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q-Derangement Identities

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Abstract

In this paper, we derive several combinatorial identities involving the q-derangement numbers (for the major index) and many other q-numbers and q-polynomials of combinatorial interest, such as the q-binomial coefficients, the q-Stirling numbers, the q-Bell numbers, the q-Pochhammer symbol, the Gaussian polynomials, the Rogers-Szegő polynomials and the Galois numbers, and the Al-Salam-Carlitz polynomials. We also obtain two determinantal identities expressing the q-derangement numbers as tridiagonal determinants and as Hessenberg determinants.

1 Introduction

The *derangement number* d_n (sequence A000166 in the *On-Line Encyclopedia of Integer Sequences*) counts the derangements (i.e., permutation with no fixed points) of an n-set. By a simple application of the principle of inclusion-exclusion, we have the formula

$$d_n = \sum_{k=0}^{n} \binom{n}{k} (n-k)!\,(-1)^k\,. \tag{1}$$

Moreover, these numbers satisfy the recurrences

$$d_{n+1} = (n+1)\,d_n + (-1)^{n+1} \tag{2}$$

$$d_{n+2} = (n+1)\,d_{n+1} + (n+1)\,d_n \tag{3}$$

with initial conditions $d_0 = 1$ and $d_1 = 0$.

The *q-derangement numbers* [32, 5] are defined by the formula

$$d_n(q) = \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q!\,(-1)^k q^{\binom{k}{2}}\,, \tag{4}$$

where $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$, and satisfy the recurrences

$$d_{n+1}(q) = [n+1]_q\, d_n(q) + (-1)^{n+1} q^{\binom{n+1}{2}} \tag{5}$$

$$d_{n+2}(q) = [n+1]_q\, d_{n+1}(q) + [n+1]_q\, q^{n+1} d_n(q) \tag{6}$$

with initial conditions $d_0(q) = 1$ and $d_1(q) = 0$.

The *major index* $\mathrm{maj}(\sigma)$ of a permutation σ of $\{1, 2, \ldots, n\}$ is the sum of all positions k for which $\sigma(k) > \sigma(k+1)$. The q-numbers $d_n(q)$ arise from the q-counting of derangements by major index and have several interesting combinatorial properties. Indeed, if $\mathscr{D}_n$ is the set of all derangements of $\{1, 2, \ldots, n\}$, then we have [32]

$$d_n(q) = \sum_{\sigma \in \mathscr{D}_n} q^{\mathrm{maj}(\sigma)}\,.$$

Moreover, considered as a polynomial in q, the q-derangement number $d_n(q)$ has non-negative integer coefficients forming a unimodal sequence [5]. These coefficients have a spiral property [33], which implies their unimodality and also the fact that the maximum coefficient of $d_n(q)$ appears exactly in the middle of the polynomial, i.e., is the coefficient of $q^{\lfloor n(n-1)/4 \rfloor}$ (as conjectured by Chen and Rota [5]). Furthermore, they have the ratio monotonicity property (for $n \geq 6$) [7] which implies log-concavity and the spiral property.

For the ordinary derangement numbers d_n (and their generalizations [3, 10, 23, 24, 25]) there are a lot of combinatorial identities. In this paper, we derive several q-analogues of these identities. They involve the q-derangement numbers $d_n(q)$ and many other q-numbers or q-polynomials, such as the q-binomial coefficients, the q-Stirling numbers and the q-Bell numbers, the q-Pochhammer symbol, the Gaussian polynomials, the Rogers-Szegő polynomials and the Galois numbers, and the Al-Salam-Carlitz polynomials. Finally, we obtain two determinantal identities expressing the q-derangement numbers as tridiagonal determinants and as Hessenberg determinants.

2 q-binomial identities

We start by recalling some basic definitions of q-number theory. For every $n \in \mathbb{N}$, we have the *q-natural number* $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and the *q-factorial number* $[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$. Then, for every $n, k \in \mathbb{N}$, we have the *q-binomial coefficients* (or *Gaussian coefficients* [12]) defined by

$$\binom{n}{k}_q = \begin{cases} \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!}, & \text{if } k \leq n; \\ 0, & \text{otherwise,} \end{cases}$$

and satisfying the recurrence

$$\binom{n+1}{k+1}_q = \binom{n}{k}_q + q^{n+1}\binom{n}{k+1}_q \tag{7}$$

with initial conditions $\binom{n}{0}_q = 1$ and $\binom{0}{k}_q = \delta_{k,0}$. Moreover, we have the relations

$$[n]_{q^{-1}} = \frac{1}{q^{n-1}}\,[n]_q\,, \qquad [n]_{q^{-1}}! = [n]_q!\, q^{-\binom{n}{2}}\,, \qquad \binom{n}{k}_{q^{-1}} = \binom{n}{k}_q q^{-k(n-k)}\,. \tag{8}$$

In this first section, we derive some q-binomial identities using an elementary approach [21] which exploits the properties of the q-binomial coefficients and the recurrences of the q-derangement numbers. In Section 5, we derive some other q-binomial identities by using the more advanced technique of the q-exponential series.

Our first result is the following:

Theorem 1. *For every $n \in \mathbb{N}$, we have the identity*

$$1 + \sum_{k=1}^{n}\binom{n}{k}_q \frac{d_{k+1}(q)}{[k]_q} = \sum_{k=0}^{n}\binom{n+1}{k+1}_q d_k(q)\,. \tag{9}$$

Proof. By recurrence (6), we have

$$\frac{d_{k+2}(q)}{[k+1]_q} = d_{k+1}(q) + q^{k+1}d_k(q)\,.$$

Consequently, we have

$$\sum_{k=0}^{n-1}\binom{n}{k+1}_q \frac{d_{k+2}(q)}{[k+1]_q} = \sum_{k=0}^{n-1}\binom{n}{k+1}_q d_{k+1}(q) + \sum_{k=0}^{n-1}\binom{n}{k+1}_q q^{k+1}d_k(q)$$

or

$$\sum_{k=1}^{n}\binom{n}{k}_q \frac{d_{k+1}(q)}{[k]_q} = \sum_{k=1}^{n}\binom{n}{k}_q d_k(q) + \sum_{k=0}^{n}\binom{n}{k+1}_q q^{k+1}d_k(q)$$

or

$$\sum_{k=1}^{n}\binom{n}{k}_q \frac{d_{k+1}(q)}{[k]_q} = \sum_{k=0}^{n}\left(\binom{n}{k}_q + q^{k+1}\binom{n}{k+1}_q\right) d_k(q) - d_0(q)\,.$$

By recurrence (7) and the initial condition $d_0(q) = 1$, we have identity (9). □

Similarly, we have the following formula.

Theorem 2. *For every $n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k}_q d_{k+1}(q) = \sum_{k=0}^{n}\binom{n+1}{k+1}_q [k]_q d_k(q) + \sum_{k=1}^{n}\binom{n}{k}_q q^{2k-1}d_{k-1}(q)\,. \tag{10}$$

Proof. By recurrence (6), we have

$$\sum_{k=0}^{n-1} \binom{n}{k+1}_q d_{k+2}(q) = \sum_{k=0}^{n-1} \binom{n}{k+1}_q [k+1]_q d_{k+1}(q) + \sum_{k=0}^{n-1} \binom{n}{k+1}_q [k+1]_q q^{k+1} d_k(q)$$

or

$$\sum_{k=1}^{n} \binom{n}{k}_q d_{k+1}(q) = \sum_{k=1}^{n} \binom{n}{k}_q [k]_q d_k(q) + \sum_{k=0}^{n} \binom{n}{k+1}_q [k+1]_q q^{k+1} d_k(q)$$

or

$$\sum_{k=0}^{n} \binom{n}{k}_q d_{k+1}(q) - d_1(q) = \sum_{k=0}^{n} \binom{n}{k}_q [k]_q d_k(q) + \sum_{k=0}^{n} \binom{n}{k+1}_q ([k]_q + q^k) q^{k+1} d_k(q)$$

or

$$\sum_{k=0}^{n} \binom{n}{k}_q d_{k+1}(q) - d_1(q) = \sum_{k=0}^{n} \left(\binom{n}{k}_q + q^{k+1} \binom{n}{k+1}_q \right) [k]_q d_k(q) + \sum_{k=0}^{n} \binom{n}{k+1}_q q^{2k+1} d_k(q)\,.$$

By recurrence (7) and the initial condition $d_1(q) = 0$, we have identity (10). □

We also have the following formula.

