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Combinatorial Identities for the Tricomi Polynomials

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Abstract

Using the technique of formal power series, we obtain some two-parameter binomial identities for the Tricomi polynomials. Moreover, we establish some relations between the Tricomi polynomials, the generalized derangement polynomials, and the Touchard polynomials. Finally, we obtain a characterization of the rising and falling factorial powers by means of a generalized binomial theorem.

1 Introduction

The *Tricomi polynomials* [21, 5, 1] are defined by the formula

$$\ell_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{x-\alpha}{k} (-1)^k \frac{x^{n-k}}{(n-k)!} \,. \tag{1}$$

They satisfy the three-term recurrence

$$(n+1)\ell_{n+1}^{(\alpha)}(x) - (\alpha+n)\ell_n^{(\alpha)}(x) + x\ell_{n-1}^{(\alpha)}(x) = 0$$

with initial values $\ell_0^{(\alpha)}(x) = 1$ and $\ell_1^{(\alpha)}(x) = \alpha$, and have ordinary generating series

$$\ell^{(\alpha)}(x;t) = \sum_{n\geq 0} \ell_n^{(\alpha)}(x)\, t^n = (1-t)^{x-\alpha} \mathrm{e}^{xt} \,. \tag{2}$$

The *rising factorials* are defined by the *Pochhammer symbol*

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1),$$

while the *multiset coefficients* are defined by $\left(\!\!\binom{x}{n}\!\!\right) = \frac{(x)_n}{n!}$. They have generating series

$$\sum_{n\geq 0}(x)_n\frac{t^n}{n!} = \frac{1}{(1-t)^x} \qquad \text{and} \qquad \sum_{n\geq 0}\left(\!\!\binom{x}{n}\!\!\right)t^n = \frac{1}{(1-t)^x}\,.$$

Notice that, by series (2), we have the relations

$$\frac{1}{(1-t)^\alpha}\cdot \ell^{(\beta)}(x;t) = \ell^{(\alpha+\beta)}(x;t) \tag{3}$$

and

$$\ell^{(\alpha)}(x;t)\cdot \ell^{(\beta)}(y;t) = \ell^{(\alpha+\beta)}(x+y;t) \tag{4}$$

corresponding to the identities

$$\sum_{k=0}^{n}\left(\!\!\binom{\alpha}{k}\!\!\right)\ell_{n-k}^{(\beta)}(x) = \ell^{(\alpha+\beta)}(x) \tag{5}$$

and

$$\sum_{k=0}^{n}\ell_k^{(\alpha)}(x)\,\ell_{n-k}^{(\beta)}(y) = \ell_n^{(\alpha+\beta)}(x+y)\,. \tag{6}$$

From a purely combinatorial point of view, it is more convenient to consider the exponential version of the Tricomi polynomials, namely the polynomials

$$\Lambda_n^{(\alpha)}(x) = n!\ell_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{x-\alpha}{k}(-1)^k k!\,x^{n-k} \tag{7}$$

satisfying the recurrence

$$\Lambda_{n+2}^{(\alpha)}(x) - (\alpha+n+1)\,\Lambda_{n+1}^{(\alpha)}(x) + (n+1)\,x\,\Lambda_n^{(\alpha)}(x) = 0$$

with the initial values $\Lambda_0^{(\alpha)}(x) = 1$ and $\Lambda_1^{(\alpha)}(x) = \alpha$, and having exponential generating series

$$\Lambda^{(\alpha)}(x;t) = \sum_{n\geq 0}\Lambda_n^{(\alpha)}(x)\,\frac{t^n}{n!} = (1-t)^{x-\alpha}\mathrm{e}^{xt}\,. \tag{8}$$

