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# Raabe's Identity and Covering Systems

Yevgenya Movshovich
University of Illinois at Urbana-Champaign
Urbana, IL 61801
USA
yevgenya@illinois.edu

**Abstract**

A connection between covering systems and Bernoulli polynomials established by Fraenkel, Beebee, and Porubský asserts that a function $\sum_{i=1}^{q} \mu_i \chi_{A_i}(k)$ is an $M_A$-covering function for a system of arithmetic sequences $\{A_i\}_{i=1}^{q}$ if and only if it satisfies a generalized Raabe identity. The connection is used to derive recurrence formulas for the Bernoulli numbers. Here we show that the generalized Raabe identity is a sum of Raabe's identities for $A_i$ multiplied by $\mu_i$, and this sum is the direct source of the above recurrence formulas. We show that many of these formulas are special cases of the original multiplication formula of Raabe. We find new applications of the connection to the covering systems.

## 1 Introduction

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we let $a(n) := a + n\mathbb{Z} = \{x \in \mathbb{Z} : x \equiv a \pmod{n}\}$. Here $a(n)$ is called an arithmetic sequence (with *common difference* $n$) or a residue class (with *modulus* $n$). Consider a system $A = \{A_1, A_2, \ldots, A_q\}$ of arithmetic sequences

$$A_i = a_i(n_i) = a_i + n_i\mathbb{Z}, \quad 0 \le a_i < n_i, \quad i = 1, \ldots, q, \quad q \ge 1. \tag{1}$$

**Definition 1.** For $i = 1, \ldots, q$, we let $\chi_{A_i}(k)$ denote the characteristic function of $A_i$ and we let $\mu_i \ne 0$ be real numbers. An $M_A$-*covering function* for a system (1) is a sum

$$M_A(k) = \sum_{i=1}^{q} \mu_i \chi_{A_i}(k), \tag{2}$$

If all $\mu_i = 1$, then $M_A(k)$ is the usual *covering function*: $w_A(k) = \sum_{i=1}^{q} \chi_{A_i}(k)$.

Observe that the function $M_A(k)$ is $N_A$-periodic with $N_A = \operatorname{lcm}(n_1, \dots, n_q)$.

**Definition 2.** A system (1) is called a *covering system* or a *cover* (of $\mathbb{Z}$) if $w_A \geq 1$.

Paul Erdős introduced the concept of the covering system in the 1930s (see, e.g., [3]) and gave a nontrivial example of such a system: $0(2), 0(3), 1(4), 3(8), 7(12), 23(24)$.

**Definition 3.** A system (1) is an *exact $M$-cover* if $w_A(k) = M$ and an *exact cover* if $M = 1$. We also say that $A$ is an exact $M$-cover of $B$ if $w_A(k) = M w_B(k)$.

In what follows, $\big(B_m\big)_{m\geq 0} = \big(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots\big)$ are the Bernoulli numbers and $B_m(t)$ are the Bernoulli polynomials (8). In 1973 Fraenkel [6] found a connection between exact covers and $B_m(t)$. Two years later Porubský [12] extended it to exact $M$-covers.

**Theorem 4** (A. Fraenkel, $M = 1$)**.** *A system of $q \geq 2$ arithmetic sequences $\{a_i(n_i)\}_{i=1}^{q}$ is an exact $M$-cover if and only if*

$$M B_m = \sum_{i=1}^{q} n_i^{m-1} B_m\Big(\frac{a_i}{n_i}\Big) \tag{3}$$

*holds for every $m \geq 0$.*

Note that for $m = 0, 1$ the identity (3) gives us two well known properties of exact $M$-covers: $\sum_{i=1}^{q} \frac{1}{n_i} = M$ (see also Corollary 8) and $\sum_{i=1}^{q} \frac{a_i}{n_i} = \frac{q-M}{2}$.

Beebee [1] observed that (3) is the case $x = 0$ of a more general identity (4).

**Theorem 5** (J. Beebee, $M = 1$)**.** *A system of $q \geq 2$ arithmetic sequences $\{a_i(n_i)\}_{i=1}^{q}$ is an exact $M$-cover if and only if*

$$M B_m(x) = \sum_{i=1}^{q} n_i^{m-1} B_m\Big(\frac{x + a_i}{n_i}\Big) \tag{4}$$

*holds for every $m \geq 0$.*

Porubský [10, 12] used Beebee's approach to extend Theorems 4 and 5 to a system (1) with an $M_A$-covering function (2) (and restrictions $q \geq 2$, $N = N_A$, which we omit).

