Combinatorial Identities
for the Tricomi Polynomials

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Abstract
Using the technique of formal power series, we obtain some two-parameter binomial identities for the Tricomi polynomials. Moreover, we establish some relations between the Tricomi polynomials, the generalized derangement polynomials, and the Touchard polynomials. Finally, we obtain a characterization of the rising and falling factorial powers by means of a generalized binomial theorem.

1 Introduction
The Tricomi polynomials [21, 5, 1] are defined by the formula

\[ \ell_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{x - \alpha}{k} (-1)^k \frac{x^{n-k}}{(n-k)!}. \]  

They satisfy the three-term recurrence

\[ (n + 1)\ell_n^{(\alpha)}(x) - (\alpha + n)\ell_n^{(\alpha)}(x) + x\ell_{n-1}^{(\alpha)}(x) = 0 \]

with initial values \( \ell_0^{(\alpha)}(x) = 1 \) and \( \ell_1^{(\alpha)}(x) = \alpha \), and have ordinary generating series

\[ \ell^{(\alpha)}(x; \ell) = \sum_{n \geq 0} \ell_n^{(\alpha)}(x) t^n = (1 - t)^{x-\alpha} e^{xt}. \]
The *rising factorials* are defined by the *Pochhammer symbol*

\[(x)_n = x(x + 1)(x + 2) \cdots (x + n - 1),\]

while the *multiset coefficients* are defined by \[\binom{x}{n} = \frac{(x)_n}{n!}.\] They have generating series

\[\sum_{n \geq 0} (x)_n \frac{t^n}{n!} = \frac{1}{(1 - t)^x} \quad \text{and} \quad \sum_{n \geq 0} \binom{x}{n} t^n = \frac{1}{(1 - t)^x}.\]

Notice that, by series (2), we have the relations

\[\frac{1}{(1 - t)^\alpha} \cdot \ell^{(\beta)}(x; t) = \ell^{(\alpha + \beta)}(x; t)\] (3)

and

\[\ell^{(\alpha)}(x; t) \cdot \ell^{(\beta)}(y; t) = \ell^{(\alpha + \beta)}(x + y; t)\] (4)

corresponding to the identities

\[\sum_{k=0}^{n} \binom{\alpha}{k} \ell^{(\beta)}_{n-k}(x) = \ell^{(\alpha + \beta)}(x)\] (5)

and

\[\sum_{k=0}^{n} \ell^{(\alpha)}_{k}(x) \ell^{(\beta)}_{n-k}(y) = \ell^{(\alpha + \beta)}_{n}(x + y).\] (6)

From a purely combinatorial point of view, it is more convenient to consider the exponential version of the Tricomi polynomials, namely the polynomials

\[\Lambda^{(\alpha)}_{n}(x) = n! \ell^{(\alpha)}_{n}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x - \alpha}{k}\right)(-1)^{k}k! x^{n-k}\] (7)

satisfying the recurrence

\[\Lambda^{(\alpha)}_{n+2}(x) - (\alpha + n + 1) \Lambda^{(\alpha)}_{n+1}(x) + (n + 1) x \Lambda^{(\alpha)}_{n}(x) = 0\]

with the initial values \[\Lambda^{(\alpha)}_{0}(x) = 1\] and \[\Lambda^{(\alpha)}_{1}(x) = \alpha,\] and having exponential generating series

\[\Lambda^{(\alpha)}(x; t) = \sum_{n \geq 0} \Lambda^{(\alpha)}_{n}(x) \frac{t^n}{n!} = (1 - t)^{-\alpha} e^{xt} \]. (8)
For the first values of $n$, we have the following polynomials:

