076218
2	A003499	A081554	A005319	$\{4, 20, 580, 19604, \ldots\}$
3	A106329	$\{0, 12, 72, 420, \ldots\}$	A075848	$\{9, 45, 1305, 44109, \ldots\}$
4	$\{4, 12, 68, 396, \ldots\}$	$\{0, 16, 96, 560, \ldots\}$	A081554	$\{16, 80, 2320, 78416, \ldots\}$
$\sqrt{2}$	–	–	–	A098706$\setminus\{0\}$

Table 1: Integer sequences connected to the regular hyperbola.

4.2 Integer sequences associated with circle chains

In order to generate only integer sequences, we state the following theorem.

Theorem 8. *If $t = (\sqrt{k-2})/2$, $k \geq 3$, sequence $(r_n)_{n\geq 0}$, $a \in \mathbb{N}^+$ and $a \cdot k$ is even; $(y_n)_{n\geq 0}$, $a(k+2)$ is even; $(\hat{y}_n)_{n\geq 0}$, $2a$ is a positive integer, then $b = 2a/\sqrt{k-2}$ and the sequences $(r_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$, and $(\hat{y}_n)_{n\geq 0}$ are integer sequences.*

Proof. For $(r_n)_{n\geq 0}$, we have $r_0 = a$, so $a \in \mathbb{N}^+$ and $r_1 = a(2t^2+1)$ is integer if and only if $(2t^2+1) = m/a$, where m is a suitable positive integer. From this, we find $t^2 = (m/a-1)/2$. For the first coefficient of (4), we have $(4t^2+2) = 2m/a = k$ where k is a suitable positive integer. Now $m = (ak)/2$ implies that ak is even and $t^2 = (k-2)/4$. Moreover, we have $t = \sqrt{k-2}/2$, where t is positive if and only if $k \geq 3$. Obviously, $b = a/t$ comes from the definition of t.

In the case where $(y_n)_{n\geq 0}$, let $m = y_1 = 2a(t^2+1)$ be an integer. Then $t^2 = m/(2a) - 1$ and $t = \sqrt{m/(2a)-1}$, where $m > 2a$ and a is a positive real number. From the coefficient $(4t^2+2) = k$ with a suitable positive integer, we have $m = a(k+2)/2$. This implies that $a(k+2)$ is even and $t^2 = (k-2)/4$, $k \geq 3$. Moreover, $t = \sqrt{k-2}/2$ and $b = a/t$.

For $(\hat{y}_n)_{n\geq 0}$, $\hat{y}_1 = 2a = m$ and $(4t^2+2) = k$, we obtain that $2a$ is even and $t = \sqrt{k-2}/2$, $k \geq 3$. □

Theorem 9. *If a^2 is an integer and $t = k/2$, $k \geq 1$, then the sequence $(\hat{x}_n^2)_{n\geq 0}$ consists of integers and $b = a/t$.*

Proof. All the coefficients and initial values of (5) are integers. □

Table 1 and Table 5 ($t = 1$), moreover, Table 2, Table 3, and Table 4 contain examples of integer sequences. For $a = 1$ and $b = 2$, then $(y_n - r_n)_{n\geq 1}$ is the bisection of Lucas sequence A002878.

a	b	t	$(\hat{x}_n^2)_{n\geq 0}$
$\sqrt{2}$	1	$\sqrt{2}$	$\{2, 18, 1602, \ldots\}$
$\sqrt{2}$	2	$\sqrt{2}/2$	$\{2, 6, 66, 902, \ldots\}$
$\sqrt{2}$	$\sqrt{2}/2$	2	$\{2, 34, 10370, \ldots\}$
$\sqrt{2}$	$2\sqrt{2}$	$1/2$	$\{2, 4, 20, 130, \ldots\}$

Table 2: Integer sequences associated with circle chains.

