Binary Recurrences for which Powers of Two are Discriminating Moduli

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Abstract

Given a sequence \( w = (w_n)_{n \geq 0} \) of distinct positive integers \( w_0, w_1, w_2, \ldots \) and any positive integer \( n \), we define the discriminator function \( D_w(n) \) to be the smallest positive integer \( m \) such that \( w_0, \ldots, w_{n-1} \) are pairwise incongruent modulo \( m \). In this paper, we classify all binary recurrent sequences \( w \) consisting of different integer terms such that \( D_w(2^e) = 2^e \) for every \( e \geq 1 \). For all of these sequences it is expected that one can actually give a fairly simple description of \( D_w(n) \) for every \( n \geq 1 \). For one infinite family of such sequences this has already been done by Faye, Luca, and Moree, and for another by Ciolan and Moree.

1 Introduction

The discriminator sequence of a sequence \( w = (w_n)_{n \geq 0} \) of distinct integers is the sequence \((D_w(n))_{n \geq 0}\) given by

\[
D_w(n) = \min\{m \geq 1 : w_0, \ldots, w_{n-1} \text{ are pairwise distinct modulo } m\}.
\]

In other words, \( D_w(n) \) is the smallest integer \( m \) that allows one to discriminate (tell apart) the integers \( w_0, \ldots, w_{n-1} \) on reducing them modulo \( m \). If not all integers are distinct, but say \( w_0, \ldots, w_k \), then we can define \( D_w(j) \) for \( j = 1, \ldots, k + 1 \). Obviously \( D_w(n) \) is non-decreasing as a function of \( n \). Note that since \( w_0, \ldots, w_{n-1} \) are in \( n \) distinct residue classes modulo \( D_w(n) \), we must have \( D_w(n) \geq n \). On the other hand clearly

\[
D_w(n) \leq \max\{w_0, \ldots, w_{n-1}\} - \min\{w_0, \ldots, w_{n-1}\} + 1.
\]

The main problem is to give an easy description or characterization of \( D_w(n) \). In many cases such a characterization does not seem to exist.

If \( w_j \) is a polynomial in \( j \), the behavior of the discriminator is fairly well understood. See Moree [7], Zieve [11], and the references therein.

An intensively studied class of sequences is that of binary recurrent sequences, cf. the book by Everest et al. [4]. For a generic binary recurrent sequence there is currently no meaningful characterization of its discriminator. An example is provided by the discriminator for the Fibonacci sequence (see Table 1). However, if we have

\[
D_w(2^e) = 2^e \quad \text{for every } e \geq 1,
\]

the discriminator behavior tends to be much simpler. It is easy to see that then \( D_w(n) < 2n \). This allows one to exclude many potential discriminator values. Indeed, in general discriminator characterizations for a fixed \( n \) proceed by excluding all integers different from \( D_w(n) \) as values. If (1) holds, then typically many powers of two occur as values (cf. Table 2). All known binary recurrent discriminators, described in Families 1 and 2 below, satisfy (1). Thus, it is natural to ask for a classification of all binary recurrent sequences \((w_n)_{n \geq 0}\) such that (1) is satisfied. Note that for any such sequence the terms \( w_n \) must be distinct.

Our main result completely answers this question.
Theorem 1. For integers $w_0, w_1, p$ and $q$, let $(w_n)_{n \geq 0}$ be the sequence defined by

$$w_{n+2} = pw_{n+1} + qw_n \quad \text{for all } n \geq 0. \quad (2)$$

If $w_0 + w_1$ is even and $k \geq 1$, then $\#\{w_n \pmod{2^k} : 0 \leq n \leq 2^k - 1\} < 2^k$.

If $(p \pmod{4}, q \pmod{4}) = (2, 3)$ and $w_0 + w_1$ is odd, then $\mathcal{D}_w(2^k) = 2^k$ for every $k \geq 1$.

If $(p \pmod{4}, q \pmod{4}) \neq (2, 3)$ and $k \geq 3$, then $\#\{w_n \pmod{2^k} : 0 \leq n \leq 2^k - 1\} < 2^k$.