\[
\begin{align*}
\Lambda_0^{(\alpha)}(x) &= (\alpha)_0 = 1 \\
\Lambda_1^{(\alpha)}(x) &= (\alpha)_1 = \alpha \\
\Lambda_2^{(\alpha)}(x) &= (\alpha)_2 - x \\
\Lambda_3^{(\alpha)}(x) &= (\alpha)_3 - (2 + 3\alpha)x \\
\Lambda_4^{(\alpha)}(x) &= (\alpha)_4 - (6 + 14\alpha + 6\alpha^2)x + 3x^2 \\
\Lambda_5^{(\alpha)}(x) &= (\alpha)_5 - (24 + 70\alpha + 50\alpha^2 + 10\alpha^3)x + (20 + 15\alpha)x^2 \\
\Lambda_6^{(\alpha)}(x) &= (\alpha)_6 - (120 + 404\alpha + 375\alpha^2 + 130\alpha^3 + 15\alpha^4)x + (130 + 165\alpha + 45\alpha^2)x^2 - 15x^3.
\end{align*}
\]

Notice that $\Lambda_n^{(\alpha)}(x)$ is a polynomial of degree $n$ in $\alpha$ and is a polynomial of degree at most $\lfloor n/2 \rfloor$ in $x$. Moreover, if $\alpha \in \mathbb{N}$, then $\Lambda_n^{(\alpha)}(x)$ is a polynomial with integer coefficients. In particular, we have $\Lambda_n^{(\alpha)}(0) = (\alpha)_n$.

Identities (3) and (4) also hold for the exponential series $\Lambda^{(\alpha)}(x; t)$ defined by (8). This time, we have the identities

\[
\sum_{k=0}^{n} \binom{n}{k} (\alpha)_k \Lambda_{n-k}^{(\beta)}(x) = \Lambda^{(\alpha+\beta)}(x)
\]

and

\[
\sum_{k=0}^{n} \binom{n}{k} \Lambda_k^{(\alpha)}(x) \Lambda_{n-k}^{(\beta)}(y) = \Lambda_n^{(\alpha+\beta)}(x+y).
\]

The Tricomi polynomials $\Lambda_n^{(\alpha)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \Lambda_{n,k}^{(\alpha)} x^k$ are the row polynomials of the (improper) Sheffer matrix ([2, p. 309] [12, 13, 7]):

\[
\Lambda^{(\alpha)} = [\Lambda_{n,k}^{(\alpha)}]_{n,k \geq 0} = \left(\frac{1}{(1-t)^\alpha}, t - \ln \frac{1}{1-t}\right)
\]

where

\[
\Lambda_{n,k}^{(\alpha)} = \sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{\min(i,k)} \binom{i}{j} \binom{n-i}{k-j} (-1)^{k-j} (\alpha)_{i-j},
\]

where the coefficients $\binom{n}{k}$ are the Stirling numbers of the first kind [9].

In this paper, we obtain some two-parameter binomial identities for the Tricomi polynomials. Moreover, we establish some relations between the Tricomi polynomials, the generalized derangement polynomials and the Touchard polynomials. Finally, we obtain a characterization of the rising and falling factorial powers by means of a generalized binomial theorem.

To obtain the mentioned two-parameter binomial identities, we will use (as we did in [14], in order to extended a similar identity involving the derangement numbers) the following theorem in the context of formal series:
Theorem 1 (Taylor’s formula). For any formal power series \( f(t) \), the exponential generating series of the successive derivatives \( D^m_t f(t) \), where \( D_t = \frac{d}{dt} \) denotes the formal derivative with respect to \( t \), is

\[
\sum_{m \geq 0} D^m_t f(t) \frac{u^m}{m!} = f(t + u). \tag{9}
\]

Notice that this theorem is valid both when \( f(t) \) is an exponential series and when \( f(t) \) is an ordinary series. Moreover, the \( m \)-derivative of an exponential series \( f(t) = \sum_{n \geq 0} f_n \frac{t^n}{n!} \) is

\[
D^m f(t) = \sum_{n \geq 0} f_{n+m} \frac{t^n}{n!}, \tag{10}
\]

while the \( m \)-derivative of an ordinary series \( f(t) = \sum_{n \geq 0} f_n t^n \) is

\[
D^m f(t) = m! \sum_{n \geq 0} \binom{m+n}{n} f_{n+m} t^n. \tag{11}
\]

2 Tricomi polynomials

We start by computing the successive derivatives of the generating series of the Tricomi polynomials.

