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Abstract

In this paper, we derive several combinatorial identities involving the $q$-derangement numbers (for the major index) and many other $q$-numbers and $q$-polynomials of combinatorial interest, such as the $q$-binomial coefficients, the $q$-Stirling numbers, the $q$-Bell numbers, the $q$-Pochhammer symbol, the Gaussian polynomials, the Rogers-Szegő polynomials and the Galois numbers, and the Al-Salam-Carlitz polynomials. We also obtain two determinantal identities expressing the $q$-derangement numbers as tridiagonal determinants and as Hessenberg determinants.

1 Introduction

The derangement number $d_n$ (sequence A000166 in the On-Line Encyclopedia of Integer Sequences) counts the derangements (i.e., permutation with no fixed points) of an $n$-set. By a simple application of the principle of inclusion-exclusion, we have the formula

$$d_n = \sum_{k=0}^{n} \binom{n}{k} (n-k)! (-1)^k.$$  \hfill (1)

Moreover, these numbers satisfy the recurrences

$$d_{n+1} = (n + 1) d_n + (-1)^n$$  \hfill (2)

$$d_{n+2} = (n + 1) d_{n+1} + (n + 1) d_n$$  \hfill (3)
with initial conditions \(d_0 = 1\) and \(d_1 = 0\).

The \(q\)-derangement numbers \([32, 5]\) are defined by the formula
\[
d_n(q) = \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q! (-1)^k q^{\binom{k}{2}},
\]
where \([n]_q = 1 + q + q^2 + \cdots + q^{n-1}\), and satisfy the recurrences
\[
d_{n+1}(q) = [n+1]_q d_n(q) + (-1)^{n+1} q^{\binom{n+1}{2}}
\]
\[
d_{n+2}(q) = [n+1]_q d_{n+1}(q) + [n+1]_q q^{n+1} d_n(q)
\]
with initial conditions \(d_0(q) = 1\) and \(d_1(q) = 0\).

The major index \(\text{maj}(\sigma)\) of a permutation \(\sigma\) of \(\{1, 2, \ldots, n\}\) is the sum of all positions \(k\) for which \(\sigma(k) > \sigma(k+1)\). The \(q\)-numbers \(d_n(q)\) arise from the \(q\)-counting of derangements by major index and have several interesting combinatorial properties. Indeed, if \(\mathcal{D}_n\) is the set of all derangements of \(\{1, 2, \ldots, n\}\), then we have \([32]\)
\[
d_n(q) = \sum_{\sigma \in \mathcal{D}_n} q^{\text{maj}(\sigma)}.
\]
Moreover, considered as a polynomial in \(q\), the \(q\)-derangement number \(d_n(q)\) has non-negative integer coefficients forming a unimodal sequence \([5]\). These coefficients have a spiral property \([33]\), which implies their unimodality and also the fact that the maximum coefficient of \(d_n(q)\) appears exactly in the middle of the polynomial, i.e., is the coefficient of \(q^{\lfloor n(n-1)/4 \rfloor}\) (as conjectured by Chen and Rota \([5]\)). Furthermore, they have the ratio monotonicity property (for \(n \geq 6\) \([7]\) which implies log-concavity and the spiral property.

For the ordinary derangement numbers \(d_n\) (and their generalizations \([3, 10, 23, 24, 25]\)) there are a lot of combinatorial identities. In this paper, we derive several \(q\)-analogues of these identities. They involve the \(q\)-derangement numbers \(d_n(q)\) and many other \(q\)-numbers or \(q\)-polynomials, such as the \(q\)-binomial coefficients, the \(q\)-Stirling numbers and the \(q\)-Bell numbers, the \(q\)-Pochhammer symbol, the Gaussian polynomials, the Rogers-Szegő polynomials and the Galois numbers, and the Al-Salam-Carlitz polynomials. Finally, we obtain two determinantal identities expressing the \(q\)-derangement numbers as tridiagonal determinants and as Hessenberg determinants.

## 2 \(q\)-binomial identities

We start by recalling some basic definitions of \(q\)-number theory. For every \(n \in \mathbb{N}\), we have the \(q\)-natural number \([n]_q = 1 + q + q^2 + \cdots + q^{n-1}\) and the \(q\)-factorial number \([n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q\). Then, for every \(n, k \in \mathbb{N}\), we have the \(q\)-binomial coefficients (or Gaussian coefficients \([12]\)) defined by
\[
\binom{n}{k}_q = \begin{cases} 
\frac{[n]_q!}{[k]_q! [n-k]_q!}, & \text{if } k \leq n; \\
0, & \text{otherwise},
\end{cases}
\]
and satisfying the recurrence

\[
\binom{n+1}{k+1}_q = \binom{n}{k}_q + q^{n+1} \binom{n}{k+1}_q
\]  

(7)

with initial conditions \( \binom{n}{0}_q = 1 \) and \( \binom{0}{k}_q = \delta_{k,0} \). Moreover, we have the relations

\[
[n]_{q^{-1}} = \frac{1}{q^{n-1}} [n]_q, \quad [n]_{q^{-1}}! = [n]_q! q^{-\binom{n}{2}}, \quad \binom{n}{k}_{q^{-1}} = \binom{n}{k}_q q^{-k(n-k)}.
\]  

(8)

In this first section, we derive some \( q \)-binomial identities using an elementary approach [21] which exploits the properties of the \( q \)-binomial coefficients and the recurrences of the \( q \)-derangement numbers. In Section 5, we derive some other \( q \)-binomial identities by using the more advanced technique of the \( q \)-exponential series.

Our first result is the following:

**Theorem 1.** For every \( n \in \mathbb{N} \), we have the identity

\[
1 + \sum_{k=1}^{n} \binom{n}{k}_q \frac{d_{k+1}(q)}{[k]_q} = \sum_{k=0}^{n} \binom{n+1}{k+1}_q d_k(q).
\]  

(9)

**Proof.** By recurrence (6), we have

\[
\frac{d_{k+2}(q)}{[k+1]_q} = d_{k+1}(q) + q^{k+1} d_k(q).
\]

Consequently, we have

\[
\sum_{k=0}^{n-1} \binom{n}{k+1}_q \frac{d_{k+2}(q)}{[k+1]_q} = \sum_{k=0}^{n-1} \binom{n}{k+1}_q d_{k+1}(q) + \sum_{k=0}^{n-1} \binom{n}{k+1}_q q^{k+1} d_k(q)
\]

or

\[
\sum_{k=1}^{n} \binom{n}{k}_q \frac{d_{k+1}(q)}{[k]_q} = \sum_{k=1}^{n} \binom{n}{k}_q d_k(q) + \sum_{k=0}^{n} \binom{n}{k+1}_q q^{k+1} d_k(q)
\]

or

\[
\sum_{k=1}^{n} \binom{n}{k}_q \frac{d_{k+1}(q)}{[k]_q} = \sum_{k=0}^{n} \left( \binom{n}{k}_q q^{k+1} \binom{n}{k+1}_q \right) d_k(q) - d_0(q).
\]

By recurrence (7) and the initial condition \( d_0(q) = 1 \), we have identity (9). \( \square \)

Similarly, we have the following formula.

**Theorem 2.** For every \( n \in \mathbb{N} \), we have the identity

\[
\sum_{k=0}^{n} \binom{n}{k}_q d_{k+1}(q) = \sum_{k=0}^{n} \binom{n+1}{k+1}_q [k]_q d_k(q) + \sum_{k=1}^{n} \binom{n}{k}_q q^{2k-1} d_{k-1}(q).
\]  

(10)
Proof. By recurrence (6), we have
\[
\sum_{k=0}^{n-1} \binom{n}{k+1} q^{k+1} = \sum_{k=0}^{n-1} \binom{n}{k+1} [k+1]q^{k+1} + \sum_{k=0}^{n-1} \binom{n}{k+1} q^{k+1} d_k(q)
\]
or
\[
\sum_{k=0}^{n} \binom{n}{k} q^{k+1} = \sum_{k=0}^{n} \binom{n}{k} [k]q^{k+1} + \sum_{k=0}^{n} \binom{n}{k+1} q^{k+1} d_k(q)
\]
or
\[
\sum_{k=0}^{n} \binom{n}{k} q^{k+1} - d_1(q) = \sum_{k=0}^{n} \left( \binom{n}{k} q^{k+1} \right) [k]q^{k+1} + \sum_{k=0}^{n} \binom{n}{k+1} q^{k+1} d_k(q)
\]
or
\[
\sum_{k=0}^{n} \binom{n}{k} q^{k+1} - d_1(q) = \sum_{k=0}^{n} \left( \binom{n}{k} + q^{k+1} \binom{n}{k+1} \right) [k]q^{k+1} + \sum_{k=0}^{n} \binom{n}{k+1} q^{k+1} d_k(q)
\]
By recurrence (7) and the initial condition \(d_1(q) = 0\), we have identity (10).