Representing the residue classes modulo $m$ by $\overline{a}$, with $0 \leq a \leq m-1$, we can reformulate property (1) as saying that the map from $\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ given by $\overline{a} \mapsto \overline{w_a}$ is a permutation for every $m$ that is a power of two.

We next describe the binary recurrent sequences for which the discriminator has been characterized. They fall into two families. Theorem 1 shows at a glance that for all of them (1) is satisfied.

Family 1. In Faye et al. [5], and its continuation by Ciolan et al. [2], the discriminator $\mathcal{D}_{U(k)}(n)$ is studied, where the Shallit sequence $U(k)$ is given by $U(k) = (U_n(k))_{n \geq 0}$ with $U_0(k) = 0$, $U_1(k) = 1$ and

$$U_{n+2}(k) = (4k + 2)U_{n+1}(k) - U_n(k)$$

for all $n \geq 0$. By Theorem 1, we have $\mathcal{D}_{U(k)}(2^e) = 2^e$ for every $e \geq 1$.

Family 2. Let $q \geq 5$ be a prime and put $q^* = (-1)^{(q-1)/2} \cdot q$. The sequence $u_q(1), u_q(2), \ldots$, with

$$u_q(j) = \frac{3^j - q^*(-1)^j}{4},$$

we call the Browkin-Salajan sequence for $q$. The sequence $u_q$ satisfies the recursive relation $u_q(j) = 2u_q(j-1) + 3u_q(j-2)$ for $j \geq 3$, with initial values

$$u_q(1) = (3 + q^*)/4 \quad \text{and} \quad u_q(2) = (9 - q^*)/4.$$

We denote its discriminator by $\mathcal{D}_q$. In the context of the discriminator, the sequence $u_5$ (2, 1, 8, 19, 62, 181, 548, 1639, 4922, . . .) was first considered by Sabin Salajan during an internship carried out in 2012 under the guidance of Moree (for this reason we call it the Salajan sequence). Disregarding signs this is sequence A084222. Moree and Zumalacárregui [8] determined $\mathcal{D}_5(n)$ (cf. Table 2).

Theorem 2. Let $n \geq 1$ be an arbitrary integer. Let $e$ be the smallest integer such that $2^e \geq n$ and $f$ be the smallest integer such that $5^f \geq 5n/4$. Then $\mathcal{D}_5(n) = \min\{2^e, 5^f\}$.

More recently Ciolan and Moree [3] completely characterized $\mathcal{D}_q$ for arbitrary primes $q > 5$. Noting that $u_q(1) + u_q(2) = 3$, one sees that Theorem 1 applies and hence $\mathcal{D}_q(2^e) = 2^e$ for every $e \geq 1$. 

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In order to prove Theorem 1, we will deal with the special case where \( w \) is a Lucas sequence first in Section 3. In the general case, we express \( w \) as a linear combination of a Lucas and a shifted Lucas sequence (Section 4). Our arguments require some consideration of the two divisibility of binomial coefficients (Section 2).

Beyond the polynomial and the recurrence sequence case there is very little known. Haque and Shallit [6] considered the discriminator for \( k \)-regular sequences. For these also property (1) is satisfied. Sun [10] made some conjectures regarding the discriminator for various sequences.

2 Preliminaries

We recall a celebrated result of Kummer, cf. Ribenboim [9, pp. 30–33].

**Theorem 3** (Kummer, 1852). Let \( p \) be a prime number. The exponent of \( p \) in \( \binom{n}{m} \) is the number of base \( p \) carries when summing \( m \) with \( n - m \) in base \( p \).

Here and in what follows we write \( \nu_2(a) \) for the exponent of 2 in the factorization of the integer \( a \).

**Lemma 4.** We have

\[
\nu_2 \left( \binom{\ell}{k} 2^{3k} \right) > \nu_2(2\ell)
\]

for all \( k \geq 1 \). Further,

\[
\nu_2 \left( \binom{2^k}{\ell} 2^\ell \right) \geq k + 3
\]

for \( \ell = 3 \) and \( \ell \geq 5 \).