Lemma 2. For every \( m \in \mathbb{N} \), we have the identity

\[
D^m_t \ell^{(\alpha)}(x; t) = m! \sum_{k=0}^{m} \binom{m}{k} \frac{x - \alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} \ell^{(\alpha+k)}(x; t) \tag{12}
\]

or, equivalently,

\[
D^m_t \Lambda^{(\alpha)}(x; t) = \sum_{k=0}^{m} \binom{m}{k} \binom{x - \alpha}{k} (-1)^k k! \frac{x^{m-k}}{(m-k)!} \Lambda^{(\alpha+k)}(x; t). \tag{13}
\]

Proof. By applying Taylor’s formula (9) to series (2), we have

\[
\sum_{m \geq 0} D^m_t \ell^{(\alpha)}(x; t) \frac{u^m}{m!} = \ell^{(\alpha)}(x; t + u)
\]

\[
= (1 - t - u)^{x-\alpha} e^{x(t+u)}
\]

\[
= (1 - t)^{x-\alpha} \left(1 - \frac{u}{1-t}\right)^{x-\alpha} e^{xt} e^{xu}
\]

\[
= \ell^{(\alpha)}(x; t) \left(1 - \frac{u}{1-t}\right)^{x-\alpha} e^{xu}
\]

\[
= \sum_{m \geq 0} \left[ \sum_{k=0}^{m} \binom{m}{k} \frac{x - \alpha}{k} (-1)^k \frac{m! \ell^{(\alpha)}(x; t) x^{m-k}}{(m-k)! (1-t)^k} \right] \frac{u^m}{m!}.
\]

Hence, by identity (3), we obtain identity (12) (and, consequently, identity (13)). □
As an immediate consequence of Lemma 2 and formulas (10) and (11), we have the following theorem.

**Theorem 3.** For every $m, n \in \mathbb{N}$, we have the identities

\[
\binom{m+n}{n} \ell_m^{(x)}(x) = \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} \ell_n^{(x)}(x) \tag{14}
\]

and

\[
\Lambda_m^{(x)}(x) = \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k k! x^{m-k} \Lambda_n^{(x)}(x). \tag{15}
\]

**Remark 4.** Notice that Agrawal [1] obtained the following different relation

\[
\binom{m+n}{n} \ell_m^{(x)}(x) = \sum_{k=0}^{\min(m,n)} \binom{\alpha-x+n}{k} \ell_{m-k}^{(\alpha)}(x) \ell_{n-k}^{(\alpha)}(x),
\]

which can also be rewritten as

\[
\Lambda_m^{(x)}(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} (\alpha-x+n) k! \Lambda_{m-k}^{(\alpha)}(x) \Lambda_{n-k}^{(\alpha)}(x).
\]

More generally, Lemma 2 implies the following two-parameter identities.

**Theorem 5.** For every $m, n \in \mathbb{N}$, we have the identity

\[
\sum_{k=0}^{n} \binom{m+k}{k} \ell_{m+k}^{(x)}(x) \ell_n^{(y)}(y) = \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} \ell_n^{(x)}(x+\alpha)
\]

**Proof.** By identity (12) and property (4), we have

\[
\ell^{(\beta)}(y, t) \frac{1}{m!} D_t^m \ell^{(\alpha)}(x; t) = \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} \ell^{(\alpha)}(x; t) \ell^{(\beta)}(y; t)
\]

\[
= \sum_{k=0}^{m} \binom{x-\alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} \ell^{(\alpha+\beta)}(x+y; t)
\]