For the first values of n, we have the following polynomials:

$$\Lambda_0^{(\alpha)}(x) = (\alpha)_0 = 1$$
$$\Lambda_1^{(\alpha)}(x) = (\alpha)_1 = \alpha$$
$$\Lambda_2^{(\alpha)}(x) = (\alpha)_2 - x$$
$$\Lambda_3^{(\alpha)}(x) = (\alpha)_3 - (2+3\alpha)x$$
$$\Lambda_4^{(\alpha)}(x) = (\alpha)_4 - (6+14\alpha+6\alpha^2)x + 3x^2$$
$$\Lambda_5^{(\alpha)}(x) = (\alpha)_5 - (24+70\alpha+50\alpha^2+10\alpha^3)x + (20+15\alpha)x^2$$
$$\Lambda_6^{(\alpha)}(x) = (\alpha)_6 - (120+404\alpha+375\alpha^2+130\alpha^3+15\alpha^4)x + (130+165\alpha+45\alpha^2)x^2 - 15x^3\,.$$

Notice that $\Lambda_n^{(\alpha)}(x)$ is a polynomial of degree n in α and is a polynomial of degree at most $\lfloor n/2 \rfloor$ in x. Moreover, if $\alpha \in \mathbb{N}$, then $\Lambda_n^{(\alpha)}(x)$ is a polynomial with integer coefficients. In particular, we have $\Lambda_n^{(\alpha)}(0) = (\alpha)_n$.

Identities (3) and (4) also hold for the exponential series $\Lambda^{(\alpha)}(x;t)$ defined by (8). This time, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k} (\alpha)_k \, \Lambda_{n-k}^{(\beta)}(x) = \Lambda^{(\alpha+\beta)}(x)$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \Lambda_k^{(\alpha)}(x) \, \Lambda_{n-k}^{(\beta)}(y) = \Lambda_n^{(\alpha+\beta)}(x+y)\,.$$

The Tricomi polynomials $\Lambda_n^{(\alpha)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \Lambda_{n,k}^{(\alpha)} \, x^k$ are the row polynomials of the (improper) Sheffer matrix ([2, p. 309] [12, 13, 7])

$$\Lambda^{(\alpha)} = \left[\Lambda_{n,k}^{(\alpha)}\right]_{n,k\geq 0} = \left(\frac{1}{(1-t)^{\alpha}}, t - \ln \frac{1}{1-t} \right)$$

where

$$\Lambda_{n,k}^{(\alpha)} = \sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{\min(i,k)} \binom{i}{j} {n-i \brack k-j} (-1)^{k-j} (\alpha)_{i-j}\,,$$

where the coefficients ${n \brack k}$ are the Stirling numbers of the first kind [9].

In this paper, we obtain some two-parameter binomial identities for the Tricomi polynomials. Moreover, we establish some relations between the Tricomi polynomials, the generalized derangement polynomials and the Touchard polynomials. Finally, we obtain a characterization of the rising and falling factorial powers by means of a generalized binomial theorem.

To obtain the mentioned two-parameter binomial identities, we will use (as we did in [14], in order to extended a similar identity involving the derangement numbers) the following theorem in the context of formal series:

Theorem 1 (Taylor's formula)**.** *For any formal power series $f(t)$, the exponential generating series of the successive derivatives $D_t^m f(t)$, where $D_t = \frac{d}{dt}$ denotes the formal derivative with respect to t, is*

$$\sum_{m\geq 0} D_t^m f(t)\, \frac{u^m}{m!} = f(t+u)\,. \tag{9}$$

Notice that this theorem is valid both when $f(t)$ is an exponential series and when $f(t)$ is an ordinary series. Moreover, the m-derivative of an exponential series $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$ is

$$D^m f(t) = \sum_{n\geq 0} f_{n+m} \frac{t^n}{n!} \tag{10}$$

while the m-derivative of an ordinary series $f(t) = \sum_{n\geq 0} f_n\, t^n$ is

$$D^m f(t) = m! \sum_{n\geq 0} \binom{m+n}{n} f_{n+m}\, t^n\,. \tag{11}$$

2 Tricomi polynomials

We start by computing the successive derivatives of the generating series of the Tricomi polynomials.