**Proof.** We use Theorem 3 with \( p = 2 \). For the first inequality, we note that it is clear for \( k = 1 \), so we may assume that \( k \geq 2 \). Write \( \ell = 2^{\ell_0}\ell_1 \) with integers \( \ell_0 \geq 0 \) and \( \ell_1 \) odd. The inequality is clear for \( \ell_0 \leq 1 \). It is also clear if \( k > (\ell_0 + 1)/3 \). So, we may assume that \( k \leq (\ell_0 + 1)/3 \). Write \( k = 2^{k_0}k_1 \), where \( k_0 \geq 0 \) and \( k_1 \) is odd. Then \( k_0 < k \leq (\ell_0 + 1)/3 < \ell_0 \).

It follows that by summing up \( k \) with \( \ell - k \), we have at least \( \ell_0 - k_0 \) carries in base 2. Thus,

\[
\nu_2 \left( \binom{\ell}{k} 2^{3k} \right) \geq (\ell_0 - k_0) + 3k > \ell_0 + 2k \geq \ell_0 + 2,
\]

which is what we wanted to prove.

We will now prove the second inequality. Assume first that \( \ell \in [3, 2^k - 1] \). Then the number of carries from summing up \( \ell \) with \( 2^k - \ell \) is, by the previous argument, \( k - \ell_0 \), where again \( \ell = 2^{\ell_0}\ell_1 \) with \( \ell_1 \) odd. Hence,

\[
\nu_2 \left( \binom{2^k}{\ell} 2^\ell \right) = k - \ell_0 + \ell.
\]
This is at least \( k + 3 \) if \( \ell \geq 3 \) is odd (since then \( \ell_0 = 0 \)). It is also at least \( k - \ell_0 + 2^\ell_0 > k + 3 \) if \( \ell_0 \geq 3 \). If \( \ell_0 = 1 \), then \( \ell > 4 \) so \( k - \ell_0 + \ell \geq k + 3 \). Finally, if \( \ell_0 = 2 \), then since \( \ell \neq 4 \), we have \( \ell \geq 8 \) (since \( 4 \mid \ell \)), so the above expression is at least \( k - 2 + 8 > k + 3 \). This was for \( \ell < 2^k \). Finally, when \( \ell = 2^k \), we have

\[
\nu_2\left(\left(\frac{2^k}{\ell}\right)\ell^k\right) = 2^k > k + 3
\]

because \( k \geq 3 \). \( \square \)

3 The Lucas sequence

A basic role in the theory of binary recurrent sequences is played by Lucas sequences.

**Theorem 5.** Let \((u_n)_{n \geq 0}\) be a Lucas sequence with \( u_0 = 0 \), \( u_1 = 1 \) and

\[
u_n = pu_{n+1} + qu_n,
\]

for all \( n \geq 0 \). Then \( \mathcal{D}_u(2^k) = 2^k \) for all \( k \geq 1 \) if and only if \((p \mod 4, q \mod 4) = (2, 3)\).

**Proof.** We look at \( \{u_0, u_1, u_2, u_3\} = \{0, 1, p, p^2 + q\} \). Since these are all the residues modulo 4, it follows that either \((p \mod 4, q \mod 4) = (2, 3)\) or \((p \mod 4, q \mod 4) = (3, 1)\). The second possibility entails \((p \mod 8, q \mod 8) \in \{(3, 1), (7, 1), (3, 5), (7, 5)\}\) and one checks computationally that none of these 4 possibilities gives that \( \{u_k \mod 8 : 0 \leq k \leq 7\} \) covers all residue classes modulo 8. Thus, we must have \((p \mod 4, q \mod 4) = (2, 3)\).

We consider the quadratic polynomial \( x^2 - px - q \) having discriminant \( \Delta = p^2 + 4q \). The equation \( x^2 - px - q = 0 \) is the characteristic equation for the Lucas sequence. We consider the cases \( \Delta = 0 \) and \( \Delta \neq 0 \) separately.

The degenerate case (\( \Delta = 0 \)). In this case \( u_n = np_0^{n-1} \) with \( p_0 = p/2 \). We have \( \{u_0, u_1, u_2, u_3\} = \{0, 1, 2p_0, 3p_0^2\} \) and since \( p_0 \) is odd, these are distinct modulo 4. We claim that \( \nu_2(u_m - u_n) = \nu_2(m - n) \) for \( m > n \). Notice that this claim implies (1).

We have \( u_m - u_n \equiv m - n \pmod{2} \). So \( \nu_2(u_m - u_n) = 0 \) if and only if \( \nu_2(m - n) = 0 \). Next assume that \( m \equiv n \pmod{2} \). Write \( m = n + 2\ell \). Then

\[
u_m - u_n = (n + 2\ell)p_0^{n+2\ell-1} - np_0^{n-1} = (n + 2\ell)p_0^{n-1}(p_0^{2\ell} - 1) + 2\ell p_0^{n-1}.
\]

We can write \( p_0^2 = 1 + 8p_1 \) with \( p_1 \) an integer. Thus,

\[
p_0^{2\ell} = (1 + 8p_1)^\ell = 1 + 8\ell p_1 + \binom{\ell}{2}(8p_1)^2 + \cdots.
\]

This in combination with (3) leads to

\[
u_m - u_n = p_0^{n-1}\left(2\ell + \sum_{k \geq 1}(n + 2\ell)\binom{\ell}{k}(8p_1)^k\right).
\]
Since by Lemma 4 for every \( k \geq 1 \) we have
\[
\nu_2 \left( \binom{\ell}{k} (8p_1)^k \right) > \nu_2 (2\ell),
\]
we conclude that
\[
\nu_2 (u_m - u_n) = \nu_2 (2\ell) = \nu_2 (m - n),
\]
thus establishing the claim.

The non-degenerate case (\( \Delta \neq 0 \)). Since \( p = 2p_0 \) and \( q \mod 4 = 3 \), it follows that
\[
\Delta = 4(p_0^2 + q) = 16\Delta_0,
\]
where \( \Delta_0 \) is an integer. Let
\[
\alpha = p_0 + 2\sqrt{\Delta_0} \quad \text{and} \quad \beta = p_0 - 2\sqrt{\Delta_0}
\]
be the roots of \( x^2 - px - q = 0 \). The Binet formula for \( u_n \) is
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\]
thus establishing the claim.
It follows that $k \geq \ell$. In particular, $z(2^k) = 2^k$.

Next we show that

$$u_{n+2^k} \equiv u_n + 2^k \pmod{2^{k+1}} \quad (6)$$

for all $k \geq 1$. One checks it easily by hand for $k = 1$ and $n = 0, 1$ as well as for $k = 2$ and $n = 0, 1, 2, 3$. Assume next $k \geq 3$. In what follows, for three algebraic integers $a, b, c$, we write $a \equiv b \pmod{c}$ if $(a - b)/c$ is an algebraic integer. We have

$$\alpha^{2^k} = (p_0 + 2\sqrt{\Delta_0})^{2^k} = p_0^{2^k} + 2^k p_0^{2^k-1} (2\sqrt{\Delta_0}) + \left(\frac{2^k}{2}\right) p_0^{2^k-2} (2\sqrt{\Delta_0})^2$$

$$+ \left(\frac{2^k}{4}\right) p_0^{2^k-4} (2\sqrt{\Delta_0})^4 + \sum_{\ell \geq 3, \ell \neq 4} \left(\frac{2^k}{\ell}\right) p_0^{2^k-\ell} (2\sqrt{\Delta_0})^\ell.$$

Then, by Lemma 4,

$$\alpha^{2^k} \equiv p_0^{2^k} + 2^k p_0^{2^k-1} (2\sqrt{\Delta_0}) + \left(\frac{2^k}{2}\right) p_0^{2^k-2} (2\sqrt{\Delta_0})^2 + \left(\frac{2^k}{4}\right) p_0^{2^k-4} (2\sqrt{\Delta_0})^4 \pmod{2^{k+3} \sqrt{\Delta_0}}.$$

Changing $\alpha$ to $\beta$, the same calculation yields

$$\beta^{2^k} \equiv p_0^{2^k} - 2^k p_0^{2^k-1} (2\sqrt{\Delta_0}) + \left(\frac{2^k}{2}\right) p_0^{2^k-2} (2\sqrt{\Delta_0})^2 + \left(\frac{2^k}{4}\right) p_0^{2^k-4} (2\sqrt{\Delta_0})^4 \pmod{2^{k+3} \sqrt{\Delta_0}}.$$

Thus,

$$\alpha^{n+2^k} - \beta^{n+2^k} \equiv \alpha^n \left(p_0^{2^k} + 2^k p_0^{2^k-1} (2\sqrt{\Delta_0}) + \left(\frac{2^k}{2}\right) p_0^{2^k-2} (2\sqrt{\Delta_0})^2 + \left(\frac{2^k}{4}\right) p_0^{2^k-4} (2\sqrt{\Delta_0})^4\right)$$

$$- \beta^n \left(p_0^{2^k} - 2^k p_0^{2^k-1} (2\sqrt{\Delta_0}) + \left(\frac{2^k}{2}\right) p_0^{2^k-2} (2\sqrt{\Delta_0})^2 + \left(\frac{2^k}{4}\right) p_0^{2^k-4} (2\sqrt{\Delta_0})^4\right)$$

$$\equiv p_0^{2^k} (\alpha^n - \beta^n) + 2^k p_0^{2^k-1} (2\sqrt{\Delta_0}) (\alpha^n + \beta^n)$$

$$+ \left(\frac{2^k}{2}\right) p_0^{2^k-2} (2\sqrt{\Delta_0})^2 (\alpha^n - \beta^n)$$

$$+ \left(\frac{2^k}{4}\right) p_0^{2^k-4} (2\sqrt{\Delta_0})^4 (\alpha^n - \beta^n) \pmod{2^{k+3} \sqrt{\Delta_0}}.$$

Dividing across by $\alpha - \beta$ (which is equal to $4\sqrt{\Delta_0}$), we obtain

$$u_{n+2^k} \equiv p_0^{2^k} u_n + 2^k p_0^{2^k-1} (u_n/2) + \left(\frac{2^k}{2}\right) p_0^{2^k-2} (4\Delta_0) u_n$$

$$+ \left(\frac{2^k}{4}\right) p_0^{2^k-4} (16\Delta_0^2) u_n \pmod{2^{k+1}}. \quad (7)$$
We have \( p_0^{2^k} \equiv 1 \pmod{2^{k+1}} \) and \( v_n/2 \equiv 1 \pmod{2} \). Finally,
\[
\frac{2^k}{2} p_0^{2^k-2} (4\Delta_0) = 2^{k+1} (2^k - 1) p_0^{2^k-2} \Delta_0 \equiv 0 \pmod{2^{k+1}},
\]
and also
\[
\frac{2^k}{4} p_0^{2^k-4} (16\Delta_0^2) = \frac{2^{k-2} (2^k - 1)(2^k-1-1)(2^k-3)}{3} 2^4 \Delta_0^2 \equiv 0 \pmod{2^{k+1}}.
\]

We thus get from (7) that (6) holds for all \( k \geq 1 \). This implies by induction on \( k \) that \( D_u(2^k) = 2^k \).

4 The general case: the proof of Theorem 1

In the previous section we dealt with the Lucas sequence (Theorem 5). We will make crucial use of that result in order to deal with a more general recurrence \( (w_n)_{n \geq 0} \) as in (2).

Proof of Theorem 1. If \( \#\{w_n \pmod{2^k} : 0 \leq n \leq 2^k - 1\} = 2^k \) for all \( k \), it holds for \( k = 1 \) in particular. Thus, \( w_0, w_1 \) have different parities which is equivalent to \( w_0 + w_1 \) being odd. This proves the first assertion. Conversely, write
\[
w_n = au_n + bu_{n+1}.
\]

Note that \( au_n + bu_{n+1} \) satisfies the same recurrence relation as \( w_n \). On setting \( n = 0 \), respectively \( n = 1 \), we find \( b = w_0 \) and \( a = w_1 - pw_0 \). Thus, \( a + b = (w_1 + w_0) - pw_0 \) is odd. By (6), we obtain
\[
w_{n+2^k} = au_{n+2^k} + bu_{n+1+2^k} \equiv a(u_n + 2^k) + b(u_{n+1} + 2^k)
\]
\[
\equiv (au_n + bu_{n+1}) + (a + b) 2^k \equiv w_n + 2^k \pmod{2^{k+1}}
\]
for \( k \geq 1 \). This shows that \( D_w(2^k) = 2^k \) for every \( k \geq 1 \).

It remains to prove the final assertion. Note that it is enough to prove it for \( k = 3 \). This can be done by doing a computer calculation modulo 8. We consider all integers \( a, b, p, q \) with \( 0 \leq a, b, p, q \leq 7 \) and compute \( \#\{w_n \pmod{8} : 0 \leq n \leq 7\} \). It turns out that if \( (p \pmod{4}, q \pmod{4}) \neq (2,3) \), then this cardinality is \( < 8 \).

5 Tables

We tabulate the discriminator for a sequence that does not (Fibonacci sequence) and a sequence that does (Sălăjan sequence) satisfy the conditions of Theorem 1. We give the prime factorization of the values. Note the big difference in behavior.
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<td>$2^3$</td>
<td>36 – 39</td>
<td>157</td>
<td>153 – 158</td>
<td>5·13·17</td>
</tr>
<tr>
<td>6</td>
<td>$3^2$</td>
<td>40 – 44</td>
<td>173</td>
<td>159 – 162</td>
<td>1171</td>
</tr>
<tr>
<td>7 – 8</td>
<td>2 · 7</td>
<td>45 – 55</td>
<td>193</td>
<td>163 – 166</td>
<td>1451</td>
</tr>
<tr>
<td>9 – 11</td>
<td>3 · 5</td>
<td>56 – 59</td>
<td>311</td>
<td>167 – 184</td>
<td>3·487</td>
</tr>
<tr>
<td>12 – 16</td>
<td>2 · 3 · 5</td>
<td>60 – 64</td>
<td>337</td>
<td>185 – 208</td>
<td>1609</td>
</tr>
<tr>
<td>17 – 20</td>
<td>5 · 7</td>
<td>65 – 68</td>
<td>409</td>
<td>209 – 281</td>
<td>3·761</td>
</tr>
</tbody>
</table>

Table 1: \[A270151\]: Discriminator for the Fibonacci sequence 1, 2, 3, 5, 8, 13, \ldots

<table>
<thead>
<tr>
<th>$n$</th>
<th>$D_S(n)$</th>
<th>$n$</th>
<th>$D_S(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>129 – 256</td>
<td>$2^8$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>257 – 512</td>
<td>$2^9$</td>
</tr>
<tr>
<td>3 – 4</td>
<td>$2^2$</td>
<td>513 – 1024</td>
<td>$2^{10}$</td>
</tr>
<tr>
<td>5 – 8</td>
<td>$2^3$</td>
<td>1025 – 2048</td>
<td>$2^{11}$</td>
</tr>
<tr>
<td>9 – 16</td>
<td>$2^4$</td>
<td>2049 – 2500</td>
<td>$5^4$</td>
</tr>
<tr>
<td>17 – 20</td>
<td>$5^2$</td>
<td>2501 – 4096</td>
<td>$2^{12}$</td>
</tr>
<tr>
<td>21 – 32</td>
<td>$2^5$</td>
<td>4097 – 8192</td>
<td>$2^{13}$</td>
</tr>
<tr>
<td>33 – 64</td>
<td>$2^6$</td>
<td>8193 – 12500</td>
<td>$5^5$</td>
</tr>
<tr>
<td>65 – 100</td>
<td>$5^3$</td>
<td>12501 – 16384</td>
<td>$2^{14}$</td>
</tr>
<tr>
<td>101 – 128</td>
<td>$2^7$</td>
<td>16385 – 32768</td>
<td>$2^{15}$</td>
</tr>
</tbody>
</table>

Table 2: Discriminator for the Sălăjan sequence 2, 1, 8, 19, 62, 181, \ldots

Table 2 demonstrates Theorem 2.

### 6 Acknowledgments

The problem of characterizing the binary recurrences \(w\) such that \(D_w(2^e) = 2^e\) for every \(e \geq 1\) was posed by Moree to many interns. Eventually it was solved, in essence, independently by de Clercq, Matthis, and Weiss (interns in 2019) and Stoumen (a student of Martirosyan). Their work was completed by Luca. Ciolan kindly commented on an earlier version.

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References


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