from which we have identity (16) (and identity (17)).
Remark 6. If \( y = -x \), then
\[
\ell_n^{(\alpha + \beta + k)}(x + y) = \ell_n^{(\alpha + \beta + k)}(0) = \binom{\alpha + \beta + k}{n}
\]
and identity (16) becomes
\[
\sum_{k=0}^{n} \binom{m+k}{k} \ell_m^{(\alpha)}(x) \ell_{m-k}^{(\beta)}(-x) = \sum_{k=0}^{m} \binom{x - \alpha}{k} \binom{x - \alpha - k}{n} (-1)^k \frac{x^{m-k}}{(m-k)!}.
\] (18)
Moreover, if \( \beta = y = -x \), then
\[
\ell_n^{(-x)}(-x) = (-1)^n \frac{x^n}{n!}
\]
and
\[
\ell_n^{(\alpha + \beta + k)}(x + y) = \ell_n^{(\alpha + \beta + k)}(0) = \binom{\alpha - x + k}{n} = (-1)^n \binom{x - \alpha - k}{n}.
\]
So, identity (16) becomes
\[
\sum_{k=0}^{n} \binom{m+k}{k} (-1)^k \frac{x^{n-k}}{(n-k)!} \ell_m^{(\alpha)}(x) = \sum_{k=0}^{m} \binom{x - \alpha}{k} \binom{x - \alpha - k}{n} (-1)^k \frac{x^{m-k}}{(m-k)!}.
\] (19)
Similarly, if \( y = 0 \), then identity (16) becomes
\[
\sum_{k=0}^{n} \binom{m+k}{k} \binom{\beta}{n-k} \ell_m^{(\alpha)}(x) = \sum_{k=0}^{m} \binom{x - \alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} \ell_n^{(\alpha + \beta + k)}(x).
\] (20)
Finally, if \( x = y = 0 \), then identity (16) becomes
\[
\sum_{k=0}^{n} \binom{m+k}{k} \binom{\alpha}{m+k} \binom{\beta}{n-k} = \binom{\alpha}{m} \binom{\alpha + \beta + m}{n}.
\] (21)
Equivalently, this identity can be easily rewritten as
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{\alpha}{m+k} \binom{\beta}{n-k} = \binom{\alpha + \beta + m}{n}.
\] (22)
For \( m = 0 \), we recover the fact that the rising factorials form a polynomial sequence of binomial type \([11, 18, 10]\), that is, that they satisfy the binomial identity
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{\alpha}{k} \binom{\beta}{n-k} = \binom{\alpha + \beta}{n}.
\]
Notice that replacing \( \alpha \) and \( \beta \) by \(-\alpha\) and \(-\beta\), respectively, then identity (22) becomes
\[
\sum_{k=0}^{n} \binom{n}{k} \alpha^{m+k} \beta^{n-k} = \alpha^m \beta^n \binom{\alpha + \beta - m}{n}
\] (23)
where the polynomials \( x^\underline{n} = x(x-1)(x-2) \cdots (x-n+1) \) are the falling factorials.
3 Generalized derangement polynomials

The generalized derangement numbers $d_n^{(\nu)}$ and the generalized arrangement numbers $a_n^{(\nu)}$ are defined [14] by the formulas

$$d_n^{(\nu)} = \sum_{k=0}^{n} \binom{\nu + n - k}{n - k} \frac{n!}{k!} (-1)^k$$

and

$$a_n^{(\nu)} = \sum_{k=0}^{n} \binom{\nu + n - k}{n - k} \frac{n!}{k!}$$

and have exponential generating series

$$d^{(\nu)}(t) = \sum_{n \geq 0} d_n^{(\nu)} \frac{t^n}{n!} = e^{-t} \frac{e^{-(1-t)^{\nu+1}}}{(1-t)^{\nu+1}}$$

and

$$a^{(\nu)}(t) = \sum_{n \geq 0} a_n^{(\nu)} \frac{t^n}{n!} = e^t \frac{e^{(1-t)^{\nu+1}}}{(1-t)^{\nu+1}}.$$ 

For $\nu = 0$, we have the ordinary derangement numbers $d_n$ [6, p. 182] (A000166 in the OEIS Sloane) and the ordinary arrangement numbers $a_n$ [6, p. 75] A000522.