Theorem 3. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{m+n+1}{m+k+1}_q \frac{[n]_q!}{[k]_q!} (-1)^k q^{\binom{m+k+1}{2}} d_k(q) = \sum_{k=0}^{n} \binom{m+n}{m+k}_q \frac{[n]_q!}{[k]_q!} (-1)^k q^{\binom{m+k+1}{2}+\binom{k}{2}}\,. \quad (11)$$

In particular, for $m = 0$, we have the identity

$$\sum_{k=0}^{n} \binom{n+1}{k+1}_q \frac{[n]_q!}{[k]_q!} (-1)^k q^{\binom{k+1}{2}} d_k(q) = \sum_{k=0}^{n} \binom{n}{k}_q \frac{[n]_q!}{[k]_q!} (-1)^k q^{k^2}. \quad (12)$$

Proof. From recurrence (5), we have

$$\frac{d_{k+1}(q)}{[k+1]_q!} = \frac{d_k(q)}{[k]_q!} + (-1)^{k+1} \frac{q^{\binom{k+1}{2}}}{[k+1]_q!}\,.$$

and consequently

$$\sum_{k=0}^{n-1} \binom{m+n}{m+k+1}_q (-1)^{k+1} [n]_q! q^{\binom{m+k+2}{2}} \frac{d_{k+1}(q)}{[k+1]_q!}$$
$$= \sum_{k=0}^{n-1} \binom{m+n}{m+k+1}_q (-1)^{k+1} [n]_q! q^{\binom{m+k+2}{2}} \frac{d_k(q)}{[k]_q!} + \sum_{k=0}^{n-1} \binom{m+n}{m+k+1}_q [n]_q! q^{\binom{m+k+2}{2}} \frac{q^{\binom{k+1}{2}}}{[k+1]_q!}\,;$$

that is,

$$\sum_{k=1}^{n}\binom{m+n}{m+k}_q\frac{[n]_q!}{[k]_q!}(-1)^k q^{\binom{m+k+1}{2}}d_k(q)+\sum_{k=0}^{n-1}\binom{m+n}{m+k+1}_q\frac{[n]_q!}{[k]_q!}(-1)^k q^{\binom{m+k+1}{2}}q^{m+k+1}d_k(q)$$
$$=\sum_{k=1}^{n}\binom{m+n}{m+k}_q\frac{[n]_q!}{[k]_q!}q^{\binom{m+k+1}{2}}q^{\binom{k}{2}},$$

or

$$\sum_{k=0}^{n}\left(\binom{m+n}{m+k}_q+q^{m+k+1}\binom{m+n}{m+k+1}_q\right)\frac{[n]_q!}{[k]_q!}(-1)^k q^{\binom{m+k+1}{2}}d_k(q)$$
$$=\binom{m+n}{m}_q[n]_q!q^{\binom{m+1}{2}}+\sum_{k=1}^{n}\binom{m+n}{m+k}_q\frac{[n]_q!}{[k]_q!}q^{\binom{m+k+1}{2}+\binom{k}{2}}.$$

Hence, by formula (7), we have identity (11). □

Using the same approach, we also have the following result.

Theorem 4. *We have the identity*

$$q\sum_{k=0}^{n}[k]_q d_k(q)+\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}=[n+1]_q d_n(q)\,. \tag{13}$$

Proof. Since $[k+1]_q=1+q[k]_q$, recurrence (5) can be rewritten as

$$d_{k+1}(q)=(1+q[k]_q)\,d_k(q)+(-1)^{k+1}q^{\binom{k+1}{2}}$$

or

$$d_{k+1}(q)-d_k(q)=q[k]_q\,d_k(q)+(-1)^{k+1}q^{\binom{k+1}{2}}\,.$$

Hence, we have

$$\sum_{k=0}^{n}d_{k+1}(q)-\sum_{k=0}^{n}d_k(q)=q\sum_{k=0}^{n}[k]_q\,d_k(q)+\sum_{k=0}^{n}(-1)^{k+1}q^{\binom{k+1}{2}}$$

or

$$\sum_{k=1}^{n+1}d_k(q)-\sum_{k=0}^{n}d_k(q)=q\sum_{k=0}^{n}[k]_q\,d_k(q)+\sum_{k=1}^{n+1}(-1)^k q^{\binom{k}{2}}$$

or

$$d_{n+1}(q)-d_0(q)=q\sum_{k=0}^{n}[k]_q\,d_k(q)+\sum_{k=0}^{n+1}(-1)^k q^{\binom{k}{2}}-1\,.$$

Hence, by the recurrence (5) once again, we have the identity

$$[n+1]_q d_n(q) + (-1)^{n+1} q^{\binom{n+1}{2}} = q\sum_{k=0}^{n} [k]_q\, d_k(q) + \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} + (-1)^{n+1} q^{\binom{n+1}{2}}$$

which simplifies to identity (13). $\square$

Similarly, we also have the following property.

Theorem 5. *We have the identity*

$$\sum_{k=0}^{n} \binom{n+1}{k+1}_q (-1)^k q^{\binom{k+1}{2}} d_k(q) = q\sum_{k=1}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k+1}{2}} [k-1]_q d_{k-1}(q) + \sum_{k=0}^{n} \binom{n}{k}_q q^{k^2}. \quad (14)$$

Proof. Once again, we start by the recurrence (5) written as

$$d_{k+1}(q) - d_k(q) = q[k]_q\, d_k(q) + (-1)^{k+1} q^{\binom{k+1}{2}}\,.$$

Then we have

$$\begin{aligned}\sum_{k=0}^{n-1} \binom{n}{k+1}_q (-1)^{k+1} q^{\binom{k+2}{2}} d_{k+1}(q) - \sum_{k=0}^{n-1} \binom{n}{k+1}_q (-1)^{k+1} q^{\binom{k+2}{2}} d_k(q) = \\ q\sum_{k=0}^{n-1} \binom{n}{k+1}_q (-1)^{k+1} q^{\binom{k+2}{2}} [k]_q\, d_k(q) + \sum_{k=0}^{n-1} \binom{n}{k+1}_q q^{\binom{k+2}{2}} q^{\binom{k+1}{2}},\end{aligned}$$

which is

$$\begin{aligned}\sum_{k=1}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k+1}{2}} d_k(q) + \sum_{k=0}^{n} \binom{n}{k+1}_q (-1)^k q^{\binom{k+1}{2}} q^{k+1} d_k(q) = \\ q\sum_{k=1}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k+1}{2}} [k-1]_q\, d_{k-1}(q) + \sum_{k=1}^{n} \binom{n}{k}_q q^{\binom{k+1}{2}} q^{\binom{k}{2}}\end{aligned}$$

or

$$\begin{aligned}&\sum_{k=0}^{n} \left(\binom{n}{k}_q + q^{k+1} \binom{n}{k+1}_q \right) (-1)^k q^{\binom{k+1}{2}} d_k(q) - 1 \\ &\quad = q\sum_{k=1}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k+1}{2}} [k-1]_q\, d_{k-1}(q) + \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k+1}{2}} q^{\binom{k}{2}} - 1\,.\end{aligned}$$

By recurrence (7), this last identity simplifies to identity (14). $\square$

3 q-Stirling identities

The *q-Stirling numbers of the second kind* are defined as the connection constants [9, 21] between the ordinary powers x^n and the q-falling factorials $x_q^{\underline{n}} = x(x-[1]_q)(x-[2]_q)\cdots(x-[n-1]_q)$, that is, as the coefficients $\left\{{n \atop k}\right\}_q$ for which

$$x^n = \sum_{k=0}^{n} \left\{{n \atop k}\right\}_q x_q^{\underline{k}}\,.$$

Equivalently, they are the numbers defined by the recurrence

$$\left\{{n+1 \atop k+1}\right\}_q = \left\{{n \atop k}\right\}_q + [k+1]_q \left\{{n \atop k+1}\right\}_q \tag{15}$$

with initial values $\left\{{n \atop 0}\right\}_q = \delta_{n,0}$ and $\left\{{0 \atop k}\right\}_q = \delta_{k,0}$.

Similarly, the *q-Stirling numbers of the first kind* are defined as the connection constants [9, 21] between the q-rising factorials $x_q^{\overline{n}} = x(x+[1]_q)(x+[2]_q)\cdots(x+[n-1]_q)$ and the ordinary powers x^n, that is, as the coefficients $\left[{n \atop k}\right]_q$ for which

$$x_q^{\overline{n}} = \sum_{k=0}^{n} \left[{n \atop k}\right]_q x^k\,.$$

Equivalently, they are the numbers defined by the recurrence

$$\left[{n+1 \atop k+1}\right]_q = \left[{n \atop k}\right]_q + [n]_q \left[{n \atop k+1}\right]_q \tag{16}$$

with initial values $\left[{n \atop 0}\right]_q = \delta_{n,0}$ and $\left[{0 \atop k}\right]_q = \delta_{k,0}$.

For the q-Stirling numbers, we have the inverse relations

$$f_n = \sum_{k=0}^{n} \left\{{n \atop k}\right\}_q g_k \qquad \Longleftrightarrow \qquad g_n = \sum_{k=0}^{n} \left[{n \atop k}\right]_q (-1)^{n-k} f_k \tag{17}$$

and

$$f_n = \sum_{k=0}^{n} \left\{{n+1 \atop k+1}\right\}_q g_k \qquad \Longleftrightarrow \qquad g_n = \sum_{k=0}^{n} \left[{n+1 \atop k+1}\right]_q (-1)^{n-k} f_k\,. \tag{18}$$

Consider the *q-Bell numbers* defined by

$$b_n(q) = \sum_{k=0}^{n} \left\{{n \atop k}\right\}_q q^{\binom{k}{2}}\,. \tag{19}$$

Although they are not the cumulative constants of the q-Stirling numbers considered above, we have the following formulas relating the q-derangement numbers and the q-Bell numbers,

Theorem 6. *We have the identities*

$$\sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\}_q (-1)^k d_k(q) = b_n(q) \tag{20}$$

$$\sum_{k=0}^{n} \left[{n+1 \atop k+1} \right]_q (-1)^k b_k(q) = d_n(q)\,. \tag{21}$$

Proof. By recurrence (5), we have

$$d_{k+1}(q) - [k+1]_q d_k(q) = (-1)^{k+1} q^{\binom{k+1}{2}}\,.$$

Hence, we have the identity

$$\sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_q (-1)^{k+1} d_{k+1}(q) - \sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_q (-1)^{k+1} [k+1]_q d_k(q) = \sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_q q^{\binom{k+1}{2}}$$

or

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q (-1)^k d_k(q) + \sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_q (-1)^k [k+1]_q d_k(q) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q q^{\binom{k}{2}}$$

or

$$\sum_{k=0}^{n} \left(\left\{ {n \atop k} \right\}_q + [k+1]_q \left\{ {n \atop k+1} \right\}_q \right) (-1)^k d_k(q) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q q^{\binom{k}{2}}$$

By recurrence (15) and definition (19), we have identity (20). Then, by this identity, we get identity (21) at once as its inverse relation (by property (18)). $\square$

To prove the next theorem, we need the following result.