We also have the following formula.

**Theorem 3.** For every \(m, n \in \mathbb{N}\), we have the identity
\[
\sum_{k=0}^{n} \binom{n}{k} q^{k+1} [k]q^{k+1} (-1)^k q^{\binom{k+1}{2}} = \sum_{k=0}^{n} \binom{n}{k} [k]q^{k+1} (-1)^k q^{\binom{k+1}{2} + \binom{2}{2}}.
\]
In particular, for \(m = 0\), we have the identity
\[
\sum_{k=0}^{n} \binom{n}{k} q^{k+1} [k]q^{k+1} (-1)^k q^{\binom{k+1}{2}} = \sum_{k=0}^{n} \binom{n}{k} [k]q^{k+1} (-1)^k q^{k^2}.
\]

**Proof.** From recurrence (5), we have
\[
\frac{d_{k+1}(q)}{[k+1]q} = \frac{d_k(q)}{[k]q} + (-1)^{k+1} \frac{q^{\binom{k+1}{2}}}{[k+1]q}.
\]
and consequently
\[
\sum_{k=0}^{n} \binom{n}{k} q^{k+1} [k]q^{k+1} (-1)^k q^{\binom{k+1}{2}} = \sum_{k=0}^{n} \binom{n}{k} [k]q^{k+1} (-1)^k q^{k^2}.
\]
that is,
\[
\sum_{k=1}^{n} \left( \frac{m+n}{m+k} \right) \frac{[n]_q!}{[k]_q!} (-1)^k q^{\left( \frac{m+k+1}{2} \right)} d_k(q) + \sum_{k=0}^{n-1} \left( \frac{m+n}{m+k+1} \right) \frac{[n]_q!}{[k]_q!} (-1)^k q^{\left( \frac{m+k+1}{2} \right)} q^{m+k+1} d_k(q)
\]
\[
= \sum_{k=1}^{n} \left( \frac{m+n}{m+k} \right) \frac{[n]_q!}{[k]_q!} q^{\left( \frac{m+k+1}{2} \right)} q^{k} ,
\]

or
\[
\sum_{k=0}^{n} \left( \frac{m+n}{m+k} \right) q^{m+k+1} q^{\left( \frac{m+k+1}{2} \right)} d_k(q)
\]
\[
= \left( \frac{m+n}{m} \right) \frac{[n]_q!}{[k]_q!} q^{\left( \frac{m+k+1}{2} \right)} + \sum_{k=1}^{n} \left( \frac{m+n}{m+k} \right) \frac{[n]_q!}{[k]_q!} q^{\left( \frac{m+k+1}{2} \right)} + (q).
\]

Hence, by formula (7), we have identity (11). \qed

Using the same approach, we also have the following result.

**Theorem 4.** We have the identity
\[
q \sum_{k=0}^{n} [k]_q d_k(q) + \sum_{k=0}^{n} (-1)^k q^{\left( \frac{k}{2} \right)} = [n+1]_q d_n(q) .
\]

**Proof.** Since \([k+1]_q = 1 + q[k]_q\), recurrence (5) can be rewritten as
\[
d_{k+1}(q) = (1 + q[k]_q) d_k(q) + (-1)^k q^{\left( \frac{k+1}{2} \right)}
\]
or
\[
d_{k+1}(q) - d_k(q) = q[k]_q d_k(q) + (-1)^k q^{\left( \frac{k+1}{2} \right)} .
\]

Hence, we have
\[
\sum_{k=0}^{n} d_{k+1}(q) - \sum_{k=0}^{n} d_k(q) = q \sum_{k=0}^{n} [k]_q d_k(q) + \sum_{k=0}^{n} (-1)^k q^{\left( \frac{k+1}{2} \right)}
\]
or
\[
\sum_{k=1}^{n+1} d_k(q) - \sum_{k=0}^{n} d_k(q) = q \sum_{k=0}^{n} [k]_q d_k(q) + \sum_{k=1}^{n+1} (-1)^k q^{\left( \frac{k}{2} \right)}
\]
or
\[
d_{n+1}(q) - d_0(q) = q \sum_{k=0}^{n} [k]_q d_k(q) + \sum_{k=0}^{n+1} (-1)^k q^{\left( \frac{k}{2} \right)} - 1 .
\]
Hence, by the recurrence (5) once again, we have the identity
\[ [n + 1]q d_n(q) + (-1)^n q^{(n+1)} = q \sum_{k=0}^{n}[k]q d_k(q) + \sum_{k=0}^{n}(-1)^k q^{(k)} + (-1)^n q^{(n+1)} \]
which simplifies to identity (13).

Similarly, we also have the following property.

**Theorem 5.** We have the identity
\[ \sum_{k=0}^{n} \left( \begin{array}{c} n+1 \\ k+1 \end{array} \right) q (-1)^k q^{(k+1)} d_k(q) = q \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (-1)^k q^{(k+1)} d_{k-1}(q) + \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{k^2}. \quad (14) \]

**Proof.** Once again, we start by the recurrence (5) written as
\[ d_{k+1}(q) - d_k(q) = q[k]q d_k(q) + (-1)^{k+1} q^{(k+1)}. \]

Then we have
\[ \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k+1 \end{array} \right) q (-1)^k q^{(k+2)} d_{k+1}(q) - \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k+1 \end{array} \right) q (-1)^k q^{(k+2)} d_k(q) = \]
\[ q \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k+1 \end{array} \right) q (-1)^k q^{(k+2)} [k]q d_k(q) + \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k+1 \end{array} \right) q^{(k+2)} q^{(k+1)}, \]
which is
\[ \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q (-1)^k q^{(k+1)} d_k(q) + \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (-1)^k q^{(k+1)} q^{k+1} d_k(q) = \]
\[ q \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q (-1)^k q^{(k+1)} [k-1]q d_{k-1}(q) + \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{(k+1)} q^{(k)} \]
or
\[ \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) + q^{k+1} \left( \begin{array}{c} n \\ k+1 \end{array} \right) \right) (-1)^k q^{(k+1)} d_k(q) - 1 \]
\[ = q \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (-1)^k q^{(k+1)} [k-1]q d_{k-1}(q) + \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{(k+1)} q^{(k)} - 1. \]

By recurrence (7), this last identity simplifies to identity (14).
3 \(q\)-Stirling identities

The \(q\)-Stirling numbers of the second kind are defined as the connection constants \([9, 21]\) between the ordinary powers \(x^n\) and the \(q\)-falling factorials \(x_q^n = x(x-[1]_q)(x-[2]_q) \cdots (x-[n-1]_q)\), that is, as the coefficients \(\left\{\begin{array}{c}n \\ k \end{array}\right\}_q\) for which

\[x^n = \sum_{k=0}^{n} \left\{\begin{array}{c}n \\ k \end{array}\right\}_q x_q^k.\]

Equivalently, they are the numbers defined by the recurrence

\[
\left\{\begin{array}{c}n+1 \\ k+1 \end{array}\right\}_q = \left\{\begin{array}{c}n \\ k \end{array}\right\}_q + [k+1]_q \left\{\begin{array}{c}n \\ k+1 \end{array}\right\}_q
\]

with initial values \(\left\{\begin{array}{c}n \\ 0 \end{array}\right\}_q = \delta_{n,0}\) and \(\left\{\begin{array}{c}0 \\ k \end{array}\right\}_q = \delta_{k,0}\).

Similarly, the \(q\)-Stirling numbers of the first kind are defined as the connection constants \([9, 21]\) between the \(q\)-rising factorials \(x_q^\pi = x(x+[1]_q)(x+[2]_q) \cdots (x+[n-1]_q)\) and the ordinary powers \(x^n\), that is, as the coefficients \([\begin{array}{c}n \\ k \end{array}]_q\) for which

\[x_q^\pi = \sum_{k=0}^{n} \left[\begin{array}{c}n \\ k \end{array}\right]_q x^k.\]

Equivalently, they are the numbers defined by the recurrence

\[
\left[\begin{array}{c}n+1 \\ k+1 \end{array}\right]_q = \left[\begin{array}{c}n \\ k \end{array}\right]_q + [n]_q \left[\begin{array}{c}n \\ k+1 \end{array}\right]_q
\]

with initial values \([\begin{array}{c}n \\ 0 \end{array}]_q = \delta_{n,0}\) and \([\begin{array}{c}0 \\ k \end{array}]_q = \delta_{k,0}\).