**Theorem 6** (Š. Porubský [10, Th. 1])**.** *Let $q, S \in \mathbb{Z}^+$ and $N = S N_A$. Then (2) is an $M_A(k)$-covering function for the system (1) if and only if*

$$N^{m-1} \sum_{k=0}^{N-1} M_A(k) B_m\Big(\frac{x + k}{N}\Big) = \sum_{i=1}^{q} \mu_i n_i^{m-1} B_m\Big(\frac{x + a_i}{n_i}\Big) \tag{5}$$

*holds for every $m \geq 0$.*

We prove that, by itself, identity (5) is a sum of Raabe identities for arithmetics sequences $a_i(n_i)$ (see Theorem 9 and Example 18). Aside from this result, our other contribution to Theorem 6 is proving its corollary, Theorem 10, where we take advantage of $N = SN_A$ rather than $N = N_A$. Theorem 10 generalizes the results of Beebee and Fraenkel and is useful in applications to covering systems. In particular, it can be used to check if a system is an exact $M$-cover of another system (see Examples 20 and 21). In Section 4 we present Porubský's proof of Theorem 6. In Theorem 16 of that section we derive identity (5) again, this time, from the asymptotic expansion of the Hurwitz zeta function. In Section 5 we give a few examples.

We begin the article by showing that many known identities for Bernoulli numbers, including those of Stern and Namias, are special cases of the multiplication formula proved by Raabe in his 1848 work "The Jacob Bernoulli function" [13].

## 2 Multiplication formula of J. L. Raabe (1848)

Raabe [13] proved his celebrated multiplication formula:

$$T^{m-1}\sum_{l=0}^{T-1} b_m\Big(\frac{t+l}{T}\Big) = b_m(t) - \frac{B_m}{m}\Big(T^m - 1\Big), \qquad m \geq 1, \tag{6}$$

for polynomials $\big(t, \frac{t^2}{2} - \frac{t}{2}, \frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{6}, \dots\big)$, called *the Bernoulli functions* in [13], [1]

$$b_m(t) = \frac{1}{m}\sum_{s=0}^{m-1} B_s \binom{m}{s} t^{m-s}. \tag{7}$$

Modern Bernoulli polynomials $B_m(t)$ are linearly related to $b_m(t)$, as follows:

$$B_m(t) = \sum_{s=0}^{m} B_s \binom{m}{s} t^{m-s} = m\, b_m(t) + B_m. \tag{8}$$

Polynomials $b_m(t)$ are solutions to Jacob Bernoulli's problem: find polynomials that sum powers of consecutive integers $\sum_{j=1}^{t-1} j^{m-1} = b_m(t)$ when $m, t > 1$ $\big($e.g., $\sum_{j=1}^{30} j^7 = b_8(31)\big)$.

To prove (6), Raabe observed that for odd $m$ the property $b_m(1-x) = (-1)^m b_m(x)$, written as a cancellation law $b_{2n+1}(1-x) + b_{2n+1}(x) = 0$, yields $b_{2n+1}(\frac{1}{2}) = 0$ and a general cancellation law, that is, multiplication formula (6) at $t = 0$ and odd $m \geq 3$:

$$b_{2n+1}(\tfrac{1}{T}) + b_{2n+1}(\tfrac{2}{T}) + \cdots + b_{2n+1}(\tfrac{T-2}{T}) + b_{2n+1}(\tfrac{T-1}{T}) = 0.$$

[1] Raabe's 1851 article [14] is often cited instead of his 1848 monograph [13] as the earliest place in the literature where the term *Bernoulli polynomial (function)* appears.

He then used impressive algebra to obtain (6) for all $t$ and $m$.

Raabe's identity (6) can be viewed as a recurrence relation for the Bernoulli numbers:

$$B_m = \frac{T^{m-1}}{1-T^m} \sum_{l=0}^{T-1} \widehat{b}_m\Big(\frac{t+l}{T}\Big) - \widehat{b}_m(t), \tag{9}$$

where $\widehat{b}_m(t) = m\, b_m(t)$, which are sometimes called *Bernoulli polynomials of an older type*. If we take $t = 0$ in the last identity, we see that it coincides with the following known identities: J. Stern [15], $T = 2$:

$$B_m = \frac{1}{2(1-2^m)} \sum_{s=0}^{m-1} B_s \binom{m}{s} 2^s \overset{(9)}{\Leftrightarrow} B_m = \frac{2^{m-1}}{1-2^m}\, \widehat{b}_m\Big(\frac{1}{2}\Big).$$

V. Namias [9], $T = 3$:

$$B_m = \frac{1}{3(1-3^m)} \sum_{s=0}^{m-1} B_s \binom{m}{s} 3^s (1+2^{m-s}) \overset{(9)}{\Leftrightarrow} B_m = \frac{3^{m-1}}{1-3^m} \sum_{l=0}^{2} \widehat{b}_m\Big(\frac{l}{3}\Big).$$

Namias [9] conjectured (and Deeba, Rodriguez [2] proved) that for arbitrary $T$

$$B_m = \frac{1}{T(1-T^m)} \sum_{s=0}^{m-1} T^s \sum_{l=1}^{T-1} B_s \binom{m}{s} l^{m-s} \overset{(9)}{\Leftrightarrow} B_m = \frac{T^{m-1}}{1-T^m} \sum_{l=0}^{T-1} \widehat{b}_m\Big(\frac{l}{T}\Big).$$

In [1, 10] these identities $\big($with our notation $\widehat{b}_m(t)$ for the sum $\sum_{s=0}^{m-1} B_s \binom{m}{s} t^{m-s}$ $\big)$ were cited as special cases of Fraenkel's identity (3). The latter is the identity (5) in the recurrence form with $t = 0$ and $M_A(k) = M$:

$$B_m\Big(M - \sum_{i=1}^{q} n_i^{m-1}\Big) = \sum_{i=1}^{q} n_i^{m-1}\, \widehat{b}_m\Big(\frac{a_i}{n_i}\Big). \tag{10}$$

# 3 Raabe's identity for arithmetic sequences

When identity (5) (with $N = T$ and $x = t$) is applied to the sequence $0(1) = \mathbb{Z}$, it becomes the multiplication formula (6) written in terms of modern Bernoulli polynomials (8):

$$T^{m-1} \sum_{l=0}^{T-1} B_m\Big(\frac{t+l}{T}\Big) = B_m(t). \tag{11}$$

A substitution $t = \frac{x+a_i}{n_i}$ in (11) yields Raabe's identity for an arithmetic sequence $a_i(n_i)$:

$$T^{m-1} \sum_{l=0}^{T-1} B_m\left(\frac{\frac{x+a_i}{n_i}+l}{T}\right) = B_m\left(\frac{x+a_i}{n_i}\right). \tag{12}$$

**Lemma 7.** *Let $A_i = a_i(n_i)$, $a_i < n_i$ and let $T \in \mathbb{Z}^+$, $N = Tn_i$, Then for any function $f$*

$$\sum_{k=0}^{N-1} \chi_{A_i}(k) f(k) = \sum_{l=0}^{T-1} f(a_i + l n_i).$$

*Proof.* The set of integers $\{0, 1, 2, \ldots, N-1\}$ is split into $T$ subsets:

$$\big\{0, 1, \ldots, n_i - 1; \quad n_i, \ldots, 2n_i - 1; \quad \ldots; \quad (T-1)n_i, \ldots, Tn_i - 1\big\}.$$

In each of these subsets there is exactly one member of $A_i$. □

**Corollary 8.** *Let (1) be a cover. Then Lemma 7 with $f = 1$ and $N = SN_A$, $S \in \mathbb{Z}^+$ yields*

$$\sum_{k=0}^{N-1} \chi_{A_i}(k) = T = \frac{N}{n_i}. \quad \textit{Thus} \quad \frac{1}{N}\sum_{k=0}^{N-1} w_A(k) = \sum_{i=1}^{q} \frac{1}{n_i} \quad \textit{and} \quad \sum_{i=1}^{q} \frac{1}{n_i} \geq 1.$$

**Theorem 9.** *Identity (5) for a system $A$ in (1) with an $M_A$-covering function (2) is the sum of $q$ identities (12) multiplied by $\mu_i$.*

*Proof.* With $f(k) = B_m\big(\frac{x+k}{N}\big)$ and $N = Tn_i$ in Lemma 7, Raabe's identity (12) for $a_i(n_i)$ admits the following form:

$$N^{m-1} \sum_{k=0}^{N-1} \chi_{A_i}(k) B_m\left(\frac{x+k}{N}\right) = n_i^{m-1} B_m\left(\frac{x+a_i}{n_i}\right). \tag{13}$$

Now if $N = SN_A$, then the sum of $q$ identities (13) multiplied by $\mu_i$ is

$$\sum_{i=1}^{q} \mu_i N^{m-1} \sum_{k=0}^{N-1} \chi_{A_i}(k) B_m\left(\frac{x+k}{N}\right) = \sum_{i=1}^{q} \mu_i n_i^{m-1} B_m\left(\frac{x+a_i}{n_i}\right), \tag{14}$$

which becomes (5) after we change the order of summation in the left side of (14). □

**Theorem 10** (Corollary to Theorem 6)**.** *A covering system $A = \{a_i(\,n_i)\}_{i=1}^{q}$ is an exact $M$-cover of a covering system $B = \{h_j(\,r_j)\}_{j=1}^{p}$ if and only if*

$$M \sum_{j=1}^{p} r_j^{m-1} B_m\left(\frac{x+h_j}{r_j}\right) = \sum_{i=1}^{q} n_i^{m-1} B_m\left(\frac{x+a_i}{n_i}\right) \tag{15}$$

*holds for every $m \geq 0$.*

*Proof.* Let $N = \operatorname{lcm}(n_1, \ldots, n_q; r_1, \ldots, r_p) = S_1 N_A = S_2 N_B$. We write (5) for systems $A$ and $B$. By the assumption, $w_A(k) = M w_B(k)$, i.e., the left side of (5) for $A$ equals that for $B$ multiplied by $M$. Hence, the right sides of (5) for $A$ and $B$ satisfy Eq. (15). □

**Theorem 11** (Special case of Theorem 10)**.** *A covering system* $A = \{a_i(\,n_i)\}_{i=1}^q$ *is an exact* $M$*-cover of an arithmetic sequence* $h(r)$ *if and only if for every* $m \geq 0$*:*

$$Mr^{m-1}B_m\left(\frac{x+h}{r}\right) = \sum_{i=1}^{q} n_i^{m-1}B_m\left(\frac{x+a_i}{n_i}\right). \tag{16}$$

*Remark* 12. Theorem 11 tells us that writing Raabe's identity (12) for $a_i(n_i)$ for all $m \geq 0$ is equivalent to exactly covering $a_i(n_i)$ by the system $\{(a_i + ln_i)(n_iT),\ l = 0, \ldots, T-1\}$.

# 4 Proof of Theorem 6

**Lemma 13.** *Let a function* $M(z)$ *be* $N$*-periodic and a function* $g(z)$ *satisfy* $\sum\limits_{z=0}^{\infty} |g(z)| < \infty$*. Then the summation in the series* $\sum M(z)\,g(z)$ *can be rearranged as follows:*

$$\sum_{z=0}^{\infty} M(z)\,g(z) = \sum_{k=0}^{N-1}\sum_{l=0}^{\infty} M(k+lN)g(k+lN) = \sum_{k=0}^{N-1} M(k)\sum_{l=0}^{\infty} g(k+lN).$$

**Corollary 14.** *Let* (2) *be an* $M_A(k)$*-covering function for the system* (1) *and let* $N = SN_A$*,* $S \in \mathbb{Z}^+$*. In addition, let* $|y| < 1$*. Then*

$$\sum_{k=0}^{N-1} M_A(k)\frac{y^k}{1-y^N} = \sum_{i=1}^{q} \mu_i \frac{y^{a_i}}{1-y^{n_i}}. \tag{17}$$

*Proof.* Because $M_A(z)$ is $N$-periodic, using geometric series expansions, the fact that $a_i < n_i$, and Lemma 13, we conclude that

$$\sum_{i=1}^{q} \mu_i \frac{y^{\,a_i}}{1-y^{n_i}} = \sum_{i=1}^{q} \mu_i \sum_{l=0}^{\infty} y^{\,a_i+ln_i} = \sum_{i=1}^{q} \mu_i \sum_{z=0}^{\infty} \chi_{A_i}(z)\,y^{\,z} = \sum_{z=0}^{\infty} M_A(z)\,y^{\,z} \tag{18}$$

$$= \sum_{k=0}^{N-1} M_A(k)\sum_{l=0}^{\infty} y^{\,k+lN} = \sum_{k=0}^{N-1} M_A(k)\frac{y^{\,k}}{1-y^N}. \qquad \square$$

$\square$

**Corollary 15** (Š. Porubský [12])**.** *Let* $N = SN_A$*,* $S \in \mathbb{Z}^+$*. Then* (2) *is an* $M_A(k)$*-covering function for the system* (1) *if and only if Eq.* (17) *holds for any* $y$ *with* $|y| < 1$*.*

We recall Euler's generating function of polynomials $B_m(t)$:

$$\frac{se^{st}}{e^s-1} = \sum_{m=0}^{\infty} \frac{B_m(t)}{m!}\, s^m. \tag{19}$$

*Proof of Theorem 6 (J. Beebee, Š. Porubský).*
(⇐): We assume that Eq. (5) holds for every $m \geq 0$. As in [1, 10], we multiply (5) by $s^m/m!$ and sum the result over $m$:

$$\sum_{k=0}^{N-1} M_A(k) \sum_{m=0}^{\infty} \frac{(sN)^m}{N\ m!} B_m\Big(\frac{x+k}{N}\Big) = \sum_{i=1}^{q} \mu_i \sum_{m=0}^{\infty} \frac{(sn_i)^m}{n_i\ m!} B_m\Big(\frac{x+a_i}{n_i}\Big).$$

By Eq. (19) this is equivalent to

$$\sum_{k=0}^{N-1} M_A(k) \frac{(sN)}{N} \frac{e^{sN(x+k)/N}}{e^{sN}-1} = \sum_{i=1}^{q} \mu_i \frac{(sn_i)}{n_i} \frac{e^{sn_i(x+a_i)/n_i}}{e^{sn_i}-1},$$

which, with $e^s = y$, becomes (17). Thus $M_A$ satisfies (2) by Corollary 15.

(⇒): Let $M_A$ satisfy Eq. (2). Starting with (17) we reverse steps in the above argument and obtain Eq. (5) for every $m \geq 0$. □

The authors of the proofs of Theorems 5 and 6 use the generating function of integers employed in (18) of Corollary 14 to Lemma 13 together with Euler's generating function (19).

Next we derive identity (5) from the asymptotic expansion of the Hurwitz zeta function using just Lemma 13.

**Theorem 16.** *Identity (5) holds for any system $A$ in (1) with an $M_A$-covering function (2).*

*Proof.* By Euler-Maclaurin summation (see, e.g., [5, Ex. 3, §467], where $L + x = a$) [2] the asymptotic series of an $L$-th tail of the Hurwitz zeta function at 2 in powers of $1/(L+x)$ is

$$\sum_{l=0}^{\infty} \frac{1}{(L+x+l)^2} \sim \sum_{k=0}^{\infty} \frac{(-1)^k B_k}{(L+x)^{k+1}}. \tag{20}$$

The asymptotic series composed with the converging series yields the series in powers of $1/L$:

$$\sum_{l=0}^{\infty} \frac{1}{(L+x+l)^2} \sim \sum_{k=0}^{\infty} \frac{(-1)^k B_k}{(L+x)^{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k B_k}{L^{k+1}} \sum_{j=0}^{\infty} \binom{k+j}{j} \frac{(-x)^j}{L^j}$$

$$\overset{k+j=m}{=} \sum_{m=0}^{\infty} \frac{(-1)^m}{L^{m+1}} \sum_{k=0}^{m} B_k \binom{m}{m-k} x^{m-k} \overset{(8)}{=} \sum_{m=0}^{\infty} \frac{(-1)^m}{L^{m+1}} B_m(x). \tag{21}$$

[2] Fichtenholtz [5] uses the notation $\beta_n$ for the Bernoulli numbers and $B_n$ for the absolute values of nonzero Bernoulli numbers. More precisely, $(\beta_n)_{n\geq 1} = \big(-\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots\big)$ and $(B_n)_{n\geq 1} = \big(\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \dots\big)$.

Let $N = SN_A,\ S \in \mathbb{Z}^+$. Lemma 13 and Eq. (21) with $L$ replaced by $L/N$ and $x$ by $(x+k)/N$ give us

$$\sum_{z=0}^{\infty} \frac{M_A(z)}{(L+x+z)^2} = \sum_{k=0}^{N-1} M_A(k) \sum_{l=0}^{\infty} \frac{1}{(L+x+k+lN)^2} \tag{22}$$

$$= \sum_{k=0}^{N-1} \frac{M_A(k)}{N^2} \sum_{l=0}^{\infty} \frac{1}{(\frac{L}{N} + \frac{x+k}{N} + l)^2} \sim \sum_{m=0}^{\infty} \frac{(-1)^m}{L^{m+1}} \sum_{k=0}^{N-1} M_A(k) N^{m-1} B_m\Big(\frac{x+k}{N}\Big).$$

Similarly, Eq. (21) with $L$ replaced by $L/n_i$ and $x$ by $(x+a_i)/n_i$ gives us

$$\sum_{z=0}^{\infty} \frac{\sum_{i=1}^{q} \mu_i \chi_{A_i}(z)}{(L+x+z)^2} = \sum_{i=1}^{q} \mu_i \sum_{l=0}^{\infty} \frac{1}{(L+x+a_i+ln_i)^2} \tag{23}$$

$$= \sum_{i=1}^{q} \frac{\mu_i}{n_i^2} \sum_{l=0}^{\infty} \frac{1}{(\frac{L}{n^i} + \frac{x+a_i}{n_i} + l)^2} \sim \sum_{m=0}^{\infty} \frac{(-1)^m}{L^{m+1}} \sum_{i=1}^{q} \mu_i n_i^{m-1} B_m \left( \frac{x+a_i}{n_i} \right).$$

Now Eqs. (22) and (23) yield Eq. (5), because the asymptotic series is unique. □

We note that, unlike Theorem 6, steps in the last theorem cannot be reversed because functions can share asymptotic series.

We conclude the section with the famous proof of a relation for moduli of the exact covers found by Mirsky and Newman, and independently, by Davenport and Rado (see, e.g., [4]).

**Theorem 17** (H. Davenport, L. Mirsky, D. Newman, R. Rado)**.** *Let a system* (1) *be an exact M-cover and let its moduli* $n_i$ *satisfy* $1 < n_1 \le n_2 \le \cdots \le n_{q-1} \le n_q$. *Then* $n_{q-1} = n_q$. [3]

*Proof.* In Eq. (18) of Corollary 14 we take $\mu_i = 1,\ i = 1, \ldots, q$ and $M_A(z) = M$ to have

$$\frac{M}{1-y} = \sum_{i=1}^{q-1} \frac{y^{a_i}}{1-y^{n_i}} + \frac{y^{a_q}}{1-y^{n_q}}. \tag{24}$$

The assumption that $n_{q-1} < n_q$ would contradict Eq. (24) when $y \to e^{2\pi i/n_q}$:

$$\lim_{y \to e^{2\pi i/n_q}} \left( \frac{M}{1-y} - \sum_{i=1}^{q-1} \frac{y^{a_i}}{1-y^{n_i}} \right) \ne \lim_{y \to e^{2\pi i/n_q}} \frac{y^{a_q}}{1-y^{n_q}} = \infty.$$

□

[3] Further results for covering systems in this direction can be found in [7, 16] and in "Covering systems and periodic arithmetical functions" (a talk given by Z. W. Sun at UIUC on April 13, 2006).

## 5 Examples

Identity (5) in the recurrence form with $t = 0$ and $M_A(k) = w_A(k)$ was given in [10, Cor. 1]:

$$B_m \sum_{i=1}^{q} \left( \frac{N^m}{n_i} - n_i^{m-1} \right) = \sum_{i=1}^{q} n_i^{m-1} \widehat{b}_m\left(\frac{a_i}{n_i}\right) - N^{m-1} \sum_{k=0}^{N-1} w_A(k)\, \widehat{b}_m\left(\frac{k}{N}\right). \tag{25}$$

The term $\sum_{i=1}^{q} \frac{N}{n_i}$ in the left side of Eq. (25) replaced $\sum_{k=0}^{N-1} w_A(k)$ (see Corollary 8).

**Example 18.** Identity (25) applied to the system $A = \{0(2), 0(3), 1(6), 5(6)\}$ with $N = 6$ is

$$B_m\Big(2^{m-1} + 3^{m-1} - 5 \cdot 6^{m-1}\Big) = 6^{m-1} \left( \widehat{b}_m\left(\tfrac{2}{6}\right) + \widehat{b}_m\left(\tfrac{3}{6}\right) + \widehat{b}_m\left(\tfrac{4}{6}\right) \right). \qquad \text{[10, Cor. 3]}$$

Since $\widehat{b}_m(t) = B_m(t) - B_m$, the above identity is equivalent to

$$B_m\Big(2^{m-1} + 3^{m-1} - 2 \cdot 6^{m-1}\Big) = 6^{m-1}\Big(B_m\left(\tfrac{2}{6}\right) + B_m\left(\tfrac{3}{6}\right) + B_m\left(\tfrac{4}{6}\right)\Big). \tag{26}$$

The latter is the sum of Raabe identities (12) for $0(2)$ with $T = \frac{6}{2}$ and for $0(3)$ with $T = \frac{6}{3}$:

$$6^{m-1}\Big(B_m + B_m\left(\tfrac{2}{6}\right) + B_m\left(\tfrac{4}{6}\right)\Big) = 2^{m-1} B_m \quad \text{and} \quad 6^{m-1}\Big(B_m + B_m\left(\tfrac{3}{6}\right)\Big) = 3^{m-1} B_m.$$

Note that in Theorem 9 identity (26) for $A$ is a sum (14) of 4 identities (12), but identity (12) applied to the arithmetic sequence $1(6)$ or $5(6)$ yields just $B_m = B_m$.

**Example 19.** Nevertheless, we could apply Eq. (25) in the original form (5) to a system $A'$ related to $A$, for example, to find $B_m\big(1/6\big)$ without knowledge of $B_m(1/2)$ and $B_m(1/3)$:

$$B_m\left(\tfrac{1}{6}\right) = B_m(1 - 2^{m-1})(1 - 3^{m-1})/(6^{m-1}\, 2). \tag{27}$$

According to Theorem 11, writing identities (26) is equivalent to covering $0(3)$ by $\{0(6),\ 3(6)\}$ and $0(2)$ by $\{0(6),\ 2(6),\ 4(6)\}$. Clearly systems $A' = \{0(6),\ 2(6), 4(6), 0(6), 3(6), 1(6), 5(6)\}$ and $A$ have the same covering function: $w_A(k) = 2$ if $k \equiv 0 \pmod 6$ and $w_A(k) = 1$ otherwise. We apply Eq. (5) to $A'$ with $M_{A'} = w_A$, $N = N_A = 6$ and $t = 0$ to get

$$6^{m-1} \sum_{k=0}^{5} B_m\Big(\frac{k}{6}\Big) + 6^{m-1} B_m(0) = 2^{m-1} B_m(0) + 3^{m-1} B_m(0) + 6^{m-1}\Big(B_m\Big(\frac{1}{6}\Big) + B_m\Big(\frac{5}{6}\Big)\Big)$$

and use $B_m\left(\frac{5}{6}\right) = (-1)^m B_m\left(\frac{1}{6}\right)$ and Raabe's identity $6^{m-1} \sum\limits_{k=0}^{5} B_m\left(\frac{k}{6}\right) = B_m$ to prove Eq. (27).

The next two examples show how their exact covers were constructed.

**Example 20.** Choi constructed an exact 2-cover that is not the union of two exact covers: $1(2), 0(3), 1(3), 2(6), 0(10), 4(10), 6(10), 8(10), 2(15), 5(30), 11(30), 12(30), 22(30), 23(30), 29(30)$ (see, e.g., [10, p. 156]). To confirm that this system is an exact 2-cover, we split its sequences $0(3), 1(3), 2(15)$ into odd and even terms and show that the modified cover is the union of an exact 2-cover $B \cup C$ of the sequence $1(2)$ and an exact 2-cover $D \cup E$ of the sequence $0(2)$:

$$B = 1(2); \qquad C = \big\{1(6), 3(6), 5(30), 11(30), 17(30), 23(30), 29(30)\big\};$$
$$D = \big\{0(6), 2(6), 4(6)\big\}; \qquad E = \big\{0(10), 4(10), 6(10), 8(10), 2(30), 12(30), 22(30)\big\}.$$

Since $B$ and $D$ are already exact covers of $1(2)$ and $0(2)$ respectively, we use Theorem 11 with $M = 1,\ x = 0$ to show that $C$ and $E$ are such covers.

First, in Theorem 11 we let the sequence $h(r)$ be $1(2)$ and the system $A$ be $C$. In that case, the right side of (16) in Theorem 11 is

$$6^{m-1}B_m\big(\frac{1}{6}\big) + 6^{m-1}B_m\big(\frac{3}{6}\big) \ + \ 6^{m-1}5^{m-1}\sum_{l=0}^{5-1} B_m\big(\frac{5+6l}{30}\big)$$
$$= \ 6^{m-1}B_m\big(\frac{1}{6}\big) + 6^{m-1}B_m\big(\frac{3}{6}\big) + 6^{m-1}B_m\big(\frac{5}{6}\big) \ = \ 2^{m-1}B_m\big(\frac{1}{2}\big),$$

which is the left side of (16). We have applied Raabe's identity (12) with $T = 5,\ x = \frac{5}{6}$ to the sum in the first row and with $T = 3,\ x = \frac{1}{2}$ in the second row.

Now, similarly, in Theorem 11 we take $h(r) = 0(2),\ A = E$ and with the help of (12) confirm (16):

$$10^{m-1}\sum_{l=0,2,3,4} B_m\big(\frac{2l}{10}\big) + 10^{m-1}3^{m-1}\sum_{l=0}^{3-1} B_m\big(\frac{2+10l}{30}\big) = 10^{m-1}\sum_{l=0}^{5-1} B_m\big(\frac{l}{5}\big) = 2^{m-1}B_m\big(\frac{0}{2}\big).$$

**Example 21.** Show that a system $F \cup G = \{2^{i-1}(2^i)\}_{i=1}^{f} \cup \{(j-1)2^f(n2^f)\}_{j=1}^{n}$ is an exact cover. First, we use (11) with $t = 0$ and $T = n$ to see that $G$ is an exact cover of $0(2^f)$:

$$\sum_{i=1}^{f}(2^i)^{m-1}B_m\big(\frac{1}{2}\big) + (2^f)^{m-1}n^{m-1}\sum_{l=0}^{n-1} B_m\big(\frac{l}{n}\big) \tag{28}$$
$$= \sum_{i=1}^{f}(2^i)^{m-1}B_m\big(\frac{1}{2}\big) + (2^f)^{m-1}B_m(0). \tag{29}$$

Then we apply Theorem 11 repeatedly using (11) with $t = 0$ and $T = 2$ to see that $2^{f-1}(2^f) \cup 0(2^f)$ is an exact cover of $0(2^{f-1})$, etc.:

$$(29) = \sum_{i=1}^{f-1}(2^i)^{m-1}B_m\big(\frac{1}{2}\big) + (2^{f-1})^{m-1}2^{m-1}\Big(B_m\big(\frac{0}{2}\big) + B_m\big(\frac{1}{2}\big)\Big)$$
$$= \sum_{i=1}^{f-1}(2^i)^{m-1}B_m\big(\frac{1}{2}\big) + (2^{f-1})^{m-1}B_m\big(\frac{0}{2}\big) = \cdots = B_m\big(\frac{0}{2}\big) = B_m.$$

Note that (29) $=B_m$ is $B_m\big(\frac{1}{2}\big)\sum_{i=1}^{f}(2^i)^{m-1} = B_m\big(1-(2^f)^{m-1}\big)$ which computes $B_m\big(\frac{1}{2}\big)$. Now (28) $=B_m$ with $\widehat{B_m(t)} = b_m(t)+B_m$ is the relation given in [10, Cor. 4]. The latter was (10) applied to the system $F\cup G$ and multiplied by $2^{f+1}(2^{m-1}-1)$.

When the covering function $w_A$ of a system $A$ is unknown or complicated, the identity (5) can still compute and estimate the averages of $w_A(k), kw_A(k), \ldots$ over $N_A$.

**Example 22.** For an odd $n\geq 3$, the system $F\cup H = \{2^{i-1}(2^i)\}_{i=1}^{n-1} \cup \{j2^{n-1}(n2^{j-1})\}_{j=1}^{n}$ [16, Ex. 1, p. 4314] is an example of a *distinct cover* of $\mathbb{Z}$ (all $n_i$ are different), [4] thus by the result of Davenport-Mirsky-Newman-Rado it is not an exact cover. To show that it covers $\mathbb{Z}$, we take $f=n-1$ and observe that $0(n2^f)=n2^f(n2^f)$ and, if $0\leq j\leq f$, the sequence $j2^f(n2^{j-1})$ covers the sequence $j2^f(n2^f)$. Hence, system $H=\{j2^{n-1}(n2^{j-1})\}_{j=1}^{n}$ covers system $G=\{(j-1)2^f(n2^f)\}_{j=1}^{n}$ of Example 21, while system $F$ is that of Example 21.

As in [16], Corollary 8 computes the arithmetic average of $w_{F\cup H}(k)$ with $N=N_{F\cup H}=2^{n-1}n$:

$$\frac{1}{N}\sum_{k=0}^{N-1} w_{F\cup H}(k) = \sum_{i=1}^{n-1}\frac{1}{2^i}+\sum_{j=1}^{n}\frac{1}{n2^{j-1}} = 1+\frac{2^n-n-1}{2^{n-1}\,n} < 1+\frac{2}{n}. \tag{30}$$

It shows that for large $n$ the system $F\cup H$ covers integers with fewer overlaps.

Note that the system $H=\{h_j(r_j)\}_{j=1}^{n}=\{j2^{n-1}(n2^{j-1})\}_{j=1}^{n}$ is not in the form of (1). In order to apply (5) with $m\geq 1$ to the system $F\cup H$, we define $h_j' = h_j - k_j r_j$ with $k_j$ so that $0\leq h_j' < r_j$, $j=1,\ldots,n$ and write $H=\{h_j'(r_j)\}_{j=1}^{n}$, where $h_n'=0$.

Identity (5) at $m=1$ computes the arithmetic average of $kw_A(k)$ for any system (1):

$$\frac{1}{N}\sum_{k=0}^{N-1} kw_A(k) = \frac{N}{2}\sum_{i=1}^{q}\frac{1}{n_i}+\sum_{i=1}^{q}\frac{a_i}{n_i}-\frac{q}{2}. \tag{31}$$

Now (31) and (30) estimate the arithmetic average of $kw_{F\cup H}(k)$ for our system:

$$\frac{1}{N}\sum_{k=0}^{N-1} k\,w_{F\cup H}(k) = \frac{N}{2}\sum_{i=1}^{q}\frac{1}{n_i}+\sum_{i=1}^{n-1}\frac{1}{2}+\sum_{j=1}^{n}\frac{h_j'}{r_j}-\frac{2n-1}{2} < \frac{N-1}{2}+\frac{N}{n}.$$

For an exact cover this average would be $\frac{N-1}{2}$.

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[4] **Erdős-Selfridge Conjecture** (See e.g., [8]). *There are no distinct covers with all $n_1,\ldots,n_q$ odd and greater than one.*

## References

[1] J. Beebee, Bernoulli numbers and exact covering systems, *Amer. Math. Monthly* **99** (1992), 946–948.

[2] E. Y. Deeba and D. M. Rodriguez, Stirling's series and Bernoulli numbers, *Amer. Math. Monthly* **98** (1991), 423–426.

[3] P. Erdős, On integers of the form $2^k + p$ and some related problems, *Summa Brasil. Math.* **2** (1950), 113–123.

[4] P. Erdős, On a problem concerning covering systems, *Mat. Lapok.* **3** (1952), 122–128.

[5] G. M. Fichtenholz, *Differential- und Integralrechnung II*, VEB Deutscher Verlag der Wissenschaften, 1990.

[6] A. S. Fraenkel, A characterization of exactly covering systems, *Discrete Math.* **4** (1973), 359–366.

[7] S. Guo and Z. W. Sun, On odd covering systems, *Adv. Appl. Math.* **35** (2005), 182–187.

[8] R. K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, 1981.

[9] V. Namias, A simple derivation of Stirling's asymptotic series, *Amer. Math. Monthly* **93** (1986), 25–29.

[10] Š. Porubský, Identities involving covering systems, *Math.Slovaca* **44** (1994), 153–162.

[11] Š. Porubský, On $m$-times covering systems, *Acta Arith.* **29** (1976), 159–169.

[12] Š. Porubský, Covering systems and generating functions, *Acta Arith.* **26** (1975), 223–231.

[13] J. L. Raabe, *Die Jacob Bernoullische Function*, Druck und Verlag von Orell, Füssli und Comp., 1848.

[14] J. L. Raabe, Zurückführung einiger Summen und bestimmten Integrale auf die Jacob Bernoullische, *J. Reine Angew. Math.* **42** (1851), 348–367.

[15] J. Stern, Beiträge zur Theorie der Bernoullischen und Eulerschen Zahlen, *Abh. Geselschaft Wiss. Göttingen* **23** (1878), 1–44.

[16] Z. W. Sun, Covering the integers by arithmetic sequences. II, *Trans. Amer. Math. Soc.* **348** (1996), 4279–4320.

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