Lemma 7. *For every $n, k \in \mathbb{N}$, we have the identity*

$$\left\{ {n+1 \atop k+1} \right\}_q = \sum_{i=k}^{n} \binom{n}{i} \left\{ {i \atop k} \right\}_q q^{i-k}\,. \tag{22}$$

Proof. Since

$$x^{n+1} = \sum_{k=1}^{n+1} \left\{ {n+1 \atop k} \right\}_q x^{\underline{k}}_{q} = \sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\}_q x^{\underline{k+1}}_{q} = \sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\}_q x(x-[1]_q)\cdots(x-[k]_q)\,,$$

we have

$$\begin{aligned} x^n &= \sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\}_q (x-[1]_q)(x-[2]_q)\cdots(x-[k]_q) \\ &= \sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\}_q (x-1)(x-1-q[1]_q)\cdots(x-1-q[k-1]_q) \end{aligned}$$

or

$$(qx+1)^n = \sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\}_q q^k x(x-[1]_q)\cdots(x-[k-1]_q) = \sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\}_q q^k x^{\underline{k}}_q .$$

Then, from this relation, we have

$$\sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\}_q q^k x^{\underline{k}}_q = \sum_{i=0}^{n} \binom{n}{i} q^i x^i = \sum_{i=0}^{n} \binom{n}{i} q^i \sum_{k=0}^{i} \left\{ {i \atop k} \right\}_q x^{\underline{k}}_q = \sum_{k=0}^{i} \left(\sum_{i=k}^{n} \binom{n}{i} \left\{ {i \atop k} \right\}_q q^i \right) x^{\underline{k}}_q .$$

By equating the coefficients of $x^{\underline{k}}_q$, we obtain identity (22). □

Now we can prove the following result.

Theorem 8. *We have the identity*

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q (-1)^k q^{n-k} d_k(q) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k(q) . \tag{23}$$

Proof. By identities (20) and (22), we have

$$b_n(q) = \sum_{k=0}^{n} \left(\sum_{i=0}^{n} \binom{n}{i} \left\{ {i \atop k} \right\}_q q^{i-k} \right) (-1)^k d_k(q) = \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{i} \left\{ {i \atop k} \right\}_q (-1)^k q^{i-k} d_k(q) .$$

Thus, if we set

$$z_n(q) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q (-1)^k q^{n-k} d_k(q) ,$$

then we have the identity

$$b_n(q) = \sum_{i=0}^{n} \binom{n}{i} z_i(q),$$

whose inverse is

$$z_n(q) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b_i(q) .$$

This is identity (23). □

We also have the following result.

Theorem 9. *We have the identity*

$$\sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\}_q (-1)^k d_{k+1}(q) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q (-1)^k q^k [k]_q d_{k-1}(q) . \tag{24}$$

Proof. By recurrence (6), we have

$$d_{k+2}(q) - [k+1]_q d_{k+1}(q) = [k+1]_q q^{k+1} d_k(q)\,.$$

Then we have

$$\sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_q (-1)^{k+1} d_{k+2}(q) - \sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_q (-1)^{k+1} [k+1]_q d_{k+1}(q) =$$
$$= \sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_q (-1)^{k+1} [k+1]_q q^{k+1} d_k(q),$$

or

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q (-1)^k d_{k+1}(q) + \sum_{k=0}^{n} \left\{ {n \atop k+1} \right\}_q (-1)^k [k+1]_q d_{k+1}(q) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q (-1)^k [k]_q q^k d_{k-1}(q),$$

or

$$\sum_{k=1}^{n} \left(\left\{ {n \atop k} \right\}_q + [k+1]_q \left\{ {n \atop k+1} \right\}_q \right) (-1)^k d_{k+1}(q) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q (-1)^k [k]_q q^k d_{k-1}(q)\,.$$

By recurrence (15), we have identity (24). $\square$

Similarly, we also have the following formula.

Theorem 10. *We have the identity*

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q (-1)^k q^{-\binom{k+1}{2}} d_k(q)^2 = \sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\}_q (-1)^k q^{-\binom{k+1}{2}} d_k(q) d_{k+1}(q)\,. \tag{25}$$

Proof. By recurrence (6), we have

$$d_{k+1}(q) d_{k+2}(q) = [k+1]_q d_{k+1}(q)^2 + [k+1]_q q^{k+1} d_k(q) d_{k+1}(q),$$

or

$$[k+1]_q d_{k+1}(q)^2 = d_{k+1}(q) d_{k+2}(q) - [k+1]_q q^{k+1} d_k(q) d_{k+1}(q)\,.$$

Hence, we have

$$\sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_q (-1)^{k+1} q^{-\binom{k+2}{2}} [k+1]_q d_{k+1}(q)^2$$
$$= \sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_q (-1)^{k+1} q^{-\binom{k+2}{2}} d_{k+1}(q) d_{k+2}(q)$$
$$- \sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_q (-1)^{k+1} q^{-\binom{k+2}{2}} [k+1]_q q^{k+1} d_k(q) d_{k+1}(q)$$

or

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q (-1)^k q^{-\binom{k+1}{2}} [k]_q d_k(q)^2$$
$$= \sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q (-1)^k q^{-\binom{k+1}{2}} d_k(q) d_{k+1}(q)$$
$$+ \sum_{k=0}^{n} \left\{ {n \atop k+1} \right\}_q [k+1]_q (-1)^k q^{-\binom{k+1}{2}} d_k(q) d_{k+1}(q) \, .$$

Since $[0]_q = 0$ and $d_1(q) = 0$, we have

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q (-1)^k q^{-\binom{k+1}{2}} [k]_q d_k(q)^2$$
$$= \sum_{k=0}^{n} \left(\left\{ {n \atop k} \right\}_q + [k+1]_q \left\{ {n \atop k+1} \right\}_q \right) (-1)^k q^{-\binom{k+1}{2}} d_k(q) d_{k+1}(q) \, .$$

Finally, by the recurrence (15), this identity simplifies to identity (25). □

Now consider the *q-Bell numbers* $B_n(q)$ defined by the recurrence

$$B_{n+1}(q) = \sum_{k=0}^{n} \binom{n}{k}_q q^{n(n-k)} B_k(q) \tag{26}$$

with initial value $B_0(q) = 1$. Notice that the q-numbers $b_n(q)$ and $B_n(q)$ are different q-analogues of the ordinary Bell numbers (A000110). For these q-Bell numbers, we have the following result.

Theorem 11. *We have the identity*

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{-\binom{k+1}{2}} B_k(q) \, d_{n-k}(q) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{-\binom{k+1}{2}} B_{k+1}(q) \, [n-k]_q! \tag{27}$$

Proof. Let $\sigma_n(q)$ be the sum on the right-hand side of identity (27). Then, by the recurrence

(26), we have

$$
\begin{aligned}
\sigma_n(q) &= \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{-\binom{k+1}{2}} [n-k]_q!\, B_{k+1}(q) \\
&= \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{-\binom{k+1}{2}} [n-k]_q! \sum_{i=0}^{k} \binom{k}{i}_q q^{k(k-i)} B_i(q) \\
&= \sum_{i=0}^{n} \left(\sum_{k=i}^{n} \binom{n}{k}_q \binom{k}{i}_q [n-k]_q! (-1)^{n-k} q^{-\binom{k+1}{2}} q^{k(k-i)} \right) B_i(q) \\
&= \sum_{i=0}^{n} \binom{n}{i}_q \left(\sum_{k=i}^{n} \binom{n-i}{k-i}_q [n-k]_q! (-1)^{n-k} q^{-\binom{k+1}{2}} q^{k(k-i)} \right) B_i(q) \\
&= \sum_{i=0}^{n} \binom{n}{i}_q \left(\sum_{k=0}^{n-i} \binom{n-i}{k}_q [n-k-i]_q! (-1)^{n-k-i} q^{-\binom{k+i+1}{2}} q^{(k+i)k} \right) B_i(q) \\
&= \sum_{i=0}^{n} \binom{n}{i}_q (-1)^{n-i} \left(\sum_{k=0}^{n-i} \binom{n-i}{k}_q [n-k-i]_q! (-1)^{k} q^{-\binom{i+1}{2}-\binom{k+1}{2}-ik} q^{(k+i)k} \right) B_i(q) \\
&= \sum_{i=0}^{n} \binom{n}{i}_q (-1)^{n-i} q^{-\binom{i+1}{2}} \left(\sum_{k=0}^{n-i} \binom{n-i}{k}_q [n-k-i]_q! (-1)^{k} q^{\binom{k}{2}} \right) B_i(q)\,.
\end{aligned}
$$

Finally, by formula (4), we have

$$
\sigma_n(q) = \sum_{i=0}^{n} \binom{n}{i}_q (-1)^{n-i} q^{-\binom{i+1}{2}} d_{n-i}(q)\, B_i(q),
$$

and this is the claimed identity. □

4 Elementary identities

Several combinatorial identities can be derived from the following property of linear recurrences of the first order: the general solution of the recurrence

$$
y_{n+1} = a_{n+1} y_n + b_{n+1}
$$

is given by

$$
y_n = a_n^* y_0 + \sum_{k=1}^{n} \frac{a_n^*}{a_k^*}\, b_k, \tag{28}
$$

where $a_n^* = a_1 a_2 \cdots a_n$, provided that $a_n \neq 0$ for all $n \in \mathbb{N}$.