For the \(q\)-Stirling numbers, we have the inverse relations

\[
f_n = \sum_{k=0}^{n} \left\{\begin{array}{c}n \\ k \end{array}\right\}_q g_k \quad \iff \quad g_n = \sum_{k=0}^{n} \left[\begin{array}{c}n \\ k \end{array}\right]_q (-1)^{n-k} f_k
\]

and

\[
f_n = \sum_{k=0}^{n} \left\{\begin{array}{c}n+1 \\ k+1 \end{array}\right\}_q g_k \quad \iff \quad g_n = \sum_{k=0}^{n} \left[\begin{array}{c}n+1 \\ k+1 \end{array}\right]_q (-1)^{n-k} f_k.
\]

Consider the \(q\)-Bell numbers defined by

\[b_n(q) = \sum_{k=0}^{n} \left\{\begin{array}{c}n \\ k \end{array}\right\}_q q_2^k.\]

Although they are not the cumulative constants of the \(q\)-Stirling numbers considered above, we have the following formulas relating the \(q\)-derangement numbers and the \(q\)-Bell numbers,
Theorem 6. We have the identities
\[
\sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k d_k(q) = b_n(q) \quad (20)
\]
\[
\sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k b_k(q) = d_n(q). \quad (21)
\]

Proof. By recurrence (5), we have
\[
d_{k+1}(q) - [k + 1]_q d_k(q) = (-1)^{k+1} q^{\frac{k+1}{2}}.
\]
Hence, we have the identity
\[
\sum_{k=0}^{n} \binom{n}{k+1} (-1)^{k+1} d_{k+1}(q) - \sum_{k=0}^{n} \binom{n}{k+1} (-1)^{k+1} [k + 1]_q d_k(q) = \sum_{k=0}^{n-1} \binom{n}{k+1} q^{\frac{k+1}{2}}
\]
or
\[
\sum_{k=1}^{n} \binom{n}{k} (-1)^k d_k(q) + \sum_{k=0}^{n-1} \binom{n}{k+1} (-1)^{k+1} [k + 1]_q d_k(q) = \sum_{k=0}^{n-1} \binom{n}{k} q^{\frac{k}{2}}
\]
or
\[
\sum_{k=0}^{n} \left( \binom{n}{k} + [k + 1]_q \binom{n}{k+1} \right) (-1)^k d_k(q) = \sum_{k=0}^{n} \binom{n}{k} q^{\frac{k}{2}}
\]
By recurrence (15) and definition (19), we have identity (20). Then, by this identity, we get identity (21) at once as its inverse relation (by property (18)).

To prove the next theorem, we need the following result.

Lemma 7. For every \( n, k \in \mathbb{N} \), we have the identity
\[
\binom{n+1}{k+1} q^{i-k} = \sum_{i=k}^{n} \binom{n}{i} \binom{i}{k} q^{i-k}. \quad (22)
\]

Proof. Since
\[
x^{n+1} = \sum_{k=1}^{n+1} \binom{n+1}{k} x_q^{k+1} = \sum_{k=0}^{n} \binom{n+1}{k+1} x^{k+1}_q = \sum_{k=0}^{n} \binom{n+1}{k+1} x(x-[1]_q) \cdots (x-[k]_q),
\]
we have
\[
x^n = \sum_{k=0}^{n} \binom{n+1}{k+1} (x-[1]_q)(x-[2]_q) \cdots (x-[k]_q)
\]
\[
= \sum_{k=0}^{n} \binom{n+1}{k+1} (x-1)(x-1-q[1]_q) \cdots (x-1-q[k-1]_q)
\]

8
or

\[(qx + 1)^n = \sum_{k=0}^{n} \binom{n+1}{k+1} q^k x (x - [1]_q) \cdots (x - [k]_q) = \sum_{k=0}^{n} \binom{n+1}{k+1} q^k x^k_q.\]

Then, from this relation, we have

\[\sum_{k=0}^{n} \binom{n+1}{k+1} q^k x^k_q = \sum_{i=0}^{n} \binom{n}{i} q^i x^i = \sum_{i=0}^{n} \binom{n}{i} q^i \sum_{k=0}^{i} \binom{i}{k} x^k_q = \sum_{k=0}^{n} \left( \sum_{i=k}^{n} \binom{i}{k} q^i \right) x^k_q.\]

By equating the coefficients of \(x^k_q\), we obtain identity (22).

Now we can prove the following result.

**Theorem 8.** We have the identity

\[\sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{n-k} d_k(q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k(q). \quad (23)\]

**Proof.** By identities (20) and (22), we have

\[b_n(q) = \sum_{k=0}^{n} \left( \sum_{i=0}^{n} \binom{n}{i} \binom{i}{k} q^{i-k} \right) (-1)^k d_k(q) = \sum_{k=0}^{n} \binom{n}{i} \sum_{i=k}^{n} \binom{i}{k} q^i (-1)^k q^{i-k} d_k(q).\]

Thus, if we set

\[z_n(q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{n-k} d_k(q),\]

then we have the identity

\[b_n(q) = \sum_{i=0}^{n} \binom{n}{i} z_i(q),\]

whose inverse is

\[z_n(q) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b_i(q).\]

This is identity (23).

We also have the following result.

**Theorem 9.** We have the identity

\[\sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k d_{k+1}(q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^k [k]_q d_{k-1}(q). \quad (24)\]
Proof. By recurrence (6), we have
\[ d_{k+2}(q) - [k + 1]_q d_{k+1}(q) = [k + 1]_q q^{k+1} d_k(q). \]

Then we have
\[
\sum_{k=0}^{n-1} \left\{ \begin{array}{c} \frac{n}{k+1} \\ q \end{array} \right\} (-1)^{k+1} d_{k+2}(q) - \sum_{k=0}^{n-1} \left\{ \begin{array}{c} \frac{n}{k+1} \\ q \end{array} \right\} (-1)^{k+1} [k + 1]_q d_{k+1}(q) = 
\sum_{k=0}^{n-1} \left\{ \begin{array}{c} \frac{n}{k+1} \\ q \end{array} \right\} (-1)^{k+1} [k + 1]_q q^{k+1} d_k(q),
\]
or
\[
\sum_{k=1}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} (-1)^k d_{k+1}(q) + \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k+1 \end{array} \right\} (-1)^k [k + 1]_q d_{k+1}(q) = \sum_{k=1}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} (-1)^k [k]_q q^k d_{k-1}(q),
\]
or
\[
\sum_{k=1}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} + [k + 1]_q \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k+1 \end{array} \right\} (-1)^k d_{k+1}(q) = \sum_{k=1}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} (-1)^k [k]_q q^k d_{k-1}(q).
\]

By recurrence (15), we have identity (24).

Similarly, we also have the following formula.

Theorem 10. We have the identity
\[
\sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} (-1)^k q^{-\binom{k+1}{2}} d_k(q)^2 = \sum_{k=0}^{n} \left\{ \begin{array}{c} n+1 \\ k+1 \end{array} \right\} (-1)^k q^{-\binom{k+1}{2}} d_k(q) d_{k+1}(q). \tag{25}
\]

Proof. By recurrence (6), we have
\[ d_{k+1}(q) d_{k+2}(q) = [k + 1]_q d_{k+1}(q)^2 + [k + 1]_q q^{k+1} d_k(q) d_{k+1}(q), \]
or
\[ [k + 1]_q d_{k+1}(q)^2 = d_{k+1}(q) d_{k+2}(q) - [k + 1]_q q^{k+1} d_k(q) d_{k+1}(q). \]

Hence, we have
\[
\sum_{k=0}^{n-1} \left\{ \begin{array}{c} n \\ k+1 \end{array} \right\} (-1)^{k+1} q^{-\binom{k+2}{2}} [k + 1]_q d_{k+1}(q)^2
\]
\[
= \sum_{k=0}^{n-1} \left\{ \begin{array}{c} n \\ k+1 \end{array} \right\} (-1)^{k+1} q^{-\binom{k+2}{2}} d_{k+1}(q) d_{k+2}(q)
\]
\[
- \sum_{k=0}^{n-1} \left\{ \begin{array}{c} n \\ k+1 \end{array} \right\} (-1)^{k+1} q^{-\binom{k+2}{2}} [k + 1]_q q^{k+1} d_k(q) d_{k+1}(q)
\]

10
or
\[
\sum_{k=1}^{n} \binom{n}{k} q^{-\frac{(k+1)}{2}} [k]_q d_k(q)^2
= \sum_{k=1}^{n} \binom{n}{k} q^{-\frac{(k+1)}{2}} d_k(q)d_{k+1}(q)
+ \sum_{k=0}^{n} \binom{n}{k+1} [k+1]_q (-1)^k q^{-\frac{(k+1)}{2}} d_k(q)d_{k+1}(q).
\]
Since \([0]_q = 0\) and \(d_1(q) = 0\), we have
\[
\sum_{k=0}^{n} \binom{n}{k} q^{-\frac{(k+1)}{2}} [k]_q d_k(q)^2
= \sum_{k=0}^{n} \left( \binom{n}{k} + [k+1]_q \binom{n}{k+1} \right) (-1)^k q^{-\frac{(k+1)}{2}} d_k(q)d_{k+1}(q).
\]
Finally, by the recurrence (15), this identity simplifies to identity (25).