a	b	t	$(r_n)_{n\geq 0}$	$(y_n)_{n\geq 0}$	$(\hat{y}_n)_{n\geq 0}$	$(\hat{x}_n^2)_{n\geq 0}$
1	2	$1/2$	—	—	$\{0,$ A025169$\}$;A111282,$n \geq 1$	A064170
1	$1/2$	2	A023039	$\{0, 10, 180, \ldots\}$	A207832	$\{1, 17, 5185, \ldots\}$
1	$1/3$	3	A078986	$\{0, 20, 760, \ldots\}$	$\{0, 2, 76, 2886, \ldots\}$	$\{1, 37, 51985, \ldots\}$
1	$2/3$	$3/2$	–	–	$\{0, 2, 22, 240 \ldots\}$	$\{1, 10, 1090, \ldots\}$
1	$1/4$	4	A099370	$\{0, 34, 2244, \ldots\}$	$\{0, 2, 132, 8710, \ldots\}$	$\{1, 65, 278785, \ldots\}$
2	1	2	A087215	A004292, $n \geq 1$	A060645	$\{4, 68, 20740, \ldots\}$
2	4	$1/2$	A005248	A201157	$\{0, 4, 12, 32, \ldots\}$	$\{4, 8, 40, 260, \ldots\}$
2	$1/2$	4	$\{2, 66, 4354, \ldots\}$	A004298, $n \geq 1$	$\{0, 4, 264, 17420, \ldots\}$	$\{4, 260, 1115140, \ldots\}$
2	$1/3$	6	$\{2, 146, 21314, \ldots\}$	$\{0, 148, 21608, \ldots\}$	$\{0, 4, 584, 85260, \ldots\}$	$\{4, 580, 12278020, \ldots\}$
2	$2/3$	3	$\{2,$ A239364$\}$	$\{0, 13, 143, \ldots\}$	$\{0, 4, 44, 480, \ldots\}$	$\{4, 148, 207940, \ldots\}$
2	$4/3$	$3/2$	A057076	$\{0, 40, 1520, \ldots\}$	$\{0, 4, 152, 5772, \ldots\}$	$\{4, 40, 4360, \ldots\}$
3	1	3	$\{3, 57, 2163, \ldots\}$	$\{0, 60, 2280, \ldots\}$	A084070	$\{9, 333, 467865, \ldots\}$
3	6	$1/2$	—	—	A099857, $n \geq 1$	$\{9, 18, 90, 585, \ldots\}$
4	1	4	$\{4, 132, 8708, \ldots\}$	$\{0, 136, 8976, \ldots\}$	$\{0, 8, 528, 34840, \ldots\}$	$\{16, 1040, 4460560, \ldots\}$
4	2	2	$\{4, 36, 644, \ldots\}$	$\{0, 40, 720, \ldots\}$	A134492	$\{16, 272, 82960, \ldots\}$
4	8	$1/2$	$\{4, 6, 14, 36, \ldots\}$	$\{0, 10, 30, 80, \ldots\}$	$\{0, 8, 24, 64, \ldots\}$	$\{16, 32, 160, \ldots\}$
1	$\sqrt{2}$	$\sqrt{2}/2$	A001075	A005320	A052530	A011922
1	$\sqrt{2}/2$	$\sqrt{2}$	A001079	A122653	A001078	$\{1, 9, 801, 78409, \ldots\}$
2	$\sqrt{2}$	$\sqrt{2}$	A087799	A004291, $n \geq 1$	A122652	$\{4, 36, 3204, \ldots\}$
2	$\sqrt{2}/2$	$2\sqrt{2}$	$\{2, 34, 1154, \ldots\}$	A004294, $n \geq 1$	A202299	$\{4, 132, 147972, \ldots\}$
2	$2\sqrt{2}$	$\sqrt{2}/2$	A003500	A001352, $n \geq 1$	A231896	$\{4, 12, 132, \ldots\}$

Table 3: Integer sequences associated with circle chains.