First of all, we have the following simple result.

Theorem 12. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\begin{aligned} d_{m+n+2}(q) = & \binom{m+n+1}{m}_q [n+1]_q! d_{m+1}(q) \\ & + [m+n+1]_q \sum_{k=0}^{n} \binom{m+n}{m+k}_q [n-k]_q! q^{k+m+1} d_{m+k}(q) \end{aligned} \tag{29}$$

In particular, for $m = 0$, we have the identity

$$d_{n+2}(q) = [n+1]_q \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q! q^{k+1} d_k(q)\,. \tag{30}$$

Proof. Let $y_n(q) = d_{m+n+1}(q)$. By recurrence (6), we have

$$y_{n+1}(q) = d_{m+n+2}(q) = [m+n+1]_q\, d_{m+n+1}(q) + [m+n+1]_q\, q^{m+n+1} d_{m+n}(q),$$

or

$$y_{n+1}(q) = [m+n+1]_q\, y_n(q) + [m+n+1]_q\, q^{m+n+1} d_{m+n}(q)\,.$$

This is a linear recurrence of the first order with coefficients $a_n = [m+n]_q$ and $b_n = [m+n]_q\, q^{m+n} d_{m+n-1}(q)$. Since

$$a_n^* = [m+n]_q \cdots [m]_q = \frac{[m+n]_q}{[m]_q} = \binom{m+n}{m}_q [n]_q!\,,$$

then the solution, being $y_0(q) = d_{m+1}(q)$, is

$$\begin{aligned} y_n(q) &= \binom{m+n}{m}_q [n]_q! d_{m+1}(q) + \sum_{k=1}^{n} \frac{[m+n]_q}{[m]_q} \frac{[m]_q}{[m+k]_q} [m+k]_q\, q^{m+k} d_{m+k-1}(q) \\ &= \binom{m+n}{m}_q [n]_q! d_{m+1}(q) + \sum_{k=1}^{n} \frac{[m+n]_q}{[m+k]_q}\, [m+k]_q\, q^{m+k} d_{m+k-1}(q) \\ &= \binom{m+n}{m}_q [n]_q! d_{m+1}(q) + \sum_{k=1}^{n} \binom{m+n}{m+k}_q [m+k]_q [n-k]_q!\, q^{m+k} d_{m+k-1}(q) \\ &= \binom{m+n}{m}_q [n]_q! d_{m+1}(q) + [m+n]_q \sum_{k=1}^{n} \binom{m+n-1}{m+k-1}_q [n-k]_q!\, q^{m+k} d_{m+k-1}(q) \\ &= \binom{m+n}{m}_q [n]_q! d_{m+1}(q) + [m+n]_q \sum_{k=0}^{n-1} \binom{m+n-1}{m+k}_q [n-k-1]_q!\, q^{m+k+1} d_{m+k}(q)\,. \end{aligned}$$

Finally, by replacing n by $n+1$, we obtain formula (29). □

Theorem 13. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\frac{d_{m+n+2}(q)}{[m+n+1]_q!} = \frac{d_{m+1}(q)}{[m]_q!} + \sum_{k=0}^{n} q^{m+k+1} \frac{d_{m+k}(q)}{[m+k]_q!} . \tag{31}$$

In particular, for $m = 0$, we have the identity

$$\frac{d_{n+2}(q)}{[n+1]_q!} = \sum_{k=0}^{n} q^{k+1} \frac{d_k(q)}{[k]_q!} . \tag{32}$$

Proof. Let $y_n(q) = \frac{d_{m+n+1}(q)}{[m+n]_q!}$. By recurrence (6), we have

$$\begin{aligned} y_{n+1}(q) = \frac{d_{m+n+2}(q)}{[m+n+1]_q!} &= \frac{[m+n+1]_q(d_{m+n+1}(q) + q^{m+n+1} d_{m+n}(q))}{[m+n+1]_q!} \\ &= \frac{d_{m+n+1}(q)}{[m+n]_q!} + q^{m+n+1} \frac{d_{m+n}(q)}{[m+n]_q!}, \end{aligned}$$

or

$$y_{n+1}(q) = y_n(q) + q^{m+n+1} \frac{d_{m+n}(q)}{[m+n]_q!} .$$

This is a linear recurrence of the first order with $a_n = 1$ and $b_n = q^{m+n} \frac{d_{m+n-1}(q)}{[m+n-1]_q!}$ for $n \geq 1$. So, by formula (28), we have the solution

$$y_n(q) = y_0(q) + \sum_{k=1}^{n} q^{m+k} \frac{d_{m+k-1}(q)}{[m+k-1]_q!} = y_0(q) + \sum_{k=0}^{n-1} q^{m+k+1} \frac{d_{m+k}(q)}{[m+k]_q!},$$

or

$$\frac{d_{m+n+1}(q)}{[m+n]_q!} = \frac{d_{m+1}(q)}{[m]_q!} + \sum_{k=0}^{n-1} q^{m+k+1} \frac{d_{m+k}(q)}{[m+k]_q!} .$$

Now by replacing n by $n+1$, we obtain identity (31). $\square$

Theorem 14. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\frac{d_{m+n+1}(q) d_{m+n+2}(q)}{q^{\binom{n+2}{2}} [m+n+1]_q!} = q^{m(n+1)} \frac{d_m(q) d_{m+1}(q)}{[m]_q!} + \sum_{k=0}^{n} q^{m(n-k)} \frac{d_{m+k+1}(q)^2}{q^{\binom{k+2}{2}} [m+k]_q!} \tag{33}$$

In particular, for $m = 0$, we have the identity

$$\frac{d_{n+1}(q) d_{n+2}(q)}{q^{\binom{n+2}{2}} [n+1]_q!} = \sum_{k=0}^{n} \frac{d_{k+1}(q)^2}{q^{\binom{k+2}{2}} [k]_q!} \tag{34}$$

Proof. Let $y_n(q) = \frac{d_{m+n}(q)d_{m+n+1}(q)}{[m+n]_q!}$. By recurrence (6), we have

$$\begin{aligned} y_{n+1}(q) &= \frac{d_{m+n+1}(q)d_{m+n+2}(q)}{[m+n+1]_q!} = \frac{d_{m+n+1}(q)\left(d_{m+n+1}(q) + q^{m+n+1}d_{m+n}(q)\right)}{[m+n]_q!} \\ &= \frac{d_{m+n+1}(q)^2}{[m+n]_q!} + q^{m+n+1}\frac{d_{m+n}(q)d_{m+n+1}(q)}{[m+n]_q!}, \end{aligned}$$

or

$$y_{n+1}(q) = q^{m+n+1}y_n(q) + \frac{d_{m+n+1}(q)^2}{[m+n]_q!}\,.$$

This is a linear recurrence of the first order with $a_n = q^{m+n}$ and $b_n = \frac{d_{m+n}(q)^2}{[m+n-1]_q!}$ for $n \geq 1$. Since $a_n^* = q^{mn+\binom{n+1}{2}}$, by formula (28), we have the solution

$$\begin{aligned} y_n(q) &= q^{mn+\binom{n+1}{2}}y_0(q) + \sum_{k=1}^{n} \frac{q^{mn+\binom{n+1}{2}}}{q^{mk+\binom{k+1}{2}}}\frac{d_{m+k}(q)^2}{[m+k-1]_q!} \\ &= q^{mn+\binom{n+1}{2}}y_0(q) + q^{\binom{n+1}{2}}\sum_{k=0}^{n-1} q^{m(n-k-1)}\frac{d_{m+k+1}(q)^2}{q^{\binom{k+2}{2}}[m+k]_q!}, \end{aligned}$$

or

$$\frac{d_{m+n}(q)d_{m+n+1}(q)}{q^{\binom{n+1}{2}}[m+n]_q!} = q^{mn}\frac{d_m(q)d_{m+1}(q)}{[m]_q!} + \sum_{k=0}^{n-1} q^{m(n-k-1)}\frac{d_{m+k+1}(q)^2}{q^{\binom{k+2}{2}}[m+k]_q!}\,.$$

Now by replacing n by $n+1$, we obtain identity (33). □

The next formula can be obtained with the same elementary approach used in Section 2.