Now consider the \(q\)-Bell numbers \(B_n(q)\) defined by the recurrence
\[
B_{n+1}(q) = \sum_{k=0}^{n} \binom{n}{k} q^{n(n-k)} B_k(q)
\]
with initial value \(B_0(q) = 1\). Notice that the \(q\)-numbers \(b_n(q)\) and \(B_n(q)\) are different \(q\)-analogues of the ordinary Bell numbers (A000110). For these \(q\)-Bell numbers, we have the following result.

**Theorem 11.** We have the identity
\[
\sum_{k=0}^{n} \binom{n}{k} q^{-\frac{(k+1)}{2}} B_k(q) d_{n-k}(q) = \sum_{k=0}^{n} \binom{n}{k} q^{-\frac{(k+1)}{2}} B_{k+1}(q) [n-k]_q!
\]
**Proof.** Let \(\sigma_n(q)\) be the sum on the right-hand side of identity (27). Then, by the recurrence
(26), we have
\[
\sigma_n(q) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{-\binom{k+1}{2}} [n-k]_q B_{k+1}(q)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{-\binom{k+1}{2}} [n-k]_q \sum_{i=0}^{k} \binom{k}{i}_q q^{k-i} B_i(q)
\]
\[
= \sum_{i=0}^{n} \left( \sum_{k=i}^{n} \binom{n}{k}_q \binom{k}{i}_q [n-k]_q (-1)^{n-k} q^{-\binom{k+1}{2}} q^{k-i} \right) B_i(q)
\]
\[
= \sum_{i=0}^{n} \frac{n}{i}_q \sum_{k=0}^{n-i} \binom{n-i}{k}_q [n-k-i]_q (-1)^{n-k-i} q^{-\binom{k+1}{2}} q^{k-i} B_i(q)
\]
\[
= \sum_{i=0}^{n} \frac{n}{i}_q (-1)^{n-i} \left( \sum_{k=0}^{n-i} \binom{n-i}{k}_q [n-k-i]_q (-1)^{k} q^{-\binom{k+1}{2}} q^{k-i} \right) B_i(q)
\]
\[
= \sum_{i=0}^{n} \frac{n}{i}_q (-1)^{n-i} q^{-\binom{i+1}{2}} \left( \sum_{k=0}^{n-i} \binom{n-i}{k}_q [n-k-i]_q (-1)^{k} q^{k-i} \right) B_i(q)
\]
Finally, by formula (4), we have
\[
\sigma_n(q) = \sum_{i=0}^{n} \frac{n}{i}_q (-1)^{n-i} q^{-\binom{i+1}{2}} d_{n-i}(q) B_i(q),
\]
and this is the claimed identity. \(\square\)

4 Elementary identities

Several combinatorial identities can be derived from the following property of linear recurrences of the first order: the general solution of the recurrence
\[
y_{n+1} = a_{n+1} y_n + b_{n+1}
\]
is given by
\[
y_n = a_n^* y_0 + \sum_{k=1}^{n} \frac{a_n}{a_k^*} b_k,
\]
where \(a_n^* = a_1 a_2 \cdots a_n\), provided that \(a_n \neq 0\) for all \(n \in \mathbb{N}\).
First of all, we have the following simple result.
Theorem 12. For every \( m, n \in \mathbb{N} \), we have the identity

\[
d_{m+n+2}(q) = \binom{m+n+1}{m}_q [n+1]_q d_{m+1}(q)
\]

\[+ [m+n+1]_q \sum_{k=0}^{n} \binom{m+n}{m+k}_q \left[ n-k \right]_q q^{k+1} d_{m+k}(q)\]

(29)

In particular, for \( m = 0 \), we have the identity

\[
d_{n+2}(q) = [n+1]_q \sum_{k=0}^{n} \binom{n}{k}_q \left[ n-k \right]_q q^{k+1} d_k(q)\]

(30)

**Proof.** Let \( y_n(q) = d_{m+n+1}(q) \). By recurrence (6), we have

\[
y_{n+1}(q) = d_{m+n+2}(q) = [m+n+1]_q d_{m+n+1}(q) + [m+n+1]_q q^{m+n+1} d_{m+n}(q),
\]

or

\[
y_{n+1}(q) = [m+n+1]_q y_n(q) + [m+n+1]_q q^{m+n+1} d_{m+n}(q).\]

This is a linear recurrence of the first order with coefficients \( a_n = [m+n]_q \) and \( b_n = [m+n]_q q^{m+n} d_{m+n-1}(q) \). Since

\[
a_n^* = [m+n]_q \cdots [m]_q = \frac{[m+n]_q}{[m]_q} = \binom{m+n}{m}_q [n]_q!,
\]

then the solution, being \( y_0(q) = d_{m+1}(q) \), is

\[
y_n(q) = \binom{m+n}{m}_q [n]_q! d_{m+1}(q) + \sum_{k=1}^{n} \frac{[m+n]_q}{[m]_q} \frac{[m]_q}{[m+k]_q} [m+k]_q q^{m+k} d_{m+k-1}(q)
\]

\[
= \binom{m+n}{m}_q [n]_q! d_{m+1}(q) + \sum_{k=1}^{n} \binom{m+n}{m+k}_q [m+k]_q [n-k]_q! q^{m+k} d_{m+k-1}(q)
\]

\[
= \binom{m+n}{m}_q [n]_q! d_{m+1}(q) + [m+n]_q \sum_{k=1}^{n} \binom{m+n-1}{m+k-1}_q [n-k]_q! q^{m+k} d_{m+k-1}(q)
\]

\[
= \binom{m+n}{m}_q [n]_q! d_{m+1}(q) + [m+n]_q \sum_{k=0}^{n-1} \binom{m+n-1}{m+k}_q [n-k-1]_q! q^{m+k+1} d_{m+k}(q).
\]

Finally, by replacing \( n \) by \( n + 1 \), we obtain formula (29). □
In particular, for \( m = 0 \), we have the identity
\[
\frac{d_{n+2}(q)}{[n+1]_q!} = \sum_{k=0}^{n} q^{k+1} \frac{d_k(q)}{[k]_q!}.
\]  \tag{32}

**Proof.** Let \( y_n(q) = \frac{d_{m+n+1}(q)}{[m+n]_q!} \). By recurrence (6), we have
\[
y_{n+1}(q) = \frac{d_{m+n+2}(q)}{[m+n+1]_q!} = \frac{[m+n+1]q(d_{m+n+1}(q) + q^{m+n+1}d_{m+n}(q))}{[m+n+1]_q!} = \frac{d_{m+n+1}(q)}{[m+n]_q!} + q^{m+n+1} \frac{d_{m+n}(q)}{[m+n]_q!},
\]
or
\[
y_{n+1}(q) = y_n(q) + q^{m+n+1} \frac{d_{m+n}(q)}{[m+n]_q!}.
\]
This is a linear recurrence of the first order with \( a_n = 1 \) and \( b_n = q^{m+n} \frac{d_{m+n+1}(q)}{[m+n+1]_q!} \) for \( n \geq 1 \). So, by formula (28), we have the solution
\[
y_n(q) = y_0(q) + \sum_{k=1}^{n} q^{m+k} \frac{d_{m+k-1}(q)}{[m+k-1]_q!} = y_0(q) + \sum_{k=0}^{n-1} q^{m+k+1} \frac{d_{m+k}(q)}{[m+k]_q!},
\]
or
\[
\frac{d_{m+n+1}(q)}{[m+n]_q!} = \frac{d_{m+1}(q)}{[m]_q!} + \sum_{k=0}^{n-1} q^{m+k+1} \frac{d_{m+k}(q)}{[m+k]_q!}.
\]
Now by replacing \( n \) by \( n + 1 \), we obtain identity (31). \qed

**Theorem 14.** For every \( m, n \in \mathbb{N} \), we have the identity
\[
\frac{d_{m+n+1}(q)d_{m+n+2}(q)}{q^{(n+2)}[m+n+1]_q!} = q^{m(n+1)} \frac{d_{m}(q)d_{m+1}(q)}{[m]_q!} + \sum_{k=0}^{n} q^{m(n-k)} \frac{d_{m+k+1}(q)^2}{q^{(k+2)}[m+k]_q!} \tag{33}
\]
In particular, for \( m = 0 \), we have the identity
\[
\frac{d_{n+1}(q)d_{n+2}(q)}{q^{(n+2)}[n+1]_q!} = \sum_{k=0}^{n} \frac{d_{k+1}(q)^2}{q^{(k+2)}[k]_q!} \tag{34}
\]
Proof. Let \( y_n(q) = \frac{d_{m+n}(q) d_{m+n+1}(q)}{[m+n]_q!} \). By recurrence (6), we have
\[
y_{n+1}(q) = \frac{d_{m+n+1}(q) d_{m+n+2}(q)}{[m+n+1]_q!} = \frac{d_{m+n+1}(q) (d_{m+n+1}(q) + q^{m+n+1} d_{m+n}(q))}{[m+n]_q!}.
\]
or
\[
y_{n+1}(q) = q^{m+n+1} y_n(q) + \frac{d_{m+n+1}(q)^2}{[m+n]_q!}.
\]
This is a linear recurrence of the first order with \( a_n = q^{m+n} \) and \( b_n = \frac{d_{m+n}(q)^2}{[m+n]_q!} \) for \( n \geq 1 \).
Since \( a_n^* = q^{m+\binom{n+1}{2}} \), by formula (28), we have the solution
\[
y_n(q) = q^{m+\binom{n+1}{2}} y_0(q) + \sum_{k=1}^{n} \frac{q^{m+\binom{k+1}{2}}}{[m+k-1]_q!} \frac{d_{m+k}(q)^2}{[m+k]_q!}.
\]
Now by replacing \( n \) by \( n+1 \), we obtain identity (33).