The generalized derangement polynomials [14] are the Appell polynomials [3, 12, 16] associated with the generalized derangement numbers, namely

$$D_n^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(\nu)} x^k = \sum_{k=0}^{n} \binom{\nu + n - k}{n - k} \frac{n!}{k!} (x-1)^k.$$ 

and have exponential generating series

$$D^{(\nu)}(x; t) = \sum_{n \geq 0} D_n^{(\nu)}(x) \frac{t^n}{n!} = e^{(x-1)\frac{t}{1-t)^{\nu+1}}}.$$ 

In particular, we have $D_n^{(\nu)}(0) = d_n^{(\nu)}$, $D_n^{(\nu)}(1) = (\nu+n)n!$ and $D_n^{(\nu)}(2) = a_n^{(\nu)}$.

The generalized derangement polynomials and the Tricomi polynomials are related in the following way\(^1\).

**Theorem 7.** For every $n \in \mathbb{N}$, we have the identity

$$D_n^{(\nu)}(x) = \Lambda_n^{(\nu+x)}(x-1).$$

In particular, for $x = 0$, $x = 1$ and $x = 2$, we have the identities

$$\Lambda_n^{(\nu)}(-1) = d_n^{(\nu)}, \quad \Lambda_n^{(\nu+1)}(0) = \binom{\nu+n}{n} n! \quad \text{and} \quad \Lambda_n^{(\nu+2)}(1) = a_n^{(\nu)}.$$ 

\(^1\)Notice that the numbers $d_n^{(\nu)}$ and $a_n^{(\nu)}$, and the polynomials $D_n^{(\nu)}(x)$ considered here are very similar to those considered in some recent papers [4, 7, 15, 8] and that all the results we obtain here can be easily adapted to these variants.
Proof. By series (8) and (30), we have
\[
\Lambda^{(\nu + x)}(x - 1; t) = (1 - t)^{x-\nu-x} e^{(x-1)t} = \frac{e^{(x-1)t}}{(1 - t)^{\nu + 1}} = D^{(\nu)}(x; t).
\]

This relation implies identity (29) at once.

Moreover, we have the following result.

**Theorem 8.** For every \( n \in \mathbb{N} \), we have the identity
\[
\sum_{k=0}^{n} \binom{n}{k} D^{(\alpha)}_k (x) \Lambda^{(\beta)}_{n-k}(y) = \sum_{k=0}^{n} \binom{n}{k} (x)_k \Lambda^{(\alpha + \beta)}_{n-k}(x + y - 1). \tag{30}
\]

In particular, for \( x = 0, 2 \) and \( y = 0 \), we have the identities
\[
\sum_{k=0}^{n} \binom{n}{k} d^{(\alpha)}_k (\beta)_{n-k} = d^{(\alpha + \beta)}_n
\]
and
\[
\sum_{k=0}^{n} \binom{n}{k} a^{(\alpha)}_k (\beta)_{n-k} = \sum_{k=0}^{n} \binom{n}{k} (k + 1)! a^{(\alpha + \beta - 2)}_{n-k}.
\]

Proof. By series (30) and (8), we have
\[
D^{(\alpha)}(x; t) \Lambda^{(\beta)}(x; t) = \frac{e^{(x-1)t}}{(1 - t)^{\alpha + 1}} \cdot (1 - t)^{y-\beta} e^{yt}
\]
\[
= \frac{1}{(1 - t)^x} \cdot (1 - t)^{x+y-1-\alpha-\beta} e^{(x+y-1)t}
\]
\[
= \frac{1}{(1 - t)^x} \cdot \Lambda^{(\alpha + \beta)}(x + y - 1; t).
\]

This relation is equivalent to identity (30).

More generally, we have the following formulas.