Theorem 15. *We have the identity*

$$\begin{aligned} \frac{d_{m+n+1}(q)^2 - q^{2\binom{m+n+1}{2}}}{[m+n+1]_q!^2} &= \frac{d_m(q)^2 - q^{2\binom{m}{2}}}{[m]_q!^2} + \\ &+ 2\sum_{k=0}^{n}(-1)^{m+k+1}q^{\binom{m+k+1}{2}}\frac{d_{m+k}(q)}{[m+k]_q![k+m+1]_q!} + \sum_{k=0}^{n}\frac{q^{2\binom{m+k}{2}}}{[m+k]_q!^2}\,. \end{aligned} \tag{35}$$

In particular, for $m = 0$, we have the identity

$$\frac{d_{n+1}(q)^2 - q^{n(n+1)}}{[n+1]_q!^2} = 2\sum_{k=0}^{n}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{d_k(q)}{[k]_q![k+1]_q!} + \sum_{k=0}^{n}\frac{q^{k(k-1)}}{[k]_q!^2}\,. \tag{36}$$

Proof. By recurrences (5), we have

$$\begin{aligned} d_{m+k+1}(q)^2 &= \left([m+k+1]_q d_{m+k}(q) + (-1)^{m+k+1}q^{\binom{m+k+1}{2}}\right)^2 \\ &= [m+k+1]_q^2 d_{m+k}(q)^2 + 2(-1)^{m+k+1}q^{\binom{m+k+1}{2}}[m+k+1]_q d_{m+k}(q) + q^{2\binom{m+k+1}{2}}\,. \end{aligned}$$

Hence, we can write

$$\frac{d_{m+k+1}(q)^2}{[m+k+1]_q!^2} = \frac{d_{m+k}(q)^2}{[m+k]_q!^2} + 2(-1)^{m+k+1} q^{\binom{m+k+1}{2}} \frac{d_{m+k}(q)}{[m+k]_q![m+k+1]_q!} + \frac{q^{2\binom{m+k+1}{2}}}{[m+k+1]_q!^2}$$

and, consequently, we have

$$\sum_{k=0}^{n} \frac{d_{m+k+1}(q)^2}{[m+k+1]_q!^2} = \sum_{k=0}^{n} \frac{d_{m+k}(q)^2}{[m+k]_q!^2} + 2\sum_{k=0}^{n} (-1)^{m+k+1} q^{\binom{m+k+1}{2}} \frac{d_{m+k}(q)}{[m+k]_q![m+k+1]_q!} + \sum_{k=0}^{n} \frac{q^{2\binom{m+k+1}{2}}}{[m+k+1]_q!^2},$$

or

$$\sum_{k=1}^{n+1} \frac{d_{m+k}(q)^2}{[m+k]_q!^2} = \sum_{k=0}^{n} \frac{d_{m+k}(q)^2}{[m+k]_q!^2} + 2\sum_{k=0}^{n} (-1)^{m+k+1} q^{\binom{m+k+1}{2}} \frac{d_{m+k}(q)}{[m+k]_q![m+k+1]_q!} + \sum_{k=1}^{n+1} \frac{q^{2\binom{m+k}{2}}}{[m+k]_q!^2}.$$

By simplifying, we get the identity

$$\frac{d_{m+n+1}(q)^2}{[m+n+1]_q!^2} = \frac{d_m(q)^2}{[m]_q!^2} + 2\sum_{k=0}^{n} (-1)^{m+k+1} q^{\binom{m+k+1}{2}} \frac{d_{m+k}(q)}{[m+k]_q![m+k+1]_q!} + \sum_{k=0}^{n} \frac{q^{2\binom{m+k}{2}}}{[m+k]_q!^2} + \frac{q^{2\binom{m+n+1}{2}}}{[m+n+1]_q!^2} - \frac{q^{2\binom{m}{2}}}{[m]_q!^2}$$

which yields identity (35) at once. □

5 q-exponential series

Many identities can be obtained by using the q-exponential generating series. Recall that the product of two q-exponential series $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{[n]_q!}$ and $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{[n]_q!}$ is given by

$$f(t)\cdot g(t) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k}_q f_k g_{n-k} \right) \frac{t^n}{[n]_q!},$$

and that the *q-derivative* (*Jackson's derivative*) $\mathfrak{D}_q$ of a q-exponential generating series $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{[n]_q!}$ is defined [16, 17, 18] by the formula

$$\mathfrak{D}_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} = \sum_{n\geq 0} f_{n+1} \frac{t^n}{[n]_q!}.$$

The *q-exponential series* (*Jackson's q-exponential*) [16]

$$E_q(t) = \sum_{n\geq 0} \frac{t^n}{[n]_q!} = \prod_{k\geq 0} \frac{1}{1+(q-1)q^k t} \tag{37}$$

is the eigenfunction of the q-derivative, that is,

$$\mathfrak{D}_q E_q(\lambda t) = \lambda E_q(t)\,.$$

In particular, since $\mathfrak{D}_q E_q(t) = E_q(t)$, we have the relation

$$E_q(qt) = (1-(1-q)t)\, E_q(t)\,. \tag{38}$$

Consequently, considering the *q-Pochhammer symbol* $(x;q)_m = (1-x)(1-qx)\cdots(1-q^{m-1}x)$, we have, for every $m\in\mathbb{N}$, the identity

$$E_q(q^m t) = \prod_{k=0}^{m-1}(1-(1-q)q^k t)\cdot E_q(t) = ((1-q)t;q)_m\, E_q(t)\,. \tag{39}$$

Moreover, the inverse of the q-exponential series is

$$E_q(t)^{-1} = \sum_{n\geq 0}(-1)^n q^{\binom{n}{2}} \frac{t^n}{[n]_q!} \tag{40}$$

and we have the identities [30]

$$E_q(-t)\, E_{q^{-1}}(t) = 1 \tag{41}$$

$$E_q(t)\, E_q(-t) = E_{q^2}\!\left(\frac{1-q}{1+q}\, t^2\right). \tag{42}$$

By definition (4) and series (40), we have at once that the q-exponential generating series of the q-derangement numbers is

$$D_q(t) = \sum_{n\geq 0} d_n(q) \frac{t^n}{[n]_q!} = \frac{E_q(t)^{-1}}{1-t}\,. \tag{43}$$

We consider the following q-polynomials:

- the *q-Pochhammer symbol*

$$(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) = \sum_{k=0}^{n}\binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^k\,, \tag{44}$$

- the *Gaussian polynomials* [13, 14, 9]

$$g_n(q;x) = (x-1)(x-q)\cdots(x-q^{n-1}) = \sum_{k=0}^{n}\binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} x^k\,,$$

- the *q-Hermite polynomials* (or *Rogers-Szegő polynomials*) ([29, 4, 1, 13], [27, p. 180])

$$H_n(q;x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k$$

and the *Galois numbers* [13, 26]

$$G_n(q) = \sum_{k=0}^{n} \binom{n}{k}_q , \tag{45}$$

- the *q-Carlitz polynomials* (or *Al-Salam-Carlitz polynomials*) ([2], [8, p. 195], [15, 6, 19])

$$U_n^{(\alpha)}(q;x) = \sum_{k=0}^{n} \binom{n}{k}_q (-\alpha)^{n-k} g_k(x) ,$$

having q-exponential generating series

$$P_q(x,t) = \sum_{n\geq 0} (x;q)_n \frac{t^n}{[n]_q!} = \frac{E_q(t)}{E_q(xt)} = E_q(t)\, E_q(xt)^{-1} \tag{46}$$

$$g_q(x,t) = \sum_{n\geq 0} g_n(q;x) \frac{t^n}{[n]_q!} = \frac{E_q(xt)}{E_q(t)} = E_q(t)^{-1} E_q(xt) \tag{47}$$

$$H_q(x,t) = \sum_{n\geq 0} H_n(q;x) \frac{t^n}{[n]_q!} = E_q(t)\, E_q(xt) \tag{48}$$

$$G_q(t) = \sum_{n\geq 0} G_n(q) \frac{t^n}{[n]_q!} = E_q(t)^2 \tag{49}$$

$$U_q(x,t) = \sum_{n\geq 0} U_n^{(\alpha)}(q;x) \frac{t^n}{[n]_q!} = \frac{E_q(xt)}{E_q(t)E_q(\alpha t)} = E_q(\alpha t)^{-1} g_q(x,t) . \tag{50}$$

Using the properties of the q-exponential series, we have at once the following results.

Theorem 16. *We have the identities*

$$\sum_{k=0}^{n} \binom{n}{k}_q d_{n-k}(q)\, (x;q)_k = \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q!\, (-1)^k q^{\binom{k}{2}} x^k \tag{51}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q d_{n-k}(q)\, x^k = \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q!\, g_k(q;x) \tag{52}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q d_{n-k}(q)\, G_k(q,x) = \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q!\, x^k \tag{53}$$

$$\sum_{k=0}^{n} \binom{n}{k}_q \alpha^{n-k} d_{n-k}(q)\, g_k(x) = \sum_{k=0}^{n} \binom{n}{k}_q \alpha^{n-k} [n-k]_q!\, U_k^{(\alpha)}(q;x) . \tag{54}$$

Proof. By series (43), (46), (47), (48), (50) and (40), we have the identities

$$D_q(t)\,P_q(x,t)=\frac{E_q(xt)^{-1}}{1-t}$$
$$D_q(t)\,E_q(x,t)=\frac{g_q(x,t)}{1-t}$$
$$D_q(t)\,H_q(x,t)=\frac{E_q(xt)}{1-t}$$
$$D_q(\alpha t)\,g_q(x,t)=\frac{U_q^{(\alpha)}(xt)}{1-\alpha t}$$

which are equivalent to identities (51), (52), (53) and (54), respectively. □

Moreover, we also have the next result.