The next formula can be obtained with the same elementary approach used in Section 2.

**Theorem 15.** We have the identity
\[
\frac{d_{m+n+1}(q)^2 - q^{2\binom{m+n+1}{2}}}{[m+n+1]_q!^2} = \frac{d_m(q)^2 - q^{2\binom{m}{2}}}{[m]_q!^2} + 
2 \sum_{k=0}^{n} (-1)^{m+k+1} q^{\binom{m+k+1}{2}} \frac{d_{m+k}(q)}{[m+k]_q!} \frac{d_{m+k+1}(q)}{[m+k+1]_q!} + \sum_{k=0}^{n} \frac{q^{\binom{k+1}{2}}}{[k]_q!}.
\]
In particular, for \( m = 0 \), we have the identity
\[
\frac{d_{n+1}(q)^2 - q^{n(n+1)}}{[n+1]_q!^2} = 2 \sum_{k=0}^{n} (-1)^{k+1} q^{\binom{k+1}{2}} \frac{d_k(q)}{[k]_q!} \frac{d_{k+1}(q)}{[k+1]_q!} + \sum_{k=0}^{n} \frac{q^{\binom{k}{2}}}{[k]_q!}.
\]

**Proof.** By recurrences (5), we have
\[
d_{m+k+1}(q)^2 = \left( [m+k+1]_q d_{m+k}(q) + (-1)^{m+k+1} q^{\binom{m+k+1}{2}} \right)^2
= [m+k+1]_q d_{m+k}(q)^2 + 2(-1)^{m+k+1} q^{\binom{m+k+1}{2}} [m+k+1]_q d_{m+k}(q) + q^{\binom{m+k+1}{2}}.
\]
Hence, we can write
\[
\frac{d_{m+k+1}(q)^2}{[m + k + 1]_q!^2} = \frac{d_{m+k}(q)^2}{[m+k]_q!^2} + 2(-1)^{m+k+1}q^{\binom{m+k+1}{2}} \frac{d_{m+k}(q)}{[m+k]_q![m+k+1]_q!} + \frac{q^2\binom{m+k+1}{2}}{[m+k+1]_q!^2}
\]
and, consequently, we have
\[
\sum_{k=0}^{n} \frac{d_{m+k+1}(q)^2}{[m + k + 1]_q!^2} = \sum_{k=0}^{n} \frac{d_{m+k}(q)^2}{[m+k]_q!^2} + 2\sum_{k=0}^{n} (-1)^{m+k+1}q^{\binom{m+k+1}{2}} \frac{d_{m+k}(q)}{[m+k]_q![m+k+1]_q!} + \sum_{k=0}^{n} \frac{q^2\binom{m+k+1}{2}}{[m+k+1]_q!^2},
\]
or
\[
\sum_{k=1}^{n+1} \frac{d_{m+k}(q)^2}{[m+k]_q!^2} = \sum_{k=0}^{n} \frac{d_{m+k}(q)^2}{[m+k]_q!^2} + 2\sum_{k=0}^{n} (-1)^{m+k+1}q^{\binom{m+k+1}{2}} \frac{d_{m+k}(q)}{[m+k]_q![m+k+1]_q!} + \sum_{k=1}^{n+1} \frac{q^2\binom{m+k}{2}}{[m+k]_q!^2}.
\]
By simplifying, we get the identity
\[
\frac{d_{m+n+1}(q)^2}{[m + n + 1]_q!^2} = \frac{d_{m}(q)^2}{[m]_q!^2} + 2\sum_{k=0}^{n} (-1)^{m+k+1}q^{\binom{m+k+1}{2}} \frac{d_{m+k}(q)}{[m+k]_q![m+k+1]_q!} + \sum_{k=0}^{n} \frac{q^2\binom{m+k}{2}}{[m+k]_q!^2} + \sum_{k=0}^{n} \frac{q^2\binom{m+n+1}{2}}{[m+n+1]_q!^2} - \frac{q^2\binom{m}{2}}{[m]_q!^2}
\]
which yields identity (35) at once. \(\square\)

5 \(q\)-exponential series

Many identities can be obtained by using the \(q\)-exponential generating series. Recall that the product of two \(q\)-exponential series \(f(t) = \sum_{n \geq 0} f_n t^n [\frac{n}{m}]_q!\) and \(g(t) = \sum_{n \geq 0} g_n [\frac{n}{m}]_q!\) is given by
\[
f(t) \cdot g(t) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} t^k f_k g_{n-k} \right) [\frac{n}{m}]_q!
\]
and that the \(q\)-derivative (Jackson’s derivative) \(\mathcal{D}_q\) of a \(q\)-exponential generating series \(f(t) = \sum_{n \geq 0} f_n [\frac{n}{m}]_q!\) is defined \([16, 17, 18]\) by the formula
\[
\mathcal{D}_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} = \sum_{n \geq 0} f_{n+1} t^n [\frac{n}{m}]_q!.
\]
The \textit{q-exponential series} (Jackson’s \textit{q-exponential}) \cite{16}

\begin{equation}
E_q(t) = \sum_{n \geq 0} \frac{t^n}{[n]_q!} = \prod_{k \geq 0} \frac{1}{1 + (q - 1)q^k t}
\end{equation}

is the eigenfunction of the \textit{q}-derivative, that is,

\[ \mathfrak{D}_q E_q(\lambda t) = \lambda E_q(t). \]

In particular, since \( \mathfrak{D}_q E_q(t) = E_q(t) \), we have the relation

\[ E_q(qt) = (1 - (1 - q)t) E_q(t). \]

Consequently, considering the \textit{q-Pochhammer symbol} \((x; q)_m = (1 - x)(1 - qx) \cdots (1 - q^{m-1}x)\), we have, for every \( m \in \mathbb{N} \), the identity

\[ E_q(q^m t) = \prod_{k=0}^{m-1} (1 - (1 - q)q^k t) \cdot E_q(t) = ((1 - q)t; q)_m E_q(t). \]

Moreover, the inverse of the \textit{q}-exponential series is

\begin{equation}
E_q(t)^{-1} = \sum_{n \geq 0} (-1)^n q^{\binom{n}{2}} \frac{t^n}{[n]_q!}
\end{equation}

and we have the identities \cite{30}

\begin{align*}
E_q(-t) E_{q^{-1}}(t) &= 1 \quad (41) \\
E_q(t) E_{q^{-1}}(-t) &= E_q^2\left(1 + \frac{1 - q}{1 + q} t^2\right). \quad (42)
\end{align*}

By definition \( (4) \) and series \( (40) \), we have at once that the \textit{q}-exponential generating series of the \textit{q-derangement numbers} is

\begin{equation}
D_q(t) = \sum_{n \geq 0} d_n(q) \frac{t^n}{[n]_q!} = \frac{E_q(t)^{-1}}{1 - t}. \quad (43)
\end{equation}

We consider the following \textit{q-polynomials}:

- the \textit{q-Pochhammer symbol}

\[ (x; q)_n = (1 - x)(1 - qx) \cdots (1 - q^{n-1}x) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k, \quad (44) \]

- the \textit{Gaussian polynomials} \cite{13, 14, 9}

\[ g_n(q; x) = (x - 1)(x - q) \cdots (x - q^{n-1}) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} x^k, \]

17
• the \(q\)-Hermite polynomials (or Rogers-Szegő polynomials) ([29, 4, 1, 13], [27, p. 180])

\[
H_n(q; x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k
\]

and the Galois numbers [13, 26]

\[
G_n(q) = \sum_{k=0}^{n} \binom{n}{k}_q \tag{45}
\]

• the \(q\)-Carlitz polynomials (or Al-Salam-Carlitz polynomials) ([2], [8, p. 195], [15, 6, 19])

\[
U_n^{(\alpha)}(q; x) = \sum_{k=0}^{n} \binom{n}{k}_q (-\alpha)^{n-k} g_k(x),
\]

having \(q\)-exponential generating series

\[
P_q(x, t) = \sum_{n \geq 0} (x; q)_n \frac{t^n}{[n]_q!} = \frac{E_q(t)}{E_q(xt)} = E_q(t) E_q(xt)^{-1} \tag{46}
\]

\[
g_q(x, t) = \sum_{n \geq 0} g_n(q; x) \frac{t^n}{[n]_q!} = \frac{E_q(xt)}{E_q(t)} = E_q(t) E_q(xt)^{-1} \tag{47}
\]

\[
H_q(x, t) = \sum_{n \geq 0} H_n(q; x) \frac{t^n}{[n]_q!} = E_q(t) E_q(xt) \tag{48}
\]

\[
G_q(t) = \sum_{n \geq 0} G_n(q) \frac{t^n}{[n]_q!} = E_q(t)^2 \tag{49}
\]

\[
U_q(x, t) = \sum_{n \geq 0} U_n^{(\alpha)}(q; x) \frac{t^n}{[n]_q!} = \frac{E_q(xt)}{E_q(t) E_q(\alpha t)} = E_q(\alpha t)^{-1} g_q(x, t). \tag{50}
\]

Using the properties of the \(q\)-exponential series, we have at once the following results.

**Theorem 16.** We have the identities

\[
\sum_{k=0}^{n} \binom{n}{k}_q d_{n-k}(q) (x; q)_k = \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q! (-1)^k q^{\binom{k}{2}} x^k
\]

\[
\sum_{k=0}^{n} \binom{n}{k}_q d_{n-k}(q) x^k = \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q! g_k(q; x) \tag{52}
\]

\[
\sum_{k=0}^{n} \binom{n}{k}_q d_{n-k}(q) G_k(q, x) = \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q! x^k \tag{53}
\]

\[
\sum_{k=0}^{n} \binom{n}{k}_q \alpha^{n-k} d_{n-k}(q) g_k(x) = \sum_{k=0}^{n} \binom{n}{k}_q \alpha^{n-k} [n-k]_q! U_k^{(\alpha)}(q; x). \tag{54}
\]
Proof. By series (43), (46), (47), (48), (50) and (40), we have the identities

\[
D_q(t) P_q(x, t) = \frac{E_q(xt)^{-1}}{1-t}
\]
\[
D_q(t) E_q(x, t) = \frac{g_q(x, t)}{1-t}
\]
\[
D_q(t) H_q(x, t) = \frac{E_q(xt)}{1-t}
\]
\[
D_q(\alpha t) g_q(x, t) = \frac{U_q^{(\alpha)}(xt)}{1-\alpha t}
\]

which are equivalent to identities (51), (52), (53) and (54), respectively.

Moreover, we also have the next result.

**Theorem 17.** We have the identity

\[
\sum_{k=0}^{n} (-1)^k \frac{d_k(q)}{[k]_q!} \frac{d_{n-k}(q^{-1})}{[n-k]_{q^{-1}}!} = 1 + (-1)^n \frac{n}{2}
\]  

(55)

or, equivalently,

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k d_k(q) d_{n-k}(q^{-1}) = 1 + (-1)^n \frac{n}{2} [n]_q!.
\]  

(56)

Proof. By identity (41), we have

\[
D_q(-t) D_{q^{-1}}(t) = \frac{E_q(-t) E_{q^{-1}}(t)}{(1-t)(1+t)} = \frac{1}{1-t^2}.
\]

Now we have

\[
D_q(-t) D_{q^{-1}}(t) = \sum_{i \geq 0} (-1)^i d_i(q) \frac{t^i}{[i]_q!} \sum_{j \geq 0} d_j(q^{-1}) \frac{t^j}{[j]_{q^{-1}}!} = \sum_{i, j \geq 0} (-1)^i \frac{d_i(q)}{[i]_q!} \frac{d_j(q^{-1})}{[j]_{q^{-1}}!} t^{i+j}.
\]

Setting \(i + j = n\) and replacing \(i\) by \(k\), we have

\[
D_q(-t) D_{q^{-1}}(t) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} (-1)^k \frac{d_k(q)}{[k]_q!} \frac{d_{n-k}(q^{-1})}{[n-k]_{q^{-1}}!} \right) t^n.
\]

Hence, we have the identity

\[
\sum_{n \geq 0} \left( \sum_{k=0}^{n} (-1)^k \frac{d_k(q)}{[k]_q!} \frac{d_{n-k}(q^{-1})}{[n-k]_{q^{-1}}!} \right) t^n = \sum_{n \geq 0} \frac{1 + (-1)^n}{2} t^n
\]

and this yields identity (55). This identity and \([n]_{q^{-1}}! = [n]_q! q^{-\binom{n}{2}}\), immediately yield identity (56).
Theorem 18. We have the identity
\[
\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k d_k(q) d_{n-k}(q) = \frac{1 + (-1)^n}{2} \sum_{k=0}^{n/2} (-1)^k q^{k^2-k} \left( \frac{1 - q}{1 + q} \right)^k \frac{[n]_q!}{[k]_q^2!}. \tag{57}
\]

Proof. By identity (42), we have
\[
D_q(t)D_q(-t) = \frac{E_q(t)^{-1}E_q(-t)^{-1}}{(1-t)(1+t)} = \frac{1}{1-t^2} E_q \left( \frac{1-q}{1+q} t^2 \right)^{-1} = \sum_{i \geq 0} t^{2i} \cdot \sum_{k \geq 0} (-1)^k q^{2k-2i} \binom{1-q}{1+q}^k \frac{q^{2k}}{[k]_q^2!}.
\]
\[
= \sum_{i,k \geq 0} (-1)^k q^{k^2-k} \left( \frac{1-q}{1+q} \right)^k \frac{[2i+2k]!}{[k]_q^2!} \frac{[2i+2k]}{[2n]_q!} t^{2i+2k}.
\]
\[
= \sum_{n \geq 0} \left( \sum_{k=0}^{n} (-1)^k q^{k^2-k} \left( \frac{1-q}{1+q} \right)^k \frac{[2n]!}{[k]_q^2!} \right) \frac{1}{2} t^{2n}.
\]
\[
= \sum_{n \geq 0} \left( \frac{1 + (-1)^n}{2} \sum_{k=0}^{n/2} (-1)^k q^{k^2-k} \left( \frac{1-q}{1+q} \right)^k \frac{[n]_q!}{[k]_q^2!} \right) \frac{t^n}{[n]_q!}.
\]
Taking the coefficients of \( \frac{t^n}{[n]_q!} \) in the first and in the last series, we have identity (57). \( \square \)

Recall that for the \( q \)-binomial coefficients we have the \( q \)-series
\[
\sum_{n \geq 0} \binom{m+n}{m}_q t^n = \frac{1}{(1-t)(1-qt)(1-q^2t) \cdots (1-q^mt)} = \frac{1}{(t;q)_{m+1}}. \tag{58}
\]
Then we have the following result.

Theorem 19. For every \( m,n \in \mathbb{N} \), we have the identities
\[
\sum_{k=0}^{n} \binom{n}{k}_q \binom{m+k}{m}_q [k]_q! q^k d_{m-k}(q) = \sum_{k=0}^{n} \binom{n}{k}_q \binom{m+k+1}{m+1}_q (-1)^{n-k} [k]_q! q^{(n-k)/2}. \tag{59}
\]
\[
\sum_{k=0}^{n} \binom{n}{k}_q \binom{\alpha+k}{k}_q [k]_q! d_{m-k}(q) = \sum_{k=0}^{n} \binom{n}{k}_q \binom{\alpha+k+1}{k}_q (-1)^{n-k} [k]_q! q^{(n-k)/2}. \tag{60}
\]

Proof. From the \( q \)-series (58), we have the \( q \)-exponential series
\[
\frac{1}{(1-t)(1-qt) \cdots (1-q^mt)} = \sum_{n \geq 0} \binom{m+n}{m}_q [n]_q! \frac{t^n}{[n]_q!}.
\]
\[
\frac{1}{(1-qt)(1-q^2t) \cdots (1-q^{m+1}t)} = \sum_{n \geq 0} \binom{m+n}{m}_q q^n \frac{t^n}{[n]_q!}.
\]

20
Then, by formula (43), we have the identity
\[ \frac{D_q(t)}{(1-qt)(1-q^2t) \cdots (1-q^{m+1}t)} = \frac{E_q(t)^{-1}}{(1-t)(1-qt)(1-q^2t) \cdots (1-q^{m+1}t)} \]
which is equivalent to identity (59). Similarly, we have the identity
\[ \frac{D_q(t)}{(1-t)^{\alpha+1}} = \frac{E_q(t)^{-1}}{(1-t)^{\alpha+2}} \]
which is equivalent to identity (60).