**Theorem 9.** For every \( m, n \in \mathbb{N} \), we have the identities
\[
\sum_{k=0}^{n} \binom{m + k}{n} \frac{d^{(\nu)}_{n-k}}{(n-k)!} e^{(\alpha)}_{m+k}(x) = \sum_{k=0}^{m} \binom{x - \alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} e^{(\alpha + \nu + k)}(x - 1) \tag{31}
\]
\[
\sum_{k=0}^{n} \binom{m + k}{n} \frac{a^{(\nu)}_{n-k}}{(n-k)!} e^{(\alpha)}_{m+k}(x) = \sum_{k=0}^{m} \binom{x - \alpha}{k} (-1)^k \frac{x^{m-k}}{(m-k)!} e^{(\alpha + \nu + k+2)}(x + 1). \tag{32}
\]
Equivalently, we have the identities
\[
\sum_{k=0}^{n} \binom{n}{k} d^{(\nu)}_{k} \Lambda_{m+n-k}(x) = \sum_{k=0}^{m} \binom{m}{k} \left( x - \alpha \right)^{k} \left( -1 \right)^{k} k! \Lambda_{m+k}^{(\alpha+\nu)}(x-1) \tag{33}
\]
\[
\sum_{k=0}^{n} \binom{n}{k} a^{(\nu)}_{k} \Lambda_{m+n-k}(x) = \sum_{k=0}^{m} \binom{m}{k} \left( x - \alpha \right)^{k} \left( -1 \right)^{k} k! \Lambda_{m+k}^{(\alpha+\nu+2)}(x+1) . \tag{34}
\]

**Proof.** From identity (12) and series (26), we have
\[
d^{(\nu)}(t) \frac{1}{m!} D_{t}^{m} \ell^{(\alpha)}(x; t) = \sum_{k=0}^{m} \binom{x - \alpha}{k} \left( -1 \right)^{k} \frac{x^{m-k}}{(m-k)!} \ell^{(\alpha+k)}(x; t)
\]
\[
= \sum_{k=0}^{m} \binom{x - \alpha}{k} \left( -1 \right)^{k} \frac{x^{m-k}}{(m-k)!} (1-t)^{x-1-\alpha+k} e^{(x-1)t}
\]
\[
= \sum_{k=0}^{m} \binom{x - \alpha}{k} \left( -1 \right)^{k} \frac{x^{m-k}}{(m-k)!} \ell^{(\alpha+k)}(x-1; t)
\]
from which we obtain identity (31). In a similar way, we also obtain identity (32). \(\square\)

Consider the **Touchard polynomials** \(T_{n}(x)\), [17, 20], and the associated polynomials \(U_{n}(x)\) defined by
\[
T_{n}(x) = \sum_{k=0}^{n} \binom{n}{k} \left( -1 \right)^{n-k} x^{k}
\]
\[
U_{n}(x) = \sum_{k=0}^{n} \binom{n}{k} \left( -1 \right)^{n-k} (x)_{k}
\]
and having exponential generating series
\[
T(x; t) = \sum_{n \geq 0} T_{n}(x) \frac{t^{n}}{n!} = e^{-t}(1+t)^{x} \tag{35}
\]
\[
U(x; t) = \sum_{n \geq 0} U_{n}(x) \frac{t^{n}}{n!} = \frac{e^{-t}}{(1-t)^{x}} . \tag{36}
\]

The following identities relate the Tricomi polynomials and the generalized derangement polynomials by means of the Touchard polynomials.

**Theorem 10.** We have the identities
\[
\Lambda_{n}^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} T_{k}(x+1) D_{n-k}^{(\alpha)}(x) \tag{37}
\]
\[
D_{n}^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} U_{k}(x+1) \Lambda_{n-k}^{(\alpha)}(x) . \tag{38}
\]
Proof. By series (8), (28) and (35), we have

\[ \Lambda^{(\alpha)}(x; t) = (1 - t)^{x-\alpha}e^{xt} = e^t(1 - t)^{x+1} \cdot \frac{e^{(x-1)t}}{(1 - t)^{\alpha + 1}} = T(x + 1; -t) \cdot D^{(\alpha)}(x; t) \]

from which we get identity (37) at once. Similarly, by series (8), (28) and (36), we have

\[ D^{(\alpha)}(x; t) = e^{\alpha - t} = \frac{e^{-t}}{(1 - t)^{x+1}} \cdot (1 - t)^{x-\alpha}e^{xt} = U(x + 1; t) \cdot \Lambda^{(\alpha)}(x; t) \]

from which we get identity (38) at once. \( \square \)

4 Final remarks

The rising factorials and the falling factorials form two polynomial sequences of binomial type and have several characterizations and combinatorial interpretations [10]. In Remark 6, we noticed that these polynomials satisfy the generalized binomial theorems (22) and (23), respectively. More generally, we have the following characterization.

**Theorem 11.** Let \( \{p_n(x)\}_{n \in \mathbb{N}} \) be a polynomial sequence, where each polynomial \( p_n(x) \) has degree \( n \). There exists a constant \( \lambda \neq 0 \) for which the binomial identity

\[ \sum_{k=0}^{n} \binom{n}{k} p_{m+k}(x) p_{n-k}(y) = p_m(x) p_n(x + y + \lambda m) \quad \forall m, n \in \mathbb{N} \quad (39) \]

holds if and only if there exists a constant \( \mu \) such that

\[ p_n(x) = (\lambda \mu)^n (x/\lambda)_n. \quad (40) \]

**Proof.** If identity (39) is true for every \( m, n \in \mathbb{N} \), then it is true also for \( m = 0 \) and \( n \in \mathbb{N} \). This implies that \( \{p_n(x)\}_{n \in \mathbb{N}} \) is a polynomial sequence of binomial type and, consequently, that it has exponential generating series

\[ p(x; t) = \sum_{n \geq 0} p_n(x) \frac{t^n}{n!} = e^{xf(t)} \]

for a given exponential series \( f(t) = \sum_{n \geq 0} f_n \frac{t^n}{n!} \) with \( f_0 = 0 \) and \( f_1 \neq 0 \). Hence, identity (39) turns out to be equivalent to the identity

\[ (D_t^m p(x; t)) p(y; t) = p_m(x) p(x + y + \lambda m; t) \quad \forall m \in \mathbb{N} \]

that is

\[ (D_t^m e^{xf(t)}) e^{yf(t)} = p_m(x) e^{(x+y+\lambda m)f(t)} \quad \forall m \in \mathbb{N}. \]
that is
\[ D^m e^{xf(t)} = p_m(x) e^{(x+\lambda m)f(t)} \quad \forall m \in \mathbb{N}. \]

For \( m = 1 \), this relation reduces to
\[ xf'(t) e^{xf(t)} = p_1(x) e^{(x+\lambda)f(t)} \]
or
\[ xf'(t) = p_1(x) e^{\lambda f(t)} \]
or
\[ \frac{f'(t)}{e^{\lambda f(t)}} = \frac{p_1(x)}{x} = \mu \]
for a constant \( \mu \). This is equivalent to \( p_1(x) = \mu x \) and \( f'(t) = \mu e^{\lambda f(t)} \). By integrating this last differential equation, we obtain
\[ f(t) = \frac{1}{\lambda} \ln \frac{1}{1 - \lambda \mu t} \]
and consequently
\[ p(x; t) = e^{x \ln \frac{1}{1 - \lambda \mu t}} = \frac{1}{(1 - \lambda \mu t)^{x/\lambda}} = \sum_{n \geq 0} (\lambda \mu)^n (x/\lambda)_n \frac{t^n}{n!} \]
from which we have identity (40). Vice versa, employing identity (22), we can say that the polynomials defined by formula (40) satisfy the binomial identity (39).

Notice that the falling factorials can be expressed by identity (40) for \( \lambda = -1 \) and \( \mu = 1 \), namely \( x^\underline{n} = (-1)^n(-x)_n \).

References


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