a	b	t	$(y_n)_{n\geq 0}$	$(\hat{y}_n)_{n\geq 0}$
$1/2$	1	$1/2$	—	A001906
$1/2$	$1/3$	$3/2$	—	$\{0,$ A004190$\}$
$1/2$	$1/4$	2	$\{0, 5, 90, \ldots\}$	A049660
$1/2$	$1/5$	$5/2$	—	$\{0,$A049660$\}$
$1/2$	$1/6$	3	$\{0, 10, 380, \ldots\}$	$\{0,$ A078987$\}$
$1/2$	$1/7$	$7/2$	—	A097836
$1/2$	$1/8$	4	$\{0, 17, 1122, \ldots\}$	$\{0,$ A097316$\}$
$1/2$	$1/9$	$9/2$	—	$\{0,$ A097839$\}$
$1/2$	$1/10$	5	$\{0, 26, 2652, \ldots\}$	$\{0,$ A097725$\}$
$1/2$	$1/11$	$11/2$	—	$\{0,$ A049670$\}$
$1/2$	$1/13$	$13/2$	—	$\{0,$ A097844$\}$

Table 4: Integer sequences associated with circle chains.

$a = b$	$(y_n)_{n\geq 0}$	$(\hat{y}_n)_{n\geq 0}$
$1/2$	A001542	A001109
$3/2$	A075848	A106328
$5/2$	$\{0, 10, 60, 350, \ldots\}$	A276598
$7/2$	$\{0,$ A273182$\}$	A054890,$n \geq 1$
$9/2$	$\{0, 18, 108, 630, \ldots\}$	A276602
$1/4$	A001109	–
$3/4$	A106328	–
$5/4$	A276598	–
$7/4$	A054890, $n \geq 1$	–

Table 5: Integer sequences connected to the regular hyperbola.

4.3 Integer sequences associated with ellipse chains

Theorem 10. *For positive integers a, b, if b divides a then the sequence $(\beta_n)_{n\geq 0}$ is an integer sequence and its recurrence is*

$$\beta_n = (4t^2 + 1)\beta_{n-1} + (4t^2 + 1)\beta_{n-2} - \beta_{n-3} \quad (n \geq 3), \tag{13}$$

and the initial values are $\beta_0 = b$, $\beta_1 = 2a\mu - b$, and $\beta_2 = 8at^2\mu + b$.

Proof. We notice that the initial values of the sequence are integers. Thus, the coefficients of recurrence relation (13) are also integers, which guarantees that all the other terms of the sequence are integers. □

Equations (10) and (13) are of the form $V_n = (\theta - 1)V_{n-1} + (\theta - 1)V_{n-2} - V_{n-3}, \quad (n \geq 3)$, and thus $W_n = V_n + V_{n-1}$ is a second-order linear recurrence $W_n = \theta W_{n-1} - W_{n-2}$. Here $\theta = 2\lambda$ and $4t^2$, respectively. We deduce an explicit form for $(V_n)_{n\geq 0}$ using $(W_n)_{n\geq 0}$ and leaving the proof to the reader in the following theorem:

Theorem 11. *For all $n \geq 2$, we have $V_n = (-1)^n V_0 + \sum_{k=1}^{n} (-1)^{n-k} W_k$.*

The explicit terms of W_k are well known.

We give some integer recurrence sequences for $(\beta_n)_{n\geq 0}$ in Table 6. If $t = 1$, so $a = b$, then our hyperbola is a rectangular hyperbola, and it holds for the integer sequences associated with ellipse chain not only for the sequence $(\beta_n)_{n\geq 0}$, but also for the sequences α_n, β_n, $\tilde{y}_n$, and $\tilde{x}_n$. See Subsection 4.1.

a	b	t	$(\beta_n)_{n\geq 0}$
a	a	1	in Table 1
2	1	2	$\{1, 19, 321, 5779, 103681, 1860499, \ldots\}$
3	1	3	$\{1, 59, 2161, 82139, 3119041, 118441499, \ldots\}$
4	1	4	$\{1, 135, 8705, 574599, 37914625, 2501790855, \ldots\}$
4	2	2	$\{2, 38, 642, 11558, 207362, 3720998, \ldots\}$

Table 6: Integer sequences associated with ellipse chains.

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