To prove the next theorem, we need the following result.

**Lemma 20.** We have the \( q \)-exponential series
\[ E_q(t)^{-2} = \sum_{n \geq 0} (-1)^n q^{\binom{n}{2}} G_n(q^{-1}) \frac{t^n}{[n]q!}. \]  
(61)

**Proof.** By formula (40) and relations (8), the coefficient of \( \frac{t^n}{[n]q!} \) in the \( q \)-exponential series \( E_q(t)^{-2} \) is
\[ \sum_{k=0}^{n} \binom{n}{k} q^{\binom{k}{2}} (-1)^{n-k} q^{\binom{n-k}{2}} = (-1)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k} q^{-k(n-k)} = (-1)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k} q^{-1}. \]

By the definition (45) of the Galois numbers, this implies identity (61). \( \square \)

Recall that the \( q \)-multiset coefficients are defined by
\[ \binom{n}{k}_q = \begin{cases} \binom{n + k - 1}{k}_q, & \text{if } k \geq 1; \\ 1, & \text{if } k = 0 \end{cases} \]
and that they have \( q \)-generating series
\[ \sum_{k \geq 0} \binom{n}{k}_q t^k = \frac{1}{(1-t)(1-qt)(1-q^2t) \cdots (1-q^{n-1}t)} = \frac{1}{(t; q)_n}. \]  
(62)

**Theorem 21.** For every \( m, n \in \mathbb{N} \), we have the identity
\[ \sum_{k=0}^{n} \binom{n}{k}_q q^{mk} d_k(q) = [n]q! \sum_{k=0}^{n} \binom{m}{k}_q (1-q)^k q^{m(n-k)}. \]  
(63)
Proof. We have the $q$-exponential series
\[
L(q; t) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} q^{mk} d_k(q) \right) \frac{t^n}{[n]_q!} = E_q(t) E_q(q^n t)^{-1}
\]
By identity (39), we have
\[
L(q; t) = \frac{E_q(t) E_q(t)^{-1}}{(1 - q^m t)((1 - q)t; q)_m} = \frac{1}{1 - q^m t} \cdot \frac{1}{((1 - q)t; q)_m}
\]
\[
= \sum_{n \geq 0} \left( [n]_q! \sum_{k=0}^{n} \binom{m}{k} q^{m(n-k)} \right) \frac{t^n}{[n]_q!}
\]
from which we have at once identity (63).

We conclude this section proving the following elementary identity involving the $q$-Pochhammer symbol.

**Theorem 22.** We have the identity
\[
\frac{d_{n+2}(q)}{[n+1]_q!} \left( \frac{x^{n+1}}{(qx; q)_{n+1}} \right) = \sum_{k=0}^{n} \frac{d_{k+1}(q) + d_k(q)}{[k]_q!} \left( \frac{x^k}{(qx; q)_k} \right) = \sum_{k=0}^{n} \frac{d_{k+1}(q) x^{k+1} + d_k(q) x^k}{[k]_q!(qx; q)_{k+1}}.
\]
Proof. By recurrence (6), we have
\[
d_{k+2}(q) x = [k+1]_q d_{k+1}(q) x + [k+1]_q q^{k+1} x d_k(q)
\]
and consequently
\[
\frac{d_{k+2}(q) x^{k+1}}{[k+1]_q! (qx; q)_{k+1}} = \frac{d_{k+1}(q)}{[k]_q!} \left( \frac{x^{k+1}}{(qx; q)_{k+1}} \right) + \frac{d_k(q)}{[k]_q!} \left( \frac{x^k}{(qx; q)_{k+1}} \right) - \sum_{k=0}^{n} \frac{d_k(q) x^k}{[k]_q!(qx; q)_k}.
\]
Then we have
\[
\sum_{k=0}^{n} \frac{d_{k+2}(q)}{[k+1]_q! (qx; q)_{k+1}} = \sum_{k=0}^{n} \frac{d_{k+1}(q)}{[k]_q!} \left( \frac{x^{k+1}}{(qx; q)_{k+1}} \right) + \sum_{k=0}^{n} \frac{d_k(q)}{[k]_q!} \left( \frac{x^k}{(qx; q)_{k+1}} \right) - \sum_{k=0}^{n} \frac{d_k(q) x^k}{[k]_q!(qx; q)_k},
\]
which is
\[
\sum_{k=1}^{n+1} \frac{d_k(q)}{[k]_q!} \left( \frac{x^k}{(qx; q)_k} \right) + \sum_{k=0}^{n} \frac{d_k(q)}{[k]_q!} \left( \frac{x^k}{(qx; q)_{k+1}} \right) = \sum_{k=0}^{n} \frac{d_{k+1}(q)}{[k]_q!} \left( \frac{x^{k+1}}{(qx; q)_{k+1}} \right) + \sum_{k=0}^{n} \frac{d_k(q) x^k}{[k]_q!(qx; q)_{k+1}},
\]
or
\[
\frac{d_{n+2}(q)}{[n+1]_q! (qx; q)_{n+1}} + \sum_{k=0}^{n} \frac{d_{k+1}(q) + d_k(q)}{[k]_q!} \left( \frac{x^k}{(qx; q)_k} \right) = \sum_{k=0}^{n} \frac{d_{k+1}(q) x^{k+1} + d_k(q) x^k}{[k]_q!(qx; q)_{k+1}}.
\]
This is the claimed identity. \qed
6 Determinantal identities

Since the $q$-derangement numbers satisfy a three-term recurrence, they can be represented in terms of tridiagonal determinants (or continuants ([20, pp. 516–525], [31])).

Theorem 23. We have the identity

$$d_n(q) = \begin{vmatrix} [0]_q & -q & -q^2 & -q^3 & \cdots & 0 \\ [1]_q & [1]_q & -q^2 & -q^3 & \cdots & 0 \\ [2]_q & [2]_q & -q^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-2]_q & [n-2]_q & \cdots & [n-2]_q & [n-1]_q & [n-1]_q \\ [n-1]_q & [n-1]_q & \cdots & [n-1]_q & [n-1]_q & \cdots & \cdots \\ \end{vmatrix}_{n \times n}.$$ \hfill (65)

Proof. The tridiagonal determinants in formula (65) satisfy recurrence (6) with the appropriate initial values. This implies at once the claimed identity. \hfill \square

The $q$-derangement numbers can also be represented in terms of Hessenberg determinants [31, p. 90], as follows.

Theorem 24. Consider the $n \times n$ lower Hessenberg matrix

$$A_n(q) = \begin{bmatrix} a_{00}(q) & -1 & 0 & 0 & \cdots & 0 \\ a_{10}(q) & a_{11}(q) & -1 & 0 & \cdots & 0 \\ a_{20}(q) & a_{21}(q) & a_{22}(q) & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1}(q) & a_{n-2,2}(q) & a_{n-2,3}(q) & a_{n-2,4}(q) & \cdots & -1 \\ a_{n-1,1}(q) & a_{n-1,2}(q) & a_{n-1,3}(q) & a_{n-1,4}(q) & \cdots & a_{n-1,n-1}(q) \end{bmatrix}$$

where

$$a_{i,j}(q) = \begin{cases} \binom{i}{j}_q a_{i-j}(q), & \text{if } i \geq j; \\ -1, & \text{if } i = j - 1; \\ 0, & \text{otherwise} \end{cases}$$

where

$$a_k(q) = \frac{q}{2q-1} (q^n - (1-q)^n) [n]_q!.$$ \hfill (66)

Then we have the identity

$$d_n(q) = \det A_n(q).$$ \hfill (67)

Proof. Let $b_n(q) = \det A_n(q)$. By expanding the determinant along the last column, we get the recurrence

$$b_{n+1}(q) = \sum_{k=0}^{n} \binom{n}{k}_q a_k(q) b_{n-k}(q)$$
with initial value \( b_0(q) = 1 \). Hence, considering the \( q \)-exponential generating series

\[
a(q; t) = \sum_{n \geq 0} a_n(q) \frac{t^n}{[n]_q!} \quad \text{and} \quad b(q; t) = \sum_{n \geq 0} b_n(q) \frac{t^n}{[n]_q!},
\]

we have the \( q \)-differential equation

\[
\mathfrak{D}_q b(q; t) = a(q; t) b(q; t).
\]

If \( b(q; t) = D_q(t) \), then \( b_0(q) = d_0(q) = 1 \), as requested, and

\[
a(q; t) = \frac{\mathfrak{D}_q b(q; t)}{b(q; t)} = \frac{\mathfrak{D}_q D_q(t)}{D_q(t)}.
\]

By series (43) and relation (38), we have

\[
\mathfrak{D}_q D_q(t) = \frac{D_q(qt) - D_q(t)}{(q - 1)t}
\]

\[
= \frac{1}{(q - 1)t} \left( \frac{E_q(qt)^{-1} - E_q(t)^{-1}}{1 - qt} \right)
\]

\[
= \frac{1}{(q - 1)t} \left( \frac{E_q(t)^{-1}}{1 - qt(1 + (q - 1)t) - E_q(t)^{-1}} \right)
\]

\[
= \frac{qt E_q(t)^{-1}}{(1 - t)(1 - qt)(1 + (q - 1)t)},
\]

which is

\[
\mathfrak{D}_q D_q(t) = \frac{qt}{(1 - qt)(1 + (q - 1)t)} D_q(t).
\]

Therefore, we have

\[
a(q; t) = \frac{qt}{(1 - qt)(1 + (q - 1)t)} = \frac{q}{2q - 1} \frac{1}{1 - qt} - \frac{q}{2q - 1} \frac{1}{1 - (1 - q)t}.
\]

This decomposition yields identity (66), and, consequently, this proves identity (67). \( \square \)

7 Final remarks

In the literature, there are also other \( q \)-analogues for the derangement numbers. For instance, we have the \( q \)-derangement numbers [11]

\[
D_n(q) = \sum_{k=0}^{n} \binom{n}{k}_q [n - k]_q! (-1)^k
\]

\[
(68)
\]
satisfying the recurrences
\begin{align}
D_{n+1}(q) &= [n+1]_q D_n(q) + (-1)^{n+1} \quad (69) \\
D_{n+2}(q) &= q [n+1]_q D_{n+1}(q) + [n+1]_q D_n(q) \quad (70)
\end{align}

with initial conditions \( D_0(q) = 1 \) and \( D_1(q) = 0 \). Moreover, they have \( q \)-exponential generating series
\[
\sum_{n \geq 0} D_n(q) \frac{t^n}{[n]_q!} = \frac{E_q(-t)}{1-t}.
\]

The \( q \)-numbers \( d_n(q) \) and \( D_n(q) \) are not independent, as shown in the next theorem.

**Theorem 25.** For every \( n \in \mathbb{N} \), we have the relation
\[
d_n(q^{-1}) = q^{-\binom{n}{2}} D_n(q). \quad (71)
\]

Moreover, we have the formulas
\begin{align}
d_n(q) &= \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \frac{(q^{n-k+1};q)_k}{(1-q)^k} \quad (72) \\
D_n(q) &= \sum_{k=0}^{n} (-1)^{n-k} \frac{(q^{n-k+1};q)_k}{(1-q)^k}. \quad (73)
\end{align}

**Proof.** By formula (4) and relations (8), we have
\[
d_n(q^{-1}) = \sum_{k=0}^{n} \binom{n}{k} [n-k]_{q^{-1}}! (-1)^k q^{-\binom{k}{2}} \\
= \sum_{k=0}^{n} \binom{n}{k} q^{-k(n-k)} [n-k]_q! q^{-\binom{n-k}{2}} (-1)^k q^{-\binom{k}{2}} \\
= q^{-\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k} [n-k]_q! (-1)^k.
\]

By formula (68), this is relation (71).

Since \([n]_q! = \frac{(q;q)_n}{(1-q)_n}\), from definition (4) we have
\[
d_n(q) = \sum_{k=0}^{n} \frac{[n]_q!}{[k]_q!} (-1)^k q^{\binom{k}{2}} = \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{(1-q)^{n-k}} (q;q)_k = \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{(1-q)^{n-k}} (q^{k+1};q)_{n-k}.
\]

This is equivalent to identity (72). Similarly, formula (68) can be rewritten as formula (73).
We also have the following result.

**Theorem 26.** We have the ordinary generating series

\[
    d(q; t) = \sum_{n \geq 0} d_n(q) t^n = \sum_{k \geq 0} \frac{(-1)^k q^{\frac{k+1}{2}}}{(1-q)^k} \frac{t^k Q(q; q^k t)}{(1+q^k t) \left(\frac{t}{1-q}; q\right)_{k+1}} \quad (74)
\]

\[
    D(q; t) = \sum_{n \geq 0} D_n(q) t^n = \sum_{k \geq 0} \frac{(-1)^k q^{\frac{k+1}{2}}}{(1-q)^k} \frac{t^k}{(1+q^k t) \left(\frac{t}{1-q}; q\right)_{k+1}} \quad (75)
\]

where

\[
    Q(q; t) = \sum_{n \geq 0} (-1)^n q^{\frac{n}{2}} t^n.
\]

**Proof.** From recurrence (5), we have

\[
    \sum_{n \geq 0} d_{n+1}(q) t^n = \sum_{n \geq 0} \frac{1 - q^{n+1}}{1-q} d_n(q) t^n + \sum_{n \geq 0} (-1)^{n+1} q^{\frac{(n+1)}{2}} t^n,
\]

which is

\[
    d(q; t) - d_0(q) = \frac{1}{1-q} (d(q; t) - q d(q; q t)) + \frac{Q(q; t) - 1}{t},
\]

or

\[
    d(q; t) - 1 = \frac{t}{1-q} d(q; t) - \frac{qt}{1-q} d(q; q t) + Q(q; t) - 1,
\]

or

\[
    \left(1 - \frac{t}{1-q}\right) d(q; t) = Q(q; t) - \frac{qt}{1-q} d(q; q t),
\]

which is

\[
    d(q; t) = \frac{Q(q; t)}{1 - \frac{t}{1-q}} - \frac{qt}{(1-q) (1 - \frac{t}{1-q})} d(q; q t).
\]

By repeatedly applying this formula, we get

\[
    d(q; t) = \sum_{k=0}^{n} \frac{(-1)^k q^{\frac{k+1}{2}}}{(1-q)^k} \frac{t^k Q(q; q^k t)}{(1+q^k t) \left(\frac{t}{1-q}; q\right)_{k+1}} + \frac{(-1)^{n+1} q^{\frac{n+2}{2}}}{(1-q)^{n+1} (1 - \frac{t}{1-q}) (1 - \frac{qt}{1-q}) \cdots (1 - \frac{q^t}{1-q})} d(q; q^{n+1} t)
\]

\[
    = \sum_{k=0}^{n} \frac{(-1)^k q^{\frac{k+1}{2}}}{(1-q)^k} \frac{t^k Q(q; q^k t)}{(1-\frac{t}{1-q}; q)_{k+1}} + \frac{(-1)^{n+1} q^{\frac{n+2}{2}}}{(1-q)^{n+1} (1 - \frac{t}{1-q}) (1 - \frac{qt}{1-q}) \cdots (1 - \frac{q^t}{1-q})} d(q; q^{n+1} t).
\]

Taking the limit for \( n \) tending to \( +\infty \), we obtain series (74).
Similarly, from recurrence (69), we have
\[
\sum_{n \geq 0} D_{n+1}(q) t^n = \sum_{n \geq 0} \frac{1 - q^{n+1}}{1 - q} D_n(q) t^n + \sum_{n \geq 0} (-1)^{n+1} t^n,
\]
which is
\[
\frac{D(q; t) - D_0(q)}{t} = \frac{1}{1 - q} (D(q; t) - qD(q; qt)) - \frac{1}{1 + t},
\]
or
\[
\left(1 - \frac{t}{1 - q}\right)D(q; t) = \frac{1}{1 + t} - \frac{qt}{1 - q} D(q; qt),
\]
or
\[
D(q; t) = \frac{1}{(1 + t)(1 - \frac{t}{1 - q})} - \frac{qt}{(1 - q)(1 - \frac{t}{1 - q})} D(q; qt).
\]
By repeatedly applying this formula, we get
\[
D(q; t) = \sum_{k=0}^{n} \left(\frac{(-1)^k q^{\frac{k+1}{2}}}{1 - q} \right) \frac{t^k}{(1 + q^k t)(1 - \frac{t}{1 - q})(1 - \frac{qt}{1 - q}) \cdots (1 - \frac{q^k t}{1 - q})}
+ \left(\frac{(-1)^{n+1} q^{\frac{n+2}{2}}}{1 - q^{n+1}} \right) \frac{t^{n+1}}{(1 - q^{n+1})(1 - \frac{t}{1 - q})(1 - \frac{qt}{1 - q}) \cdots (1 - \frac{q^{n+1} t}{1 - q})} D(q; q^{n+1} t)
= \sum_{k=0}^{n} \left(\frac{(-1)^k q^{\frac{k+1}{2}}}{1 - q} \right) \frac{t^k}{(1 + q^k t)(\frac{t}{1 - q}; q)_{k+1}}
+ \left(\frac{(-1)^{n+1} q^{\frac{n+2}{2}}}{1 - q^{n+1}} \right) \frac{t^{n+1}}{(\frac{t}{1 - q}; q)_{n+1}} D(q; q^{n+1} t).
\]
Taking the limit for \( n \) tending to \(+\infty\), we obtain series (75). \( \square \)

References


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