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Arithmetic Subderivatives: p-adic Discontinuity and Continuity

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Abstract

In a previous paper, we proved that the arithmetic subderivative D_S is discontinuous at any rational point with respect to the ordinary absolute value. In the present paper, we study this question with respect to the p-adic absolute value. In particular, we show that D_S is in this sense continuous at the origin if S is finite or $p \notin S$.

1 Introduction

Let $0 \neq x \in \mathbb{Q}$. There exists a unique sequence $(\nu_p(x))_{p\in\mathbb{P}}$ of integers (with only finitely many nonzero terms) such that

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$$x = (\operatorname{sgn} x) \prod_{p \in \mathbb{P}} p^{\nu_p(x)}. \tag{1}$$

Here $\mathbb{P}$ stands for the set of primes, and $\operatorname{sgn} x = x/|x|$. Define that $\operatorname{sgn} 0 = 0$ and $\nu_p(0) = \infty$ for all $p \in \mathbb{P}$. In addition to the ordinary axioms of ