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Leibniz-Additive Functions on UFD's

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Abstract

Recall that an arithmetic function f is called an L-additive function with respect to a completely multiplicative function h if $f(mn) = f(m)h(n) + f(n)h(m)$ holds for all m and n. We study L-additive functions in the fields of fractions of unique factorization domains (UFD). In particular, we describe all L-additive functions over given UFD such that these functions can be extended to its field of fractions. We find the exact formula for an L-additive function in the terms of prime elements. For a given L-additive function $f(x)$ we study the properties of the sequence $(f^{(k)}(x))_{k\geq 1}$ and solutions of the equation $f(x) = \alpha x$. As corollaries we obtain results about the arithmetic derivative and partial arithmetic derivatives.

1 Introduction

Recall [2, 12, 13] that the arithmetic derivative is a function $D : \mathbb{N} \to \mathbb{N}$, such that

1. $D(p) = 1$ for all prime p;
2. D satisfies the Leibniz rule: $D(mn) = D(m)n + D(n)m$.

It is known that the arithmetic derivative is not a linear function:

$$D(2+3) = D(5) = 1 \text{ and } D(2) + D(3) = 2.$$

One can define the arithmetic derivative for a rational number [13]. Since D is not a linear function, it is difficult to solve even equations $D(x) = 2a$ and $D(D(x)) = 1$. Ufnarovski and Åhlander [13] showed that the first equation has a solution for any natural a if Goldbach's conjecture is true and the second equation has infinite number of solutions if twin prime conjecture is true. Equations of the form $D(x) = ax + 1$ are connected to the conjecture about Guiga numbers as was shown by Grau and Oller-Marcén [4].

Note that the partial arithmetic derivative $D_p(n)$ and the arithmetic subderivative $D_S(n)$ of an integer number satisfy the Leibniz rule [11].

Also the analogues of arithmetic derivative were considered over different sets of elements [13]. Haukkanen et al. [7] discussed the arithmetic derivative on non-unique factorization domains. Emmons et al. [3] described all functions $f : \mathbb{Z}_n \to \mathbb{Z}_n$ that satisfy the Leibniz rule. In particular, all values of such functions are divisors of zero. Kovič [10] constructed functions defined on Gaussian rationals that satisfy the Leibniz rule.

Let h be a completely multiplicative function. According to Merikoski et al. [9, 11] a function $f : \mathbb{N} \to \mathbb{N}$ is called L-additive with respect to h if $f(mn) = f(m)h(n) + f(n)h(m)$ holds for all $m, n \in \mathbb{N}$.

The aim of this paper is to study L-additive functions in the fields of fractions over unique factorization domains (Gaussian integers, Eisenstein integers and etc.).

2 Preliminary results

The notation and terminology agree with the book [1]. We refer the reader to this book for the results on ring theory. Through $\mathbb{N}$, $\mathbb{P}$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Q}$, and $\mathbb{Q}_+$ we denote here the sets of natural, prime, integer, non-negative integer, rational, and non-negative rational numbers respectively.

Let Φ be an integral domain [1, p. 368]. Recall that the field of fractions $\mathrm{Frac}(\Phi)$ of Φ is the set $\{\frac{a}{b} \mid a \in \Phi, b \in \Phi \setminus \{0\}$ with $\frac{a}{b} = \frac{c}{d}$ if $\exists k \neq 0$ such that $a = kc$ and $b = kd\}$ with two operations:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot b}{b \cdot d} \text{ and } \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \ \forall a, b, c, d \in \Phi \text{ and } b, d \neq 0.$$

Note that $\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$ and $\mathrm{Frac}(\mathbb{Z}[i]) = \mathbb{Q}(i)$.

If p is a prime in a UFD, then all elements associated with it are also primes. Let $\mathbb{P}(\Phi)$ be some maximal by inclusion set of non-associated primes in Φ. Note that a UFD can have infinite number of maximal by inclusion sets of non-associated primes. Now the unique factorization means that every non-zero element of Φ has the following unique factorization:

$$y = u \cdot \prod_{p \in \mathbb{P}(\Phi)} p^{v_p(y)}$$

where u is a unit, $v_p(y) \in \mathbb{Z}_+$, and $v_p(y) \neq 0$ for finitely many $p \in \mathbb{P}(\Phi)$.

Recall that the characteristic of an integral domain Φ is defined to be the smallest number of times one must use the ring's multiplicative identity in a sum to get the additive identity. If this sum never reaches the additive identity, then Φ is said to have characteristic zero. Note that if the characteristic of an integral domain is not equal to 0, then it is a prime. Also note that the characteristics of an integral domain and its field of fractions coincide.

Let Φ be a UFD such that its group of units G is finitely generated. Then

$$G \simeq A_1 \times A_2 \times \cdots \times A_n \times B_1 \times B_2 \times \cdots \times B_k$$

with $n, k \geq 0$, $A_i \simeq \mathbb{Z}$, and $B_j \simeq \mathbb{Z}_{n_j}$ where n_j is a power of a prime for all j. Let $A(\Phi)$ be the set containing exactly one generating element a_i of each subgroup A_i and $B(\Phi)$ be the set containing exactly one generating element b_j of each subgroup B_j such that its order is divisible by the characteristic of Φ. If the characteristic of Φ is equal to 0, then $B(\Phi)$ is empty. Define $R(\Phi) = A(\Phi) \cup B(\Phi)$.

If the characteristic of Φ is equal to 0, then all its units u can be uniquely represented

$$u = u_0 \cdot \prod_{i=1}^{n} a_i^{\gamma_i}$$

with $\gamma_i \in \mathbb{Z}$ and u_0 is an element of a finite order.

If the characteristic of Φ is equal to $p \neq 0$, then all its units u can be uniquely represented

$$u = u_0 \cdot \prod_{i=1}^{n} a_i^{\gamma_i} \cdot \prod_{j=1}^{m} b_j^{\delta_j}$$

with $\gamma_i \in \mathbb{Z}, \delta_j \in \{0, 1, \ldots, n_j - 1\}$, and u_0 is an element of a finite order t with $\gcd(t, p) = 1$.

Let $\mathbb{P}^*(\Phi) = \mathbb{P}(\Phi) \cup R(\Phi)$. So the factorization of an element $q \neq 0$ in Φ can be written uniquely in the following form

$$q = u_0 \cdot \prod_{r \in R(\Phi)} r^{v_r(q)} \cdot \prod_{p \in \mathbb{P}(\Phi)} p^{v_p(q)} = u_0 \cdot \prod_{p \in \mathbb{P}^*(\Phi)} p^{v_p(q)}.$$

This means that every element $q \neq 0$ of $\mathrm{Frac}(\Phi)$ can be written uniquely in the following form

$$q = u_0 \cdot \prod_{p \in \mathbb{P}^*(\Phi)} p^{v_p(q)}$$

where $v_p(q) \in \mathbb{Z}$ for all $p \in \mathbb{P}^*(\Phi)$ and $v_p(q) \neq 0$ only for a finite number of $p \in \mathbb{P}^*(\Phi)$.

Let $m, n \in \mathrm{Frac}(\Phi) \setminus \{0\}$. It is easy to check that $v_p(m \cdot n) = v_p(m) + v_p(n)$ for any $p \in \mathbb{P}(\Phi)$. Note that $v_{a_i}(m \cdot n) = v_{a_i}(m) + v_{a_i}(n)$. If characteristic of Φ is p, then $v_{b_i}(m \cdot n) \equiv v_{b_i}(m) + v_{b_i}(n) \pmod{n_i}$, where $n_i = p^\alpha$ for some α. This means that $v_{b_i}(m \cdot n) \equiv v_{b_i}(m) + v_{b_i}(n) \pmod{p}$. All these mean that $v_r(m \cdot n) = v_r(m) + v_r(n)$ in $\mathrm{Frac}(\Phi)$.

A *Gaussian integer* is a complex number $a + bi$ with $a, b \in \mathbb{Z}$ and $i^2 = -1$. Note that Gaussian integers form a UFD. Any Gaussian prime divides some integer prime wherein in Gaussian integers $2 = -i(1+i)^2$ with $1+i$ is a prime; every integer prime of the form $4k+1$ is a product of two non-associated Gaussian primes of the form $a+bi$ and $a-bi$; every integer prime of the form $4k+3$ is a Gaussian prime. Note that the group of units of $\mathbb{Z}[i]$ is the cyclic group of order 4. So $R(\mathbb{Z}[i]) = \emptyset$. We can chose $\mathbb{P}^*(\mathbb{Z}[i])$ in the following way: natural primes of the form $4k+3$; $1+i$; numbers of the form $a+bi$ and $a-bi$ with $a < b$, $a, b \in \mathbb{N}$ and a^2+b^2 is a natural prime of the form $4k+1$. Then $\mathbb{P}^*(\mathbb{Z}[i]) = \{1+i, 3, 1+2i, 1-2i, 7, 2+3i, 2-3i, 11, \ldots\}$.

An *Eisenstein integer* is a complex number $a + b\omega$ with $a, b \in \mathbb{Z}$ and $\omega = \frac{-1+i\sqrt{3}}{2}$ is a solution of the equation $\omega^2 + \omega + 1 = 0$. Note that Eisenstein integers form a UFD. Any Eisenstein prime divides some integer prime, with $3 = (1+2\omega)^2$ where $1+2\omega$ is an Eisenstein prime. An integer prime $p \equiv 2 \pmod{3}$ is an Eisenstein prime. The remaining integer primes are the products of two non-associated Eisenstein primes. Note that the group of units of $\mathbb{Z}[\omega]$ is the cyclic group of order 6. So $R(\mathbb{Z}[\omega]) = \emptyset$. We can chose $\mathbb{P}^*(\mathbb{Z}[\omega])$ in the following way: an integer prime $p \equiv 2 \pmod{3}$; $1 + 2\omega$ and numbers of the form $a + b\omega$ and $a + b\omega^2$ with $a \in \mathbb{N}, |a| < |b|$ and $a^2 - ab + b^2$ is a natural prime number $p \equiv 1 \pmod{3}$. Then $\mathbb{P}^*(\mathbb{Z}[\omega]) = \{2, 1 + 2\omega, 5, 1 + 3\omega, 1 + 3\omega^2, 11, \ldots\}$.

Note that the norm of a Gaussian integer is $N_1(a + bi) = a^2 + b^2$. The norm of an Eisenstein integer is $N_2(a + b\omega) = a^2 - ab + b^2$.

Recall that $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \,|\, a, b \in \mathbb{Z}\}$ is a UFD. Note that its group of units is generated by $\{-1, \sqrt{2} + 1\}$ and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$. So in this case we can choose $R(\mathbb{Z}[\sqrt{2}]) = \{\sqrt{2} + 1\}$.

3 Main results

The following well-known function classes play an important role in our paper.

Definition 1. Let K be a ring. A function $h : K \to K$ is called

(1) *Completely multiplicative*, if $h(1) = 1$ and $h(mn) = h(m)h(n)$ holds for all $m, n \in K$.

(2) *Completely additive*, if $h(mn) = h(m) + h(n)$ holds for all $m, n \in K$.

Definition 2 ([11])**.** Let K be a ring and $h : K \to K$ be a completely multiplicative function. A function $f : K \to K$ is called *L-additive with respect to* h if $f(mn) = f(m)h(n) + f(n)h(m)$ holds for all $m, n \in K$.

If $h(n) \equiv 1$, then every L-additive function with respect to h is completely additive. Every L-additive function with respect to $h(x) = x$ will be called *L-additive*.

Theorem 3. *Let Φ be an integral domain and $h, f : \Phi \to \Phi$ be completely multiplicative and L-additive with respect to h functions. Then there exist unique functions $\bar{h}, \bar{f} : \mathrm{Frac}(\Phi) \to \mathrm{Frac}(\Phi)$, such that $\bar{h}$ is completely multiplicative, $\bar{f}$ is L-additive with respect to $\bar{h}$, and $h(x) = \bar{h}(x)$ and $f(x) = \bar{f}(x) \, \forall x \in \Phi$ iff $h(x) \neq 0 \, \forall x \in \Phi \setminus \{0\}$.*

Let $q : \mathrm{Frac}(\Phi) \to \mathrm{Frac}(\Phi)$ be completely multiplicative. In these terms a function $V : \mathrm{Frac}(\Phi) \to \mathrm{Frac}(\Phi)$ is L-additive with respect to q where $V(x) = \frac{\bar{f}(x) \cdot q(x)}{\bar{h}(x)}$ for $x \neq 0$ and $V(0) = 0$.

Proof. Assume that $h(x) \neq 0 \, \forall x \in \Phi \setminus \{0\}$. Then for all $x \in \mathrm{Frac}(\Phi)$ there exist $m, n \in \Phi$ with $x = \frac{m}{n}, n \neq 0$. Let $\bar{h}(x) = \frac{h(m)}{h(n)}$ and $\bar{f}(x) = \frac{f(m)h(n) - f(n)h(m)}{h^2(n)}$.

Let $x = \frac{m_1}{n_1} = \frac{m_2}{n_2}$. Then there is $k \neq 0$ such that $m_1 = km_2$ and $n_1 = kn_2$. The following equality shows that $\bar{h}$ is well-defined:

$$\bar{h}\left(\frac{m_1}{n_1}\right) = \frac{h(m_1)}{h(n_1)} = \frac{h(km_2)}{h(kn_2)} = \frac{h(k) \cdot h(m_2)}{h(k) \cdot h(n_2)} = \bar{h}\left(\frac{m_2}{n_2}\right).$$

Let $x = x_1 \cdot x_2$, $x = \frac{m}{n}$, $x_1 = \frac{m_1}{n_1}$ and $x_2 = \frac{m_2}{n_2}$. Then $m = m_1 \cdot m_2 \cdot k$ and $n = n_1 \cdot n_2 \cdot k$. The following equality shows that $\bar{h}$ is completely multiplicative:

$$\bar{h}(x_1) \cdot \bar{h}(x_2) = \frac{h(m_1)}{h(n_1)} \cdot \frac{h(m_2)}{h(n_2)} \cdot \frac{h(k)}{h(k)} = \frac{h(k \cdot m_1 \cdot m_2)}{h(k \cdot n_1 \cdot n_2)} = \frac{h(m)}{h(n)} = \bar{h}(x).$$

Let $\bar{g}$ be a completely multiplicative extension of h. Then

$$1 = \bar{g}(1) = \bar{g}(x \cdot x^{-1}) = \bar{g}(x) \cdot \bar{g}(x^{-1}).$$

Therefore $\bar{g}(x^{-1}) = \frac{1}{\bar{g}(x)} = \frac{1}{h(x)} \, \forall x \in \Phi \setminus \{0\}$. The following shows that $\bar{h}$ is unique:

$$\bar{g}(x) = \bar{g}\left(\frac{m}{n}\right) = \bar{g}(m) \cdot \bar{g}(n^{-1}) = \frac{h(m)}{h(n)} = \bar{h}(x).$$

The following equality shows that $\bar{f}$ is well-defined:

$$\begin{aligned}
\bar{f}\left(\frac{km}{kn}\right) &= \frac{f(km) \cdot h(kn) - f(kn) \cdot h(km)}{h^2(kn)} \\
&= \frac{(f(k)h(m) + f(m)h(k)) \cdot h(k)h(n) - (f(k)h(n) + f(n)h(k)) \cdot h(k)h(m)}{h^2(kn)} \\
&= \frac{f(k)h(m)h(k)h(n) + f(m)h^2(k)h(n) - f(k)h(n)h(k)h(m) - f(n)h^2(k)h(m)}{h^2(k)h^2(n)} \\
&= \frac{h^2(k)(f(m)h(n) - f(n)h(m))}{h^2(k)h^2(n)} = \frac{f(m)h(n) - f(n)h(m)}{h^2(n)} = \bar{f}(x).
\end{aligned}$$

Let $x = x_1 \cdot x_2$, $x = \frac{m}{n}$, $x_1 = \frac{m_1}{n_1}$, and $x_2 = \frac{m_2}{n_2}$. Then $m = m_1 \cdot m_2 \cdot k$ and $n = n_1 \cdot n_2 \cdot k$. The following equality shows that $\bar{f}$ is L-additive with respect to $\bar{h}$:

$$\bar{h}(x_1)\bar{f}(x_2) + \bar{h}(x_2)\bar{f}(x_1) = \bar{h}\left(\frac{m_1}{n_1}\right)\bar{f}\left(\frac{m_2}{n_2}\right) + \bar{h}\left(\frac{m_2}{n_2}\right)\bar{f}\left(\frac{m_1}{n_1}\right)$$
$$= \frac{h(m_1)}{h(n_1)} \cdot \left(\frac{f(m_2)h(n_2) - f(n_2)h(m_2)}{h^2(n_2)}\right) + \frac{h(m_2)}{h(n_2)} \cdot \left(\frac{f(m_1)h(n_1) - f(n_1)h(m_1)}{h^2(n_1)}\right)$$
$$= \frac{f(m_2)h(m_1)h(n_2)h(n_1) - f(n_2)h(n_1)h(m_1)h(m_2)}{h^2(n_1) \cdot h^2(n_2)}$$
$$+ \frac{f(m_1)h(m_2)h(n_2)h(n_1) - f(n_1)h(n_2)h(m_1)h(m_2)}{h^2(n_1) \cdot h^2(n_2)}$$

$$= \frac{h(n_1n_2)(h(m_1)f(m_2) + h(m_2)f(m_1))}{h^2(n_1n_2)} - \frac{h(m_1m_2)(h(n_1)f(n_2) + h(n_2)f(n_1))}{h^2(n_1n_2)}$$
$$= \frac{h(n_1n_2)f(m_1m_2) - h(m_1m_2)f(n_1n_2)}{h^2(n_1n_2)} = \bar{f}\left(\frac{m_1 \cdot m_2}{n_1 \cdot n_2}\right) = \bar{f}\left(\frac{k \cdot m_1 \cdot m_2}{k \cdot n_1 \cdot n_2}\right) = \bar{f}(x).$$

Let $\bar{g}$ be an L-additive with respect to $\bar{h}$ extension of $\bar{f}$:

$$0 = \bar{g}(1) = \bar{g}(xx^{-1}) = \bar{g}(x)\bar{h}(x^{-1}) + \bar{g}(x^{-1})\bar{h}(x) = \frac{f(x)}{h(x)} + \bar{g}(x^{-1})h(x).$$

Thus $\bar{g}(x^{-1}) = -\frac{f(x)}{h^2(x)} \forall x \in \Phi \setminus \{0\}$ and

$$\bar{g}\left(\frac{m}{n}\right) = \bar{g}(m)\bar{h}\left(\frac{1}{n}\right) + \bar{g}\left(\frac{1}{n}\right)\bar{h}(m) = \frac{f(m)}{h(n)} - \frac{f(n)h(m)}{h^2(n)} = \frac{f(m)h(n) - f(n)h(m)}{h^2(n)} = \bar{f}\left(\frac{m}{n}\right).$$

Therefore the function $\bar{f}$ is unique.

Let us prove the converse statement. Assume that f and h can be extended to $\mathrm{Frac}(\Phi)$. Suppose that there exists $x \in \Phi \setminus \{0\}$ with $h(x) = 0$. Then $x^{-1} \in \mathrm{Frac}(\Phi)$. Therefore $1 = \bar{h}(x \cdot x^{-1}) = \bar{h}(x) \cdot \bar{h}(x^{-1}) = 0$. This is a contradiction.

Let us prove that $V(x)$ is L-additive with respect to q. Let $m, n \in \mathrm{Frac}(\Phi)$. Assume that $m, n \neq 0$. Then

$$V(mn) = \frac{\bar{f}(mn) \cdot q(mn)}{\bar{h}(mn)} = \frac{\left(\bar{f}(m)\bar{h}(n) + \bar{f}(n)\bar{h}(m)\right) q(mn)}{\bar{h}(m) \cdot \bar{h}(n)}$$
$$= \frac{\bar{f}(m)\bar{h}(n)q(mn)}{\bar{h}(m)\bar{h}(n)} + \frac{\bar{f}(n)\bar{h}(m)q(mn)}{\bar{h}(m)\bar{h}(n)} = \frac{\bar{f}(m)q(m)q(n)}{\bar{h}(m)} + \frac{\bar{f}(n)q(n)q(m)}{\bar{h}(n)}$$
$$= V(m)q(n) + V(n)q(m).$$

Assume now that $m = 0$. Note that $q(0) = 0$. Then

$$V(mn) = V(0) = 0 = 0q(n) + V(n)0 = V(m)q(n) + V(n)q(m).$$

The case $n = 0$ is the same. Hence $V(x)$ is L-additive with respect to q. □

Example 4. Let

$$h(n) = \begin{cases} 1, & n = 1; \\ 0, & n \neq 1; \end{cases} \quad \text{and} \quad f(n) = \begin{cases} 1, & n \in \mathbb{P}; \\ 0, & n \notin \mathbb{P}. \end{cases}$$

Note that f is L-additive with respect to h and we cannot extend f and h from $\mathbb{Z}$ to $\mathbb{Q}$.

Corollary 5. *A function $f : \mathrm{Frac}(\Phi) \to \mathrm{Frac}(\Phi)$ is L-additive iff $f(0) = 0$ and $\frac{f(x)}{x}$ is completely additive.*

Proof. Since every completely additive function is L-additive with respect to $h(x) \equiv 1$ and every L-additive function is L-additive with respect to $h(x) = x$, the statement of corollary directly follows from the last statement of Theorem 3. ☐

Corollary 6 ([6, Theorem 3.1])**.** *A function $D_S^f(n)$ is L-additive iff $\sum_{p \in S} \frac{f_p(n)}{p}$ is completely additive where $D_S^f(n)$ is as defined in [6].*

Proposition 7. *Let Φ be an integral domain and f, $g : \mathrm{Frac}(\Phi) \to \mathrm{Frac}(\Phi)$ be L-additive functions. Then*

1. *$h(x) = \alpha \cdot f(x)$ is L-additive.*
2. *$h(x) = f(x) + g(x)$ is L-additive.*
3. *$f\left(\frac{a}{b}\right) = \frac{f(a)b - f(b)a}{b^2}$.*
4. *$f(a^n) = n \cdot a^{n-1} \cdot f(a)$, $\forall n \in \mathbb{Z}$.*

Proof.

1. Let $m, n \in \mathrm{Frac}(\Phi)$. Then $f(mn) = f(m)n + f(n)m$. So $h(mn) = \alpha \cdot (f(n)m + f(m)n) = (\alpha \cdot f(n))m + (\alpha \cdot f(m))n = h(n)m + h(m)n$. Hence $h(x) = \alpha \cdot f(x)$ is *L-additive*.

2. Let $m, n \in \mathrm{Frac}(\Phi)$. Then $f(mn) = f(m)n + f(n)m$ and $g(mn) = g(m)n + g(n)m$. Now $h(mn) = f(m)n + g(m)n + f(n)m + g(n)m = n(f(m) + g(m)) + m(f(n) + g(n)) = h(m)n + h(n)m$. So $h(x) = f(x) + g(x)$ *is L-additive.*

3. From $f(1) = 0$ it follows that $f(1) = f\left(n \cdot \frac{1}{n}\right) = \frac{f(n)}{n} + f\left(\frac{1}{n}\right) n = 0$. So $f\left(\frac{1}{n}\right) = -\frac{f(n)}{n^2}$. Thus $f\left(\frac{a}{b}\right) = \frac{f(a)}{b} + f\left(\frac{1}{b}\right) a = \frac{f(a)b - f(b)a}{b^2}$.

4. Let us prove this statement by induction. Note that $0 = f(1) = f(a^0) = 0a^{-1}f(a)$ and $f(a) = 1a^0 f(a)$. Assume that the statement holds for $n \in \mathbb{N}$. Let us prove this statement for $n+1:$ $f(a^{n+1}) = f(a \cdot a^n) = f(a)a^n + f(a^n)a = f(a)a^n + nf(a)a^{n-1}a = (n+1)f(a)a^n$. Now we prove this statement for a negative $n:$ $0 = f(1) = f(a^n \cdot a^{-n}) = f(a^n)a^{-n} + a^n f(a^{-n}) = \frac{nf(a)}{a} + a^n f(a^{-n})$ and therefore $f(a^{-n}) = -n \cdot a^{-(n+1)} f(a)$.

☐

Let $S \subseteq \mathbb{P}^*(\Phi)$. We shall call the function

$$D_S(x) = x \cdot \sum_{p \in S} \frac{v_p(x)}{p}$$

an arithmetic subderivative in Φ. If $|S| = 1$, then we shall call it *a partial arithmetic derivative* and if $S = \mathbb{P}^*(\Phi)$, then we shall call it the *arithmetic derivative in* Φ. These functions for integers were studied, for example, by Merikoski et al. [11]. Note that the function D_S is also referred to as "arithmetic type derivative" [5].

Corollary 8. *Let* Φ *be a UFD such that its group of units is finitely generated and* $S \subseteq \mathbb{P}^*(\Phi)$. *Then* $D_S(x)$ *is L-additive.*

Proof. Recall that $v_p(x)$ is completely additive. Then $x \cdot v_p(x)$ is a L-additive function by Corollary 5. Since $v_p(x) \neq 0$ only for finite number of $p \in \mathbb{P}^*(\Phi)$, $D_S(x) = x \cdot \sum_{p \in S} \frac{v_p(x)}{p}$ is L-additive by 1 and 2 of Proposition 7. □

Theorem 9. *Let* Φ *be a UFD such that its group of units is finitely generated. A function* $g : \mathrm{Frac}(\Phi) \setminus \{0\} \to \mathrm{Frac}(\Phi)$ *is completely additive iff*

$$g(n) = \sum_{p \in \mathbb{P}^*(\Phi)} (v_p(n) g(p)) \ \forall n \in \mathrm{Frac}(\Phi).$$

Proof. Let g be a completely additive function. Note that for all $n \in \mathrm{Frac}(\Phi)$ holds

$$n = u_0 \cdot \prod_{p \in \mathbb{P}^*(\Phi)} p^{v_p(n)},$$

where u_0 is a unit of Φ of a finite order k (if $\mathrm{char}(\mathrm{Frac}(\Phi)) = p \neq 0$, then $\gcd(k, p) = 1$). Note that $g(1 \cdot 1) = g(1) + g(1)$. Therefore $g(1) = 0$. Hence

$$g(1) = g(u_0^k) = k g(u_0) = 0.$$

Since $\gcd(k, p) = 1$ and Φ is an integral domain, we see that $g(u_0) = 0$.

Since g is completely additive, it is clear that $g(x^k) = k \cdot g(x)$ for all $k \in \mathbb{N}$. We proved that $g(x^0) = g(1) = 0 = 0 \cdot g(x)$. From $0 = g(1) = g(x^k \cdot x^{-k}) = g(x^k) + g(x^{-k})$ it follows that $g(x^{-k}) = (-k) \cdot g(x)$. Thus $g(x^k) = k \cdot g(x)$ for all $k \in \mathbb{Z}$. Then

$$g(n) = g\left(u \cdot \prod_{\substack{p \in \mathbb{P}^*(\Phi), \\ v_p(n) \neq 0}} p^{v_p(n)} \right) = g(u) + \sum_{\substack{p \in \mathbb{P}^*(\Phi), \\ v_p(n) \neq 0}} (v_p(n) g(p)) = \sum_{p \in \mathbb{P}^*(\Phi)} (v_p(n) g(p)).$$

To prove the converse let

$$g(n) = \sum_{p \in \mathbb{P}^*(\Phi)} (v_p(n) g(p)).$$

Then

$$g(m\cdot n)=\sum_{p\in\mathbb{P}^*(\Phi)}(v_p(m\cdot n)g(p))=\sum_{\substack{p\in\mathbb{P}^*(\Phi)\\ v_p(n)\neq 0\text{ or }v_p(m)\neq 0}}(v_p(m\cdot n)g(p))$$
$$=\sum_{\substack{p\in\mathbb{P}^*(\Phi)\\ v_p(n)\neq 0\text{ or }v_p(m)\neq 0}}((v_p(n)+v_p(m))g(p))$$
$$=\sum_{\substack{p\in\mathbb{P}^*(\Phi)\\ v_p(n)\neq 0}}(v_p(n)g(p))+\sum_{\substack{p\in\mathbb{P}^*(\Phi)\\ v_p(m)\neq 0}}(v_p(m)g(p))=g(n)+g(m).$$

This means that g is completely additive. □

Corollary 10. *Let Φ be a UFD such that its group of units is finitely generated and $f:\mathrm{Frac}(\Phi)\to\mathrm{Frac}(\Phi)$ be an L-additive function. Then*

$$f(x)=x\cdot\sum_{p\in\mathbb{P}^*(\Phi),\,v_p(x)\neq 0}\frac{f(p)v_p(x)}{p}.$$

Corollary 11 ([13, Theorem 1])**.** *Let D be the arithmetic derivative of rational numbers. Then*

$$D(x)=x\cdot\sum_{p\in\mathbb{P},\,v_p(x)\neq 0}\frac{v_p(x)}{p}=D_{\mathbb{P}}(x).$$

Corollary 12. *Let F be a finite field and $f:F\to F$ be an L-additive function. Then $f(x)=0$ for all $x\in F$.*

Proof. Since F is a finite field, there is a prime p such that the characteristic of F is p and $|F|=p^\alpha$. It is well-known that the group of units of F is a cyclic group of order $|F|-1$. Hence $R(F)=\emptyset$. Since every non-zero element in F has the inverse, $\mathbb{P}(F)=\emptyset$. Thus $\mathbb{P}^*(F)=\emptyset$. Now $f(x)=0$ for all $x\in F$ by Corollary 10. □

Example 13. Let us show that the arithmetic derivatives in integers, Gaussian integers, and Eisenstein integers of the same number can be different. For example,

$$D_{\mathbb{P}}(6)=5,\ D_{\mathbb{P}(\mathbb{Z}[i])}(6)=8-6i,\text{ and }D_{\mathbb{P}(\mathbb{Z}[\omega])}(6)=\tfrac{15+6\omega}{1+2\omega}=3-4\sqrt{3}i.$$

Definition 14. We shall say that a UFD Γ of characteristic 0 *satisfies* $(*)$ if its group of units is finitely generated and for any prime $p\in\Gamma$ there is $m\in\mathbb{Z}$ with $p\mid m$.

In Definition 14 by $\mathbb{Z}$ we mean the smallest subring of Γ which contains 1. Note that $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ satisfy $(*)$, but $\mathbb{Z}[x]$ does not satisfy $(*)$.

Lemma 15. *Let Γ satisfy $(*)$ and $f:\mathrm{Frac}(\Gamma)\to\mathrm{Frac}(\Gamma)$. If $f(x)\in\Gamma$ for all $x\in\Gamma$, then the denominator of $\frac{f(p)}{p}$ is equal to 1 or p for all $p\in\mathbb{P}^*(\Gamma)$.*

Proof. Note that $f(p) \in \Gamma$ for all prime $p \in \mathbb{P}(\Gamma)$. Then we have either $f(p)$ is divisible by p and the denominator of $\frac{f(p)}{p}$ is 1 or $f(p)$ is not divisible by p and the denominator of $\frac{f(p)}{p}$ is p. If $p \in R(\Gamma)$, then $\frac{f(p)}{p} \in \Gamma$. Therefore its denominator is 1. □

Let $f^{(0)}(x) = x$ and $f^{(i)}(x) = f(f^{(i-1)}(x))$ for every $i \in \mathbb{N}$.

Theorem 16. *Let Γ satisfy $(*), p \in \mathbb{P}(\Gamma)$ and $f : \mathrm{Frac}(\Gamma) \to \mathrm{Frac}(\Gamma)$ be an L-additive function with $f(x) \in \Gamma$ for all $x \in \Gamma$. Then*

$$v_p(f^{(k)}(x)) \geq \max\{n \in \mathbb{Z} \,|\, n \leq v_p(x) \,\mathrm{and}\, p \mid n\} \, \forall x \in \Gamma.$$

Proof. From Corollary 10 it follows that

$$f(x) = x \cdot \left(\frac{v_p(x) f(p)}{p} + \sum_{p_i \in \mathbb{P}^*(\Gamma),\, p_i \neq p,\, v_{p_i}(x) \neq 0} \frac{v_{p_i}(x) f(p_i)}{p_i} \right) =$$

$$= x \cdot \left(\frac{v_p(x) f(p)}{p} + \frac{A}{B} \right) = x \cdot \left(\frac{v_p(x) f(p) B + Ap}{p \cdot B} \right).$$

Since $\mathbb{Z} \subseteq \Gamma$, we may assume that A, $B \in \Gamma$ and $\gcd(B, p) = 1$ by Lemma 15. Note that $v_p(x) f(p) B + Ap \in \Gamma$. Therefore $v_p(f(x)) \geq v_p(x) - 1$ and if $p \mid v_p(x)$, then $v_p(f(x)) \geq v_p(x)$.

Since Γ satisfies $(*)$, we see that $\{n \in \mathbb{Z} \,|\, n \leq v_p(x) \,\mathrm{and}\, p \mid n\} \neq \emptyset$. This is a set of integers bounded from above. Hence it has the greatest element $\beta = \max\{n \in \mathbb{Z} \mid n \leq v_p(x) \,\mathrm{and}\, p \mid n\}$.

Suppose that $v_p(f^{(k)}(x)) < \beta$ for some k. Note that $v_p(f^{(m)}(x)) \in \mathbb{Z}$ and $v_p(f^{(m+1)}(x)) - v_p(f^{(m)}(x)) \geq -1$ for all m. Hence there is n with $v_p(f^{(n)}(x)) = \beta$ and $v_p(f^{(n+1)}(x)) = \beta - 1$. Since $p \mid \beta$, we have the contradiction. □

Corollary 17. *Let $f \in \{D_{\mathbb{P}^*(\Gamma)}, D_p, D_S\}$ and $x \in \mathrm{Frac}(\Gamma)$. Then there are finitely many different denominators of numbers in the sequence $(f^{(k)}(x))_{k \geq 1}$.*

Definition 18. A function $W : \mathrm{Frac}(\Gamma) \to \mathbb{Q}_+$ is called *norm-like* if

0) $W(x) = 0 \Leftrightarrow x = 0$.

1) $W(a) \cdot W(b) = W(ab)$ for all $a, b \in \mathrm{Frac}(\Gamma)$.

2) $W(x) \in \mathbb{N} \cup \{0\}$ for all $x \in \Gamma$.

3) $W(x) = n$ has a finite number of solutions in Γ for all $n \in \mathbb{N}$.

Lemma 19. *The absolute value of rational number $|\cdot| : \mathrm{Frac}(\mathbb{Z}) \to \mathbb{Q}_+$, the norm of Gaussian rational $N_1 : \mathrm{Frac}(\mathbb{Z}[i]) \to \mathbb{Q}_+$ and the norm of Eisenstein rational $N_2 : \mathrm{Frac}(\mathbb{Z}[\omega]) \to \mathbb{Q}_+$ are norm-like functions.*

Proof. Obviously, all these functions have properties $0-2$ and the absolute value of rational number has property 3.

Let us prove that the equation $N_1(x) = n$ has a finite number of solutions in $\mathbb{Z}[i]$. Let $x = a + bi$. Then $N_1(x) = a^2 + b^2 = n$ with $a, b \in \mathbb{Z}$. So $0 \le |a| \le \sqrt{n}$ and $0 \le |b| \le \sqrt{n}$. Therefore $N_1(x) = n$ has a finite number of solutions.

Let us prove that the equation $N_2(y) = n$ has a finite number of solutions in $\mathbb{Z}[\omega]$. Let $y = c + d\omega$. Then $N_2(y) = a^2 + b^2 - ab = m$ with $a, b \in \mathbb{Z}$. So $(a-b)^2 + ab = m$. Let $a = v + k$ and $b = v - k$. Therefore $(v + k - v + k)^2 + v^2 - k^2 = 3k^2 + v^2 = m$. Note that the denominators of v and k can be equal to 1 or 2. Since $0 \le |v| < \sqrt{m}$ and $0 \le |k| \le \sqrt{\frac{m}{3}}$, we see that $N_2(y) = m$ has a finite number of solutions. □

Theorem 20. *Let Γ satisfy $(*)$, $g : \mathrm{Frac}(\Gamma) \to \mathrm{Frac}(\Gamma)$ be an L-additive function with $g(x) \in \Gamma$ for all $x \in \Gamma$ and W be a norm-like function. If $x \in \mathrm{Frac}(\Gamma)$, then either $W(g^{(k)}(x)) \to +\infty$ or the sequence $(g^{(k)}(x))_{k\ge1}$ is periodic starting from some k.*

Proof. Let $x \in \mathrm{Frac}(\Gamma)$. Assume that $g^{(k)}(x) = 0$ for some k. Then $g^{(n)}(x) = 0$ for all $n \ge k$. Hence Theorem 20 for such x is proved. Now assume that $g^{(k)}(x) \neq 0$ for all k. Note that if $a, b \neq 0$, then

$$W(a) = W\left(b \cdot \frac{a}{b}\right) = W(b) \cdot W\left(\frac{a}{b}\right) \Rightarrow W\left(\frac{a}{b}\right) = \frac{W(a)}{W(b)}.$$

Let $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ be the denominator of x. Then by Theorem 16 there are $\beta_1, \beta_2, \dots, \beta_n$ and $A_k \in \Gamma$ such that $g^{(k)}(x)$ can be written as $\frac{A_k}{p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_n^{\beta_n}}$ (this fraction is not necessary irreducible).

Suppose that $W(g^{(k)}(x)) \not\to +\infty$. This means that

$$\exists A : \forall M \, \exists m > M : W(g^{(m)}(x)) \le A.$$

The last inequality is equivalent to $W(A_m) \le A \cdot W(p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_n^{\beta_n})$.

Note that $B = \lceil A \rceil \cdot W(p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_n^{\beta_n}) \in \mathbb{N}$ and there is a finite number of $y \in \Gamma$ with $W(y) \le B$ by 3) of Definition 18. Therefore for infinitely many m the number $W(A_m)$ belongs to the same finite set. Hence for infinitely many m all $W(A_m)$ are equal by Pigeonhole principle. So there are m_1 and m_2 with $A_{m_1} = A_{m_2}$. Now $g^{(m_1)}(x) = g^{(m_2)}(x)$ where $m_2 > m_1$. Thus $g^{(m_1 + i\cdot(m_2 - m_1))}(x) = g^{(m_1)}(x)$ and $m_2 - m_1 = T$ is a period of $g^{(k)}(x)$. □

Corollary 21. *Let $f \in \{D_{\mathbb{P}^*(\Gamma)}, D_p, D_S\}$.*

(a) If $x \in \mathbb{Q}$, then either $|f^{(k)}(x)| \to +\infty$ or $(f^{(k)}(x))_{k\ge1}$ is periodic starting from some natural number k.

(b) If $x \in \mathbb{Q}(i)$, then either $N_1(f^{(k)}(x)) \to +\infty$ or $(f^{(k)}(x))_{k\ge1}$ is periodic starting from some natural number k.

(c) *If $x \in \mathbb{Q}(\omega)$, then either $N_2(f^{(k)}(x)) \to +\infty$ or $(f^{(k)}(x))_{k\geq 1}$ is periodic starting from some natural number k.*

Theorem 22. *Let Φ be a UFD such that its group of units is finitely generated, $\alpha \in \mathrm{Frac}(\Phi)$ and $f : \mathrm{Frac}(\Phi) \to \mathrm{Frac}(\Phi)$ be an L-additive function.*

1. *Let x_0 be a non-zero solution of*
$$f(x) = \alpha x \tag{1}$$
Then $x = y \cdot x_0$ for any solution x of (1) where y is a solution of $f(y) = 0$.

2. *Let $S = \{p \in \mathbb{P}^*(\Phi) \,|\, f(p) \neq 0\}$. If $v_p(x) = 0$ for all $p \in S$, then $f(x) = 0$.*

3. *There exists a bijection between non-zero solutions l of (1) with $P = \{p \mid v_p(l) \neq 0\} \subseteq S$ and integer solutions $\{v_p(x) \mid p \in \mathbb{P}^*(\Phi)\}$ of*
$$\alpha = \sum_{p\in\mathbb{P}^*(\Phi)} \left(v_p(x) \cdot \frac{f(p)}{p} \right).$$

4. *Assume that $\mathrm{Frac}(\Phi) = \mathbb{Q}$. Let $\frac{f(p)}{p} = \frac{c_p}{z_p}$ be an irreducible fraction for all $p \in S$. The equation (1) has a solution x iff $\beta = \alpha \cdot \delta \in \mathbb{Z}$ is divisible by $\gcd\left(\frac{\delta f(p)}{p} \,|\, p \in P\right)$ where $P = \{p \in \mathbb{P} \,|\, v_p(x) \neq 0\}$ and $\delta = \mathrm{lcm}(z_p \,|\, p \in P)$.*

Proof. 1. Let $x = x_0 \cdot y$, with $f(y) = 0$. Since f is an L-additive function, we see that $f(x) = f(y \cdot x_0) = f(y)x_0 + f(x_0)y = f(x_0)y = (\alpha x_0)y = \alpha x$. Hence x is a solution of (1).

Assume now that x_1 is a non-zero solution of (1). Let us show that $x_1 = y \cdot x_0$ with $f(y) = 0$. Let $y = \frac{x_1}{x_0}$. Then $f(y) = f\left(\frac{x_1}{x_0}\right) = \frac{f(x_1)x_0 - f(x_0)x_1}{x_0^2} = \frac{\alpha x_1 x_0 - \alpha x_0 x_1}{x_0^2} = 0$.

2. By Corollary 10 we have that
$$f(x) = x \cdot \sum_{p\in\mathbb{P}^*(\Phi),\, v_p(x)\neq 0} \frac{f(p)v_p(x)}{p} = 0.$$

3. By Corollary 10 we have that
$$f(l) = \alpha \cdot l = l \cdot \sum_{p\in\mathbb{P}^*(\Phi),\, v_p(l)\neq 0} \frac{f(p)v_p(l)}{p} \Leftrightarrow \alpha = \sum_{p\in\mathbb{P}^*(\Phi),\, v_p(l)\neq 0} v_p(l) \cdot \frac{f(p)}{p}.$$

4. By *3)* we have that

$$\alpha = \frac{f(x)}{x} = \sum_{p\in P}\left(v_p(x)\frac{f(p)}{p}\right) = \sum_{p\in P}\frac{v_p(x)c_p}{z_p}.$$

Note that

$$\beta = \alpha\cdot\delta = \sum_{p\in P}\left(\frac{c_p\delta}{z_p}\cdot v_p(x)\right) = \sum_{p\in P}\left(\frac{f(p)\delta}{p}\cdot v_p(x)\right) \in \mathbb{Z}.$$

This equation is a linear Diophantine equation. It is well-known that this equation has a solution iff β is divisible by $\gcd\left(\frac{\delta f(p)}{p}\,|\,p\in P\right)$. Then we know all $v_p(x)$. Hence we know $\pm x$.

□

Corollary 23 ([8, Theorem 3])**.** *Let $p\in\mathbb{P}$ and $\alpha\in\mathbb{Q}$. The equation $D_p(x)=\alpha x$ has a nontrivial solution iff $\alpha p\in\mathbb{Z}$. Then all nontrivial solutions are of the form $x = cp^{\alpha p}$, where $p \nmid c\in\mathbb{Q}\setminus\{0\}$. Conversely, all numbers of this from are nontrivial solutions.*

Corollary 24 ([13, Theorem 18])**.** *Let $\alpha=\frac{a}{b}$ be a rational number with* $\gcd(a,b)=1$, $b>0$. *Then the equation $D(x)=\alpha x$ has non-zero rational solutions iff b is a product of different primes or $b=1$.*

Theorem 25. *Let a field F be a finite algebraic extension of $\mathbb{Q}$. If $f: F\to F$ is an L-additive linear function, then $f(x)\equiv 0$.*

Proof. According to Artin's theorem on primitive elements, every finite algebraic extension of $\mathbb{Q}$ is simple, i.e., there exists $\alpha\in F$ with $F=\mathbb{Q}(\alpha)$. Let g be a minimal polynomial of α over $\mathbb{Q}$. We may assume that all coefficients of g are integer. It is known that g does not have multiple zeros.

Since $f(xy)=f(x)y+f(y)x$ and $f(x+y)=f(x)+f(y)$, we see that

$$f(ny) = \underbrace{f(y)+f(y)+\cdots+f(y)}_{n} = n\cdot f(y) \text{ and } f(ny)=f(n)y+f(y)n.$$

Therefore $f(n)y=0$. Hence $f(n)=0$ for all $n\in\mathbb{N}$. Note that $f(-n)=-f(n)=0$. Thus $f(x)=0$ for all $x\in\mathbb{Z}$.

Let $g(x)=a_nx^n+\cdots+a_1x+a_0$. Then

$$0=f(0)=f(g(\alpha))=\sum_{i=0}^{n} f(a_i\alpha^i)=\sum_{i=0}^{n} a_if(\alpha^i)=\sum_{i=1}^{n} a_i(i\alpha^{i-1})f(\alpha)=g'(\alpha)f(\alpha).$$

Since $g'(\alpha)\neq 0$, we see that $f(\alpha)=0$. Therefore $f(x)=0$ for every $x\in\mathbb{Z}[\alpha]$. It is easy to see that for every $x\in F$ there are $m\in\mathbb{Z}[\alpha]$ and $n\in\mathbb{Z}$ with $x=\frac{m}{n}$. So $f(x)=0$ by 3 of Proposition 7. Thus $f(x)=0$ for all $x\in\mathbb{Q}(\alpha)=F$. □

4 Final remarks

In this paper we studied L-additive functions over unique factorization domains. So it is natural to ask the following question.

Question 26. Let J be a factorization domain which is not UFD. Are there any non-zero L-additive functions over J?

Haukkanen et al. [7] discussed some ideas about this question.

According to Theorem 25 there are no L-additive linear non-zero functions over any finite extension of $\mathbb{Q}$.

Question 27. Describe all UFD J such that there are no L-additive linear non-zero functions over J.

By Corollary 12 there are no non-zero L-additive functions over a finite field. According to Emmons et al. [3] every value of an L-additive function over $\mathbb{Z}_n$ is a divisor of zero.

Question 28. Let f be an L-additive function over a finite ring. Is it true that all values of f are divisors of zero?

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