Theorem 17. *We have the identity*

$$\sum_{k=0}^{n}(-1)^k\frac{d_k(q)}{[k]_q!}\,\frac{d_{n-k}(q^{-1})}{[n-k]_{q^{-1}}!}=\frac{1+(-1)^n}{2} \tag{55}$$

or, equivalently,

$$\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k d_k(q)\,d_{n-k}(q^{-1})=\frac{1+(-1)^n}{2}\,[n]_q!\,. \tag{56}$$

Proof. By identity (41), we have

$$D_q(-t)\,D_{q^{-1}}(t)=\frac{E_q(-t)\,E_{q^{-1}}(t)}{(1-t)(1+t)}=\frac{1}{1-t^2}\,.$$

Now we have

$$D_q(-t)\,D_{q^{-1}}(t)=\sum_{i\ge0}(-1)^i d_i(q)\frac{t^i}{[i]_q!}\sum_{j\ge0}d_j(q^{-1})\frac{t^j}{[j]_{q^{-1}}!}=\sum_{i,j\ge0}(-1)^i\frac{d_i(q)}{[i]_q!}\,\frac{d_j(q^{-1})}{[j]_{q^{-1}}!}\,t^{i+j}\,.$$

Setting $i+j=n$ and replacing i by k, we have

$$D_q(-t)\,D_{q^{-1}}(t)=\sum_{n\ge0}\left(\sum_{k=0}^{n}(-1)^k\frac{d_k(q)}{[k]_q!}\,\frac{d_{n-k}(q^{-1})}{[n-k]_{q^{-1}}!}\right)t^n\,.$$

Hence, we have the identity

$$\sum_{n\ge0}\left(\sum_{k=0}^{n}(-1)^k\frac{d_k(q)}{[k]_q!}\,\frac{d_{n-k}(q^{-1})}{[n-k]_{q^{-1}}!}\right)t^n=\sum_{n\ge0}\frac{1+(-1)^n}{2}\,t^n$$

and this yields identity (55). This identity and $[n]_{q^{-1}}!=[n]_q!\,q^{-\binom{n}{2}}$, immediately yield identity (56). □

Theorem 18. *We have the identity*

$$\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k d_k(q)\,d_{n-k}(q)=\frac{1+(-1)^n}{2}\sum_{k=0}^{n/2}(-1)^k q^{k^2-k}\left(\frac{1-q}{1+q}\right)^k\frac{[n]_q!}{[k]_{q^2}!}\,. \tag{57}$$

Proof. By identity (42), we have

$$\begin{aligned}
D_q(t)D_q(-t)&=\frac{E_q(t)^{-1}E_q(-t)^{-1}}{(1-t)(1+t)}\\
&=\frac{1}{1-t^2}\;E_{q^2}\Big(\frac{1-q}{1+q}\,t^2\Big)^{-1}\\
&=\sum_{i\geq 0}t^{2i}\cdot\sum_{k\geq 0}(-1)^k q^{2\binom{k}{2}}\left(\frac{1-q}{1+q}\right)^k\frac{t^{2k}}{[k]_{q^2}!}\\
&=\sum_{i,k\geq 0}(-1)^k q^{k^2-k}\left(\frac{1-q}{1+q}\right)^k\frac{[2i+2k]_q!}{[k]_{q^2}!}\frac{t^{2i+2k}}{[2i+2k]_q!}\\
&=\sum_{n\geq 0}\left(\sum_{k=0}^{n}(-1)^k q^{k^2-k}\left(\frac{1-q}{1+q}\right)^k\frac{[2n]_q!}{[k]_{q^2}!}\right)\frac{t^{2n}}{[2n]_q!}\\
&=\sum_{n\geq 0}\left(\frac{1+(-1)^n}{2}\sum_{k=0}^{n/2}(-1)^k q^{k^2-k}\left(\frac{1-q}{1+q}\right)^k\frac{[n]_q!}{[k]_{q^2}!}\right)\frac{t^n}{[n]_q!}\,.
\end{aligned}$$

Taking the coefficients of $\frac{t^n}{[n]_q!}$ in the first and in the last series, we have identity (57). $\square$

Recall that for the q-binomial coefficients we have the q-series

$$\sum_{n\geq 0}\binom{m+n}{m}_q t^n=\frac{1}{(1-t)(1-qt)(1-q^2t)\cdots(1-q^mt)}=\frac{1}{(t;q)_{m+1}}\,. \tag{58}$$

Then we have the following result.

Theorem 19. *For every $m,n\in\mathbb{N}$, we have the identities*

$$\sum_{k=0}^{n}\binom{n}{k}_q\binom{m+k}{m}_q[k]_q!q^k d_{n-k}(q)=\sum_{k=0}^{n}\binom{n}{k}_q\binom{m+k+1}{m+1}_q(-1)^{n-k}[k]_q!q^{\binom{n-k}{2}} \tag{59}$$

$$\sum_{k=0}^{n}\binom{n}{k}_q\binom{\alpha+k}{k}[k]_q!d_{n-k}(q)=\sum_{k=0}^{n}\binom{n}{k}_q\binom{\alpha+k+1}{k}(-1)^{n-k}[k]_q!q^{\binom{n-k}{2}}\,. \tag{60}$$

Proof. From the q-series (58), we have the q-exponential series

$$\begin{aligned}
\frac{1}{(1-t)(1-qt)\cdots(1-q^mt)}&=\sum_{n\geq 0}\binom{m+n}{m}_q[n]_q!\frac{t^n}{[n]_q!}\\
\frac{1}{(1-qt)(1-q^2t)\cdots(1-q^{m+1}t)}&=\sum_{n\geq 0}\binom{m+n}{m}_q[n]_q!q^n\frac{t^n}{[n]_q!}\,.
\end{aligned}$$

Then, by formula (43), we have the identity

$$\frac{D_q(t)}{(1-qt)(1-q^2t)\cdots(1-q^{m+1}t)} = \frac{E_q(t)^{-1}}{(1-t)(1-qt)(1-q^2t)\cdots(1-q^{m+1}t)}$$

which is equivalent to identity (59). Similarly, we have the identity

$$\frac{D_q(t)}{(1-t)^{\alpha+1}} = \frac{E_q(t)^{-1}}{(1-t)^{\alpha+2}}$$

which is equivalent to identity (60). □

To prove the next theorem, we need the following result.

Lemma 20. *We have the q-exponential series*

$$E_q(t)^{-2} = \sum_{n\geq 0} (-1)^n q^{\binom{n}{2}} \, G_n(q^{-1}) \, \frac{t^n}{[n]_q!} \, . \tag{61}$$

Proof. By formula (40) and relations (8), the coefficient of $\frac{t^n}{[n]_q!}$ in the q-exponential series $E_q(t)^{-2}$ is

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} (-1)^{n-k} q^{\binom{n-k}{2}} = (-1)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k}_q q^{-k(n-k)} = (-1)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k}_{q^{-1}} .$$

By the definition (45) of the Galois numbers, this implies identity (61). □

Recall that the *q-multiset coefficients* are defined by

$$\left(\!\!\binom{n}{k}\!\!\right)_q = \begin{cases} \displaystyle\binom{n+k-1}{k}_q, & \text{if } k \geq 1; \\ 1, & \text{if } k = 0 \end{cases}$$

and that they have q-generating series

$$\sum_{k\geq 0} \left(\!\!\binom{n}{k}\!\!\right)_q t^k = \frac{1}{(1-t)(1-qt)(1-q^2t)\cdots(1-q^{n-1}t)} = \frac{1}{(t;q)_n} \, . \tag{62}$$

Theorem 21. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k}_q q^{mk} \, d_k(q) = [n]_q! \sum_{k=0}^{n} \left(\!\!\binom{m}{k}\!\!\right)_q (1-q)^k q^{m(n-k)} \, . \tag{63}$$

Proof. We have the q-exponential series

$$L(q;t) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k}_q q^{mk}\, d_k(q) \right) \frac{t^n}{[n]_q!} = E_q(t) D_q(q^m t) = \frac{E_q(t) E_q(q^m t)^{-1}}{1 - q^m t}\,.$$

By identity (39), we have

$$\begin{aligned} L(q;t) &= \frac{E_q(t) E_q(t)^{-1}}{(1-q^m t)((1-q)t;q)_m} = \frac{1}{1-q^m t} \cdot \frac{1}{((1-q)t;q)_m} \\ &= \sum_{n\geq 0} \left([n]_q! \sum_{k=0}^{n} \left(\!\!\binom{m}{k}\!\!\right)_q (1-q)^k q^{m(n-k)} \right) \frac{t^n}{[n]_q!} \end{aligned}$$

from which we have at once identity (63). □

We conclude this section proving the following elementary identity involving the q-Pochhammer symbol.