Lemma 2. *For every $m \in \mathbb{N}$, we have the identity*

$$D_t^m \ell^{(\alpha)}(x;t) = m! \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!}\, \ell^{(\alpha+k)}(x;t) \tag{12}$$

or, equivalently,

$$D_t^m \Lambda^{(\alpha)}(x;t) = \sum_{k=0}^{m} \binom{m}{k} \binom{x-\alpha}{k} (-1)^k k!\, x^{m-k}\, \Lambda^{(\alpha+k)}(x;t)\,. \tag{13}$$

Proof. By applying Taylor's formula (9) to series (2), we have

$$\begin{aligned}
\sum_{m\geq 0} D_t^m \ell^{(\alpha)}(x;t)\, \frac{u^m}{m!} &= \ell^{(\alpha)}(x;t+u) \\
&= (1-t-u)^{x-\alpha}\, \mathrm{e}^{x(t+u)} \\
&= (1-t)^{x-\alpha} \left(1 - \frac{u}{1-t}\right)^{x-\alpha} \mathrm{e}^{xt}\mathrm{e}^{xu} \\
&= \ell^{(\alpha)}(x;t) \left(1 - \frac{u}{1-t}\right)^{x-\alpha} \mathrm{e}^{xu} \\
&= \sum_{m\geq 0} \left[\sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{m!\, x^{m-k}}{(m-k)!} \frac{\ell^{(\alpha)}(x;t)}{(1-t)^k} \right] \frac{u^m}{m!}\,.
\end{aligned}$$

Hence, by identity (3), we obtain identity (12) (and, consequently, identity (13)). □

As an immediate consequence of Lemma 2 and formulas (10) and (11), we have the following theorem.

Theorem 3. *For every $m, n \in \mathbb{N}$, we have the identities*

$$\binom{m+n}{n} \ell_{m+n}^{(\alpha)}(x) = \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} \, \ell_n^{(\alpha+k)}(x) \tag{14}$$

and

$$\Lambda_{m+n}^{(\alpha)}(x) = \sum_{k=0}^{m} \binom{m}{k} \binom{x-\alpha}{k} (-1)^k k! \, x^{m-k} \, \Lambda_n^{(\alpha+k)}(x) \,. \tag{15}$$

Remark 4. Notice that Agrawal [1] obtained the following different relation

$$\binom{m+n}{n} \ell_{m+n}^{(\alpha)}(x) = \sum_{k=0}^{\min(m,n)} \left(\!\!\binom{\alpha - x + n}{k}\!\!\right) \ell_{m-k}^{(\alpha+n+k)}(x) \, \ell_{n-k}^{(\alpha-m+k)}(x),$$

which can also be rewritten as

$$\Lambda_{m+n}^{(\alpha)}(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} (\alpha - x + n)_k \, k! \, \Lambda_{m-k}^{(\alpha+n+k)}(x) \, \Lambda_{n-k}^{(\alpha-m+k)}(x) \,.$$

More generally, Lemma 2 implies the following two-parameter identities.

Theorem 5. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{m+k}{k} \ell_{m+k}^{(\alpha)}(x) \, \ell_{n-k}^{(\beta)}(y) = \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} \, \ell_n^{(\alpha+\beta+k)}(x+y) \,. \tag{16}$$

Equivalently, we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} \Lambda_{m+k}^{(\alpha)}(x) \, \Lambda_{n-k}^{(\beta)}(y) = \sum_{k=0}^{m} \binom{m}{k} \binom{x-\alpha}{k} (-1)^k k! \, x^{m-k} \Lambda_n^{(\alpha+\beta+k)}(x+y) \,. \tag{17}$$

Proof. By identity (12) and property (4), we have

$$\begin{aligned} \ell^{(\beta)}(y;t) \, \frac{1}{m!} D_t^m \ell^{(\alpha)}(x;t) &= \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} \, \ell^{(\alpha+k)}(x;t) \, \ell^{(\beta)}(y;t) \\ &= \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} \, \ell^{(\alpha+\beta+k)}(x+y;t) \end{aligned}$$

from which we have identity (16) (and identity (17)). ∎

Remark 6. If $y=-x$, then

$$\ell_n^{(\alpha+\beta+k)}(x+y)=\ell_n^{(\alpha+\beta+k)}(0)=\left(\!\!\binom{\alpha+\beta+k}{n}\!\!\right)$$

and identity (16) becomes

$$\sum_{k=0}^{n}\binom{m+k}{k}\ell_{m+k}^{(\alpha)}(x)\,\ell_{n-k}^{(\beta)}(-x)=\sum_{k=0}^{m}\left(\!\!\binom{\alpha+\beta+k}{n}\!\!\right)\binom{x-\alpha}{k}(-1)^k\frac{x^{m-k}}{(m-k)!}\,. \tag{18}$$

Moreover, if $\beta=y=-x$, then

$$\ell_n^{(-x)}(-x)=(-1)^n\frac{x^n}{n!}$$

and

$$\ell_n^{(\alpha+\beta+k)}(x+y)=\ell_n^{(\alpha+\beta+k)}(0)=\left(\!\!\binom{\alpha-x+k}{n}\!\!\right)=(-1)^n\binom{x-\alpha-k}{n}.$$

So, identity (16) becomes

$$\sum_{k=0}^{n}\binom{m+k}{k}(-1)^k\frac{x^{n-k}}{(n-k)!}\,\ell_{m+k}^{(\alpha)}(x)=\sum_{k=0}^{m}\binom{x-\alpha}{k}\binom{x-\alpha-k}{n}(-1)^k\frac{x^{m-k}}{(m-k)!}\,. \tag{19}$$

Similarly, if $y=0$, then identity (16) becomes

$$\sum_{k=0}^{n}\binom{m+k}{k}\left(\!\!\binom{\beta}{n-k}\!\!\right)\ell_{m+k}^{(\alpha)}(x)=\sum_{k=0}^{m}\binom{x-\alpha}{k}(-1)^k\frac{x^{m-k}}{(m-k)!}\,\ell_n^{(\alpha+\beta+k)}(x)\,. \tag{20}$$

Finally, if $x=y=0$, then identity (16) becomes

$$\sum_{k=0}^{n}\binom{m+k}{k}\left(\!\!\binom{\alpha}{m+k}\!\!\right)\left(\!\!\binom{\beta}{n-k}\!\!\right)=\left(\!\!\binom{\alpha}{m}\!\!\right)\left(\!\!\binom{\alpha+\beta+m}{n}\!\!\right). \tag{21}$$

Equivalently, this identity can be easily rewritten as

$$\sum_{k=0}^{n}\binom{n}{k}(\alpha)_{m+k}(\beta)_{n-k}=(\alpha)_m(\alpha+\beta+m)_n\,. \tag{22}$$

For $m=0$, we recover the fact that the rising factorials form a polynomial sequence of binomial type [11, 18, 10], that is, that they satisfy the binomial identity

$$\sum_{k=0}^{n}\binom{n}{k}(\alpha)_k(\beta)_{n-k}=(\alpha+\beta)_n\,.$$

Notice that replacing α and β by $-\alpha$ and $-\beta$, respectively, then identity (22) becomes

$$\sum_{k=0}^{n}\binom{n}{k}\alpha^{\underline{m+k}}\,\beta^{\underline{n-k}}=\alpha^{\underline{m}}\,(\alpha+\beta-m)^{\underline{n}} \tag{23}$$

where the polynomials $x^{\underline{n}}=x(x-1)(x-2)\cdots(x-n+1)$ are the *falling factorials*.

3 Generalized derangement polynomials

The *generalized derangement numbers* $d_n^{(\nu)}$ and the *generalized arrangement numbers* $a_n^{(\nu)}$ are defined [14] by the formulas

$$d_n^{(\nu)} = \sum_{k=0}^{n} \binom{\nu+n-k}{n-k} \frac{n!}{k!} (-1)^k \tag{24}$$

$$a_n^{(\nu)} = \sum_{k=0}^{n} \binom{\nu+n-k}{n-k} \frac{n!}{k!} \tag{25}$$

and have exponential generating series

$$d^{(\nu)}(t) = \sum_{n\geq 0} d_n^{(\nu)} \frac{t^n}{n!} = \frac{\mathrm{e}^{-t}}{(1-t)^{\nu+1}} \tag{26}$$

$$a^{(\nu)}(t) = \sum_{n\geq 0} a_n^{(\nu)} \frac{t^n}{n!} = \frac{\mathrm{e}^{t}}{(1-t)^{\nu+1}}\,. \tag{27}$$

For $\nu = 0$, we have the ordinary derangement numbers d_n [6, p. 182] (A000166 in the OEIS Sloane) and the ordinary arrangement numbers a_n [6, p. 75] A000522.

The *generalized derangement polynomials* [14] are the *Appell polynomials* [3, 12, 16] associated with the generalized derangement numbers, namely

$$D_n^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(\nu)} x^k = \sum_{k=0}^{n} \binom{\nu+n-k}{n-k} \frac{n!}{k!} (x-1)^k$$

and have exponential generating series

$$D^{(\nu)}(x;t) = \sum_{n\geq 0} D_n^{(\nu)}(x) \frac{t^n}{n!} = \frac{\mathrm{e}^{(x-1)t}}{(1-t)^{\nu+1}}\,. \tag{28}$$

In particular, we have $D_n^{(\nu)}(0) = d_n^{(\nu)}$, $D_n^{(\nu)}(1) = \binom{\nu+n}{n} n!$ and $D_n^{(\nu)}(2) = a_n^{(\nu)}$.

The generalized derangement polynomials and the Tricomi polynomials are related in the following way[1].

Theorem 7. *For every $n \in \mathbb{N}$, we have the identity*

$$D_n^{(\nu)}(x) = \Lambda_n^{(\nu+x)}(x-1)\,. \tag{29}$$

In particular, for $x = 0$, $x = 1$ and $x = 2$, we have the identities

$$\Lambda_n^{(\nu)}(-1) = d_n^{(\nu)}\,, \qquad \Lambda_n^{(\nu+1)}(0) = \binom{\nu+n}{n} n! \qquad \textit{and} \qquad \Lambda_n^{(\nu+2)}(1) = a_n^{(\nu)}\,.$$

[1]Notice that the numbers $d_n^{(\nu)}$ and $a_n^{(\nu)}$, and the polynomials $D_n^{(\nu)}(x)$ considered here are very similar to those considered in some recent papers [4, 7, 15, 8] and that all the results we obtain here can be easily adapted to these variants.

Proof. By series (8) and (30), we have

$$\Lambda^{(\nu+x)}(x-1;t) = (1-t)^{x-1-\nu-x} \mathrm{e}^{(x-1)t} = \frac{\mathrm{e}^{(x-1)t}}{(1-t)^{\nu+1}} = D^{(\nu)}(x;t)\,.$$

This relation implies identity (29) at once. □

Moreover, we have the following result.

Theorem 8. *For every $n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} D_k^{(\alpha)}(x)\, \Lambda_{n-k}^{(\beta)}(y) = \sum_{k=0}^{n} \binom{n}{k} (x)_k\, \Lambda_{n-k}^{(\alpha+\beta)}(x+y-1)\,. \tag{30}$$

In particular, for $x = 0, 2$ and $y = 0$, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k} d_k^{(\alpha)}\, (\beta)_{n-k} = d_n^{(\alpha+\beta)}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} a_k^{(\alpha)}\, (\beta)_{n-k} = \sum_{k=0}^{n} \binom{n}{k} (k+1)!\, a_{n-k}^{(\alpha+\beta-2)}\,.$$

Proof. By series (30) and (8), we have

$$\begin{aligned} D^{(\alpha)}(x;t)\, \Lambda^{(\beta)}(x;t) &= \frac{\mathrm{e}^{(x-1)t}}{(1-t)^{\alpha+1}} \cdot (1-t)^{y-\beta} \mathrm{e}^{yt} \\ &= \frac{1}{(1-t)^{x}} \cdot (1-t)^{x+y-1-\alpha-\beta} \mathrm{e}^{(x+y-1)t} \\ &= \frac{1}{(1-t)^{x}} \cdot \Lambda^{(\alpha+\beta)}(x+y-1;t)\,. \end{aligned}$$

This relation is equivalent to identity (30). □

More generally, we have the following formulas.

Theorem 9. *For every $m, n \in \mathbb{N}$, we have the identities*

$$\sum_{k=0}^{n} \binom{m+k}{k} \frac{d_{n-k}^{(\nu)}}{(n-k)!}\, \ell_{m+k}^{(\alpha)}(x) = \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!}\, \ell_n^{(\alpha+\nu+k)}(x-1) \tag{31}$$

$$\sum_{k=0}^{n} \binom{m+k}{k} \frac{a_{n-k}^{(\nu)}}{(n-k)!}\, \ell_{m+k}^{(\alpha)}(x) = \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!}\, \ell_n^{(\alpha+\nu+k+2)}(x+1)\,. \tag{32}$$

Equivalently, we have the identities

$$\sum_{k=0}^{n}\binom{n}{k}d_k^{(\nu)}\,\Lambda_{m+n-k}^{(\alpha)}(x)=\sum_{k=0}^{m}\binom{m}{k}\binom{x-\alpha}{k}(-1)^k k!\,x^{m-k}\,\Lambda_n^{(\alpha+\nu+k)}(x-1) \tag{33}$$

$$\sum_{k=0}^{n}\binom{n}{k}a_k^{(\nu)}\,\Lambda_{m+n-k}^{(\alpha)}(x)=\sum_{k=0}^{m}\binom{m}{k}\binom{x-\alpha}{k}(-1)^k k!\,x^{m-k}\,\Lambda_n^{(\alpha+\nu+k+2)}(x+1)\,. \tag{34}$$

Proof. From identity (12) and series (26), we have

$$\begin{aligned}
d^{(\nu)}(t)\,\frac{1}{m!}D_t^m\ell^{(\alpha)}(x;t) &= \sum_{k=0}^{m}\binom{x-\alpha}{k}(-1)^k\frac{x^{m-k}}{(m-k)!}\,d^{(\nu)}(t)\,\ell^{(\alpha+k)}(x;t)\\
&= \sum_{k=0}^{m}\binom{x-\alpha}{k}(-1)^k\frac{x^{m-k}}{(m-k)!}\,(1-t)^{x-1-\alpha+\nu-k}\,\mathrm{e}^{(x-1)t}\\
&= \sum_{k=0}^{m}\binom{x-\alpha}{k}(-1)^k\frac{x^{m-k}}{(m-k)!}\,\ell^{(\alpha+\nu+k)}(x-1;t)
\end{aligned}$$

from which we obtain identity (31). In a similar way, we also obtain identity (32). □

Consider the *Touchard polynomials* $T_n(x)$, [17, 20], and the associated polynomials $U_n(x)$ defined by

$$T_n(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}x^{\underline{k}}$$

$$U_n(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(x)_k$$

and having exponential generating series

$$T(x;t)=\sum_{n\geq 0}T_n(x)\frac{t^n}{n!}=\mathrm{e}^{-t}(1+t)^x \tag{35}$$

$$U(x;t)=\sum_{n\geq 0}U_n(x)\frac{t^n}{n!}=\frac{\mathrm{e}^{-t}}{(1-t)^x}\,. \tag{36}$$

The following identities relate the Tricomi polynomials and the generalized derangement polynomials by means of the Touchard polynomials.

Theorem 10. *We have the identities*

$$\Lambda_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\,T_k(x+1)\,D_{n-k}^{(\alpha)}(x) \tag{37}$$

$$D_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}U_k(x+1)\,\Lambda_{n-k}^{(\alpha)}(x)\,. \tag{38}$$