Theorem 22. *We have the identity*

$$\frac{d_{n+2}(q)}{[n+1]_q!} \frac{x^{n+1}}{(qx;q)_{n+1}} + \sum_{k=0}^{n} \frac{d_{k+1}(q) + d_k(q)}{[k]_q!} \frac{x^k}{(qx;q)_k} = \sum_{k=0}^{n} \frac{d_{k+1}(q)\, x^{k+1} + d_k(q)\, x^k}{[k]_q!(qx;q)_{k+1}}\,. \tag{64}$$

Proof. By recurrence (6), we have

$$\begin{aligned} d_{k+2}(q)\, x &= [k+1]_q\, d_{k+1}(q)\, x + [k+1]_q\, q^{k+1} x\, d_k(q) \\ &= [k+1]_q\, d_{k+1}(q)\, x + [k+1]_q\, d_k(q) - [k+1]_q\, (1 - q^{k+1}x)\, d_k(q), \end{aligned}$$

and consequently

$$\frac{d_{k+2}(q)}{[k+1]_q!} \frac{x^{k+1}}{(qx;q)_{k+1}} = \frac{d_{k+1}(q)}{[k]_q!} \frac{x^{k+1}}{(qx;q)_{k+1}} + \frac{d_k(q)}{[k]_q!} \frac{x^k}{(qx;q)_{k+1}} - \frac{d_k(q)}{[k]_q!} \frac{x^k}{(qx;q)_k}\,.$$

Then we have

$$\sum_{k=0}^{n} \frac{d_{k+2}(q)}{[k+1]_q!} \frac{x^{k+1}}{(qx;q)_{k+1}} = \sum_{k=0}^{n} \frac{d_{k+1}(q)}{[k]_q!} \frac{x^{k+1}}{(qx;q)_{k+1}} + \sum_{k=0}^{n} \frac{d_k(q)}{[k]_q!} \frac{x^k}{(qx;q)_{k+1}} - \sum_{k=0}^{n} \frac{d_k(q)}{[k]_q!} \frac{x^k}{(qx;q)_k}\,,$$

which is

$$\sum_{k=1}^{n+1} \frac{d_{k+1}(q)}{[k]_q!} \frac{x^k}{(qx;q)_k} + \sum_{k=0}^{n} \frac{d_k(q)}{[k]_q!} \frac{x^k}{(qx;q)_k} = \sum_{k=0}^{n} \frac{d_{k+1}(q)}{[k]_q!} \frac{x^{k+1}}{(qx;q)_{k+1}} + \sum_{k=0}^{n} \frac{d_k(q)}{[k]_q!} \frac{x^k}{(qx;q)_{k+1}}\,,$$

or

$$\frac{d_{n+2}(q)}{[n+1]_q!} \frac{x^{n+1}}{(qx;q)_{n+1}} + \sum_{k=0}^{n} \frac{d_{k+1}(q) + d_k(q)}{[k]_q!} \frac{x^k}{(qx;q)_k} = \sum_{k=0}^{n} \frac{d_{k+1}(q)\, x^{k+1} + d_k(q)\, x^k}{[k]_q!(qx;q)_{k+1}}\,.$$

This is the claimed identity. □

6 Determinantal identities

Since the q-derangement numbers satisfy a three-term recurrence, they can be represented in terms of *tridiagonal determinants* (or *continuants* ([20, pp. 516–525], [31])).

Theorem 23. *We have the identity*

$$d_n(q) = \begin{vmatrix} [0]_q & -q & & & & \\ [1]_q & [1]_q & -q^2 & & & \\ & [2]_q & [2]_q & -q^3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & [n-2]_q & [n-2]_q & -q^{n-1} \\ & & & & [n-1]_q & [n-1]_q \end{vmatrix}_{n\times n}. \tag{65}$$

Proof. The tridiagonal determinants in formula (65) satisfy recurrence (6) with the appropriate initial values. This implies at once the claimed identity. □

The q-derangement numbers can also be represented in terms of *Hessenberg determinants* [31, p. 90], as follows.

Theorem 24. *Consider the $n \times n$ lower Hessenberg matrix*

$$A_n(q) = \begin{bmatrix} a_{00}(q) & -1 & 0 & 0 & \cdots & 0 \\ a_{10}(q) & a_{11}(q) & -1 & 0 & \cdots & 0 \\ a_{20}(q) & a_{21}(q) & a_{22}(q) & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1}(q) & a_{n-2,2}(q) & a_{n-2,3}(q) & a_{n-2,4}(q) & \cdots & -1 \\ a_{n-1,1}(q) & a_{n-1,2}(q) & a_{n-1,3}(q) & a_{n-1,4}(q) & \cdots & a_{n-1,n-1}(q) \end{bmatrix}$$

where

$$a_{i,j}(q) = \begin{cases} \binom{i}{j}_q a_{i-j}(q), & \text{if } i \geq j; \\ -1, & \text{if } i = j-1; \\ 0, & \text{otherwise} \end{cases}$$

where

$$a_k(q) = \frac{q}{2q-1}\left(q^n - (1-q)^n\right)[n]_q!\,. \tag{66}$$

Then we have the identity

$$d_n(q) = \det A_n(q)\,. \tag{67}$$

Proof. Let $b_n(q) = \det A_n(q)$. By expanding the determinant along the last column, we get the recurrence

$$b_{n+1}(q) = \sum_{k=0}^{n} \binom{n}{k}_q a_k(q) b_{n-k}(q)$$

with initial value $b_0(q) = 1$. Hence, considering the q-exponential generating series

$$a(q;t) = \sum_{n\geq 0} a_n(q) \frac{t^n}{[n]_q!} \qquad \text{and} \qquad b(q;t) = \sum_{n\geq 0} b_n(q) \frac{t^n}{[n]_q!}\,,$$

we have the q-differential equation

$$\mathfrak{D}_q b(q;t) = a(q;t)\, b(q;t)\,.$$

If $b(q;t) = D_q(t)$, then $b_0(q) = d_0(q) = 1$, as requested, and

$$a(q;t) = \frac{\mathfrak{D}_q b(q;t)}{b(q;t)} = \frac{\mathfrak{D}_q D_q(t)}{D_q(t)}\,.$$

By series (43) and relation (38), we have

$$\begin{aligned}
\mathfrak{D}_q D_q(t) &= \frac{D_q(qt) - D_q(t)}{(q-1)t} \\
&= \frac{1}{(q-1)t} \left(\frac{E_q(qt)^{-1}}{1-qt} - \frac{E_q(t)^{-1}}{1-t} \right) \\
&= \frac{1}{(q-1)t} \left(\frac{E_q(t)^{-1}}{(1-qt)(1+(q-1)t)} - \frac{E_q(t)^{-1}}{1-t} \right) \\
&= \frac{qt\, E_q(t)^{-1}}{(1-t)(1-qt)(1+(q-1)t)},
\end{aligned}$$

which is

$$\mathfrak{D}_q D_q(t) = \frac{qt}{(1-qt)(1+(q-1)t)}\, D_q(t)\,.$$

Therefore, we have

$$a(q;t) = \frac{qt}{(1-qt)(1+(q-1)t)} = \frac{q}{2q-1}\,\frac{1}{1-qt} - \frac{q}{2q-1}\,\frac{1}{1-(1-q)t}\,.$$

This decomposition yields identity (66), and, consequently, this proves identity (67). $\square$

7 Final remarks

In the literature, there are also other q-analogues for the derangement numbers. For instance, we have the q-derangement numbers [11]

$$D_n(q) = \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q!\,(-1)^k \tag{68}$$

satisfying the recurrences

$$D_{n+1}(q) = [n+1]_q\, D_n(q) + (-1)^{n+1} \tag{69}$$
$$D_{n+2}(q) = q\,[n+1]_q\, D_{n+1}(q) + [n+1]_q\, D_n(q) \tag{70}$$

with initial conditions $D_0(q) = 1$ and $D_1(q) = 0$. Moreover, they have q-exponential generating series

$$\sum_{n\geq 0} D_n(q) \frac{t^n}{[n]_q!} = \frac{E_q(-t)}{1-t}\,.$$

The q-numbers $d_n(q)$ and $D_n(q)$ are not independent, as shown in the next theorem.

Theorem 25. *For every $n \in \mathbb{N}$, we have the relation*

$$d_n(q^{-1}) = q^{-\binom{n}{2}} D_n(q)\,. \tag{71}$$

Moreover, we have the formulas

$$d_n(q) = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \frac{(q^{n-k+1};q)_k}{(1-q)^k} \tag{72}$$
$$D_n(q) = \sum_{k=0}^{n} (-1)^{n-k} \frac{(q^{n-k+1};q)_k}{(1-q)^k}\,. \tag{73}$$

Proof. By formula (4) and relations (8), we have

$$\begin{aligned} d_n(q^{-1}) &= \sum_{k=0}^{n} \binom{n}{k}_{q^{-1}} [n-k]_{q^{-1}}!\,(-1)^k q^{-\binom{k}{2}} \\ &= \sum_{k=0}^{n} \binom{n}{k}_q q^{-k(n-k)} [n-k]_q! q^{-\binom{n-k}{2}}\,(-1)^k q^{-\binom{k}{2}} \\ &= q^{-\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k}_q [n-k]_q!\,(-1)^k\,. \end{aligned}$$

By formula (68), this is relation (71).

Since $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$, from definition (4) we have

$$d_n(q) = \sum_{k=0}^{n} \frac{[n]_q!}{[k]_q!} (-1)^k q^{\binom{k}{2}} = \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{(1-q)^{n-k}} \frac{(q;q)_n}{(q;q)_k} = \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{(1-q)^{n-k}} (q^{k+1};q)_{n-k}\,.$$

This is equivalent to identity (72). Similarly, formula (68) can be rewritten as formula (73). $\square$

We also have the following result.

Theorem 26. *We have the ordinary generating series*

$$d(q;t) = \sum_{n\geq 0} d_n(q)\, t^n = \sum_{k\geq 0} \frac{(-1)^k q^{\binom{k+1}{2}}}{(1-q)^k} \frac{t^k Q(q;q^k t)}{\left(\frac{t}{1-q};q\right)_{k+1}} \tag{74}$$

$$D(q;t) = \sum_{n\geq 0} D_n(q)\, t^n = \sum_{k\geq 0} \frac{(-1)^k q^{\binom{k+1}{2}}}{(1-q)^k} \frac{t^k}{(1+q^k t)\left(\frac{t}{1-q};q\right)_{k+1}} \tag{75}$$

where

$$Q(q;t) = \sum_{n\geq 0} (-1)^n q^{\binom{n}{2}}\, t^n\,.$$

Proof. From recurrence (5), we have

$$\sum_{n\geq 0} d_{n+1}(q)\, t^n = \sum_{n\geq 0} \frac{1-q^{n+1}}{1-q}\, d_n(q)\, t^n + \sum_{n\geq 0} (-1)^{n+1} q^{\binom{n+1}{2}}\, t^n,$$

which is

$$\frac{d(q;t)-d_0(q)}{t} = \frac{1}{1-q}(d(q;t)-qd(q;qt)) + \frac{Q(q;t)-1}{t},$$

or

$$d(q;t) - 1 = \frac{t}{1-q} d(q;t) - \frac{qt}{1-q}\, d(q;qt) + Q(q;t) - 1,$$

or

$$\left(1 - \frac{t}{1-q}\right) d(q;t) = Q(q;t) - \frac{qt}{1-q}\, d(q;qt),$$

which is

$$d(q;t) = \frac{Q(q;t)}{1-\frac{t}{1-q}} - \frac{qt}{(1-q)\left(1-\frac{t}{1-q}\right)}\, d(q;qt)\,.$$

By repeatedly applying this formula, we get

$$\begin{aligned} d(q;t) &= \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}}}{(1-q)^k} \frac{t^k\, Q(q;q^k t)}{\left(1-\frac{t}{1-q}\right)\left(1-\frac{qt}{1-q}\right)\cdots\left(1-\frac{q^k t}{1-q}\right)} \\ &\qquad + \frac{(-1)^{n+1} q^{\binom{n+2}{2}}}{(1-q)^{n+1}} \frac{t^{n+1}}{\left(1-\frac{t}{1-q}\right)\left(1-\frac{qt}{1-q}\right)\cdots\left(1-\frac{q^n t}{1-q}\right)}\, d(q;q^{n+1}t) \\ &= \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}}}{(1-q)^k} \frac{t^k\, Q(q;q^k t)}{\left(\frac{t}{1-q};q\right)_{k+1}} + \frac{(-1)^{n+1} q^{\binom{n+2}{2}}}{(1-q)^{n+1}} \frac{t^{n+1}}{\left(\frac{t}{1-q};q\right)_{n+1}}\, d(q;q^{n+1}t)\,. \end{aligned}$$

Taking the limit for n tending to $+\infty$, we obtain series (74).

Similarly, from recurrence (69), we have

$$\sum_{n\geq 0} D_{n+1}(q)\, t^n = \sum_{n\geq 0} \frac{1-q^{n+1}}{1-q}\, D_n(q)\, t^n + \sum_{n\geq 0} (-1)^{n+1} t^n,$$

which is

$$\frac{D(q;t)-D_0(q)}{t} = \frac{1}{1-q}(D(q;t)-qD(q;qt)) - \frac{1}{1+t},$$

or

$$\left(1-\frac{t}{1-q}\right) D(q;t) = \frac{1}{1+t} - \frac{qt}{1-q} D(q;qt),$$

or

$$D(q;t) = \frac{1}{(1+t)\big(1-\frac{t}{1-q}\big)} - \frac{qt}{(1-q)\big(1-\frac{t}{1-q}\big)} D(q;qt).$$

By repeatedly applying this formula, we get

$$\begin{aligned} D(q;t) &= \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}}}{(1-q)^k} \frac{t^k}{(1+q^k t)\big(1-\frac{t}{1-q}\big)(1-\frac{qt}{1-q}) \cdots (1-\frac{q^k t}{1-q})} \\ &\qquad + \frac{(-1)^{n+1} q^{\binom{n+2}{2}}}{(1-q)^{n+1}} \frac{t^{n+1}}{\big(1-\frac{t}{1-q}\big)(1-\frac{qt}{1-q}) \cdots (1-\frac{q^n t}{1-q})} D(q;q^{n+1}t) \\ &= \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}}}{(1-q)^k} \frac{t^k}{(1+q^k t)\big(\frac{t}{1-q};q\big)_{k+1}} \\ &\qquad + \frac{(-1)^{n+1} q^{\binom{n+2}{2}}}{(1-q)^{n+1}} \frac{t^{n+1}}{\big(\frac{t}{1-q};q\big)_{n+1}} D(q;q^{n+1}t)\,. \end{aligned}$$

Taking the limit for n tending to $+\infty$, we obtain series (75). □

References

[1] W. A. Al-Salam and L. Carlitz, A q-analog of a formula of Toscano, *Boll. Un. Mat. Ital.* (3) **12** (1957), 414–417.

[2] W. A. Al-Salam and L. Carlitz, Some orthogonal q-polynomials, *Math. Nachr.* **30** (1965), 47-61.

[3] S. Capparelli, M. M. Ferrari, E. Munarini, and N. Zagaglia Salvi, A generalization of the "*Problème des Rencontres*", *J. Integer Sequences* **21** (2018), Article 18.2.8.

[4] L. Carlitz, Some polynomials related to the theta functions, *Ann. Mat. Pura Appl.* **41** (1955), 359–373.

[5] W. Y. C. Chen and G.-C. Rota, q-analogs of the inclusion-exclusion principle and permutations with restricted position, *Discrete Math.* **104** (1992), 7–22.

[6] W. Y. C. Chen, H. L. Saad, and L. H. Sun, An operator approach to the Al-Salam–Carlitz polynomials, *J. Math. Phys.* **51** (2010), Paper 043502.

[7] W. Y. C. Chen and E. X. W. Xia, The ratio monotonicity of the q-derangement numbers, *Discrete Math.* **311** (2011), 393–397.

[8] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978.

[9] O. M. D'Antona and E. Munarini, A combinatorial interpretation of the connection constants for persistent sequences of polynomials, *European J. Combin.* **26** (2005), 1105–1118.

[10] M. M. Ferrari and E. Munarini, Decomposition of some Hankel matrices generated by the generalized rencontres polynomials, *Linear Algebra Appl.* **567** (2019), 180-–201.

[11] A. M. Garsia and J. Remmel, A combinatorial interpretation of q-derangement and q-Laguerre numbers, *Europ. J. Combin.* **1** (1980), 47–5.

[12] C. F. Gauss, Summatio quarumdam serierum singularium, *Comm. soc. reg. sci. Gottingensis rec.* **1** (1808), 147–186. Reprinted in Gauss's *Werke* **2** (1863), 9–45.

[13] J. Goldman and G.-C. Rota, The number of subspaces of a vector space, *Recent Progress in Combinatorics* (Proc. Third Waterloo Conf. on Combinatorics, 1968) Academic Press, 1969, pp. 75–83.

[14] J. Goldman and G.-C. Rota, On the foundations of combinatorial theory. IV. Finite vector spaces and Eulerian generating functions, *Studies in Appl. Math.* **49** (1970), 239–258.

[15] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, 2005.

[16] F. H. Jackson, A basic-sine and cosine with symbolical solutions of certain differential equations, *Proc. Edinburgh Math. Soc.* **22** (1904), 28–39.

[17] F. H. Jackson, On q-functions and a certain difference operator, *Trans. Roy. Soc. Edin.* **46** (1908), 253–281.

[18] F. H. Jackson, q-Difference equations, *Am. J. Math.* **32** (1910), 305–314.

[19] D. Kim, On combinatorics of Al-Salam Carlitz polynomials, *European J. Combin.* **18** (1997), 295–302.

[20] T. Muir, *The Theory of Determinants in the Historical Order of Developement*, Dover, 1960.

[21] E. Munarini, Shifting property for Riordan, Sheffer and connection constants matrices, *J. Integer Sequencesf* **20** (2017), Article 17.8.2.

[22] E. Munarini, Some q-binomial identities involving the generalized q-Fibonacci numbers, *Integers* **18** (2018), Paper No. A76.

[23] E. Munarini, Combinatorial identities for Appell polynomials, *Appl. Anal. Discrete Math.* **12** (2018), 362–388.

[24] E. Munarini, Callan-like identities, *Online J. Anal. Comb.* **14** (2019), Article #06.

[25] E. Munarini, An operational approach to the generalized rencontres polynomials, *Kragujevac J. Math.* **46** (2022), 461–469.

[26] A. Nijenhuis, A. E. Solow, and H. S. Wilf, Bijective methods in the theory of finite vector spaces, *J. Combin. Theory Ser. A* **37** (1984), 80–84.

[27] S. Roman, *The Umbral Calculus*, Academic Press, 1984.

[28] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, 2020. Available at `https://oeis.org/`.

[29] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloquium Publication **23**, Amer. Math. Soc., 1975.

[30] M. R. Ubriaco, Time evolution in quantum mechanics on the quantum line, *Phys. Lett. A* **163** (1992), 1–4.

[31] R. Vein and P. Dale, *Determinants and Their Applications in Mathematical Physics*, Springer-Verlag, 1999.

[32] M. L. Wachs, On q-derangement numbers, *Proc. Amer. Math. Soc.* **106** (1989), 273–278.

[33] X.-D. Zhang, On the spiral property of the q-derangement numbers, *Discrete Math.* **159** (1996), 295–298.

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