Proof. By series (8), (28) and (35), we have

$$\Lambda^{(\alpha)}(x;t) = (1-t)^{x-\alpha}\mathrm{e}^{xt} = \mathrm{e}^{t}(1-t)^{x+1}\cdot\frac{\mathrm{e}^{(x-1)t}}{(1-t)^{\alpha+1}} = T(x+1;-t)\cdot D^{(\alpha)}(x;t)$$

from which we get identity (37) at once. Similarly, by series (8), (28) and (36), we have

$$D^{(\alpha)}(x;t) = \frac{\mathrm{e}^{(x-1)t}}{(1-t)^{\alpha+1}} = \frac{\mathrm{e}^{-t}}{(1-t)^{x+1}}\cdot(1-t)^{x-\alpha}\mathrm{e}^{xt} = U(x+1;t)\cdot\Lambda^{(\alpha)}(x;t)$$

from which we get identity (38) at once. □

4 Final remarks

The rising factorials and the falling factorials form two polynomial sequences of binomial type and have several characterizations and combinatorial interpretations [10]. In Remark 6, we noticed that these polynomials satisfy the generalized binomial theorems (22) and (23), respectively. More generally, we have the following characterization.

Theorem 11. *Let $\{p_n(x)\}_{n\in\mathbb{N}}$ be a polynomial sequence, where each polynomial $p_n(x)$ has degree n. There exists a constant $\lambda\neq 0$ for which the binomial identity*

$$\sum_{k=0}^{n}\binom{n}{k}p_{m+k}(x)\,p_{n-k}(y) = p_m(x)\,p_n(x+y+\lambda\,m) \qquad \forall m,n\in\mathbb{N} \tag{39}$$

holds if and only if there exists a constant μ such that

$$p_n(x) = (\lambda\mu)^n(x/\lambda)_n\,. \tag{40}$$

Proof. If identity (39) is true for every $m,n\in\mathbb{N}$, then it is true also for $m=0$ and $n\in\mathbb{N}$. This implies that $\{p_n(x)\}_{n\in\mathbb{N}}$ is a polynomial sequence of binomial type and, consequently, that it has exponential generating series

$$p(x;t) = \sum_{n\geq 0}p_n(x)\,\frac{t^n}{n!} = \mathrm{e}^{xf(t)}$$

for a given exponential series $f(t)=\sum_{n\geq 0}f_n\frac{t^n}{n!}$ with $f_0=0$ and $f_1\neq 0$. Hence, identity (39) turns out to be equivalent to the identity

$$(D_t^m p(x;t))\,p(y;t) = p_m(x)\,p(x+y+\lambda m;t) \qquad \forall m\in\mathbb{N}$$

that is

$$(D_t^m \mathrm{e}^{xf(t)})\,\mathrm{e}^{yf(t)} = p_m(x)\,\mathrm{e}^{(x+y+\lambda m)f(t)} \qquad \forall m\in\mathbb{N}\,.$$

that is

$$D_t^m \mathrm{e}^{xf(t)} = p_m(x)\,\mathrm{e}^{(x+\lambda m)f(t)} \qquad \forall m \in \mathbb{N}\,.$$

For $m = 1$, this relation reduces to

$$xf'(t)\,\mathrm{e}^{xf(t)} = p_1(x)\,\mathrm{e}^{(x+\lambda)f(t)}$$

or

$$xf'(t) = p_1(x)\,\mathrm{e}^{\lambda f(t)}$$

or

$$\frac{f'(t)}{\mathrm{e}^{\lambda f(t)}} = \frac{p_1(x)}{x} = \mu$$

for a constant μ. This is equivalent to $p_1(x) = \mu\, x$ and $f'(t) = \mu\,\mathrm{e}^{\lambda f(t)}$. By integrating this last differential equation, we obtain

$$f(t) = \frac{1}{\lambda} \ln \frac{1}{1 - \lambda\mu t}$$

and consequently

$$p(x;t) = \mathrm{e}^{\frac{x}{\lambda} \ln \frac{1}{1-\lambda\mu t}} = \frac{1}{(1-\lambda\mu t)^{x/\lambda}} = \sum_{n\geq 0} (\lambda\mu)^n (x/\lambda)_n \frac{t^n}{n!}$$

from which we have identity (40). Vice versa, employing identity (22), we can say that the polynomials defined by formula (40) satisfy the binomial identity (39). $\square$

Notice that the falling factorials can be expressed by identity (40) for $\lambda = -1$ and $\mu = 1$, namely $x^{\underline{n}} = (-1)^n(-x)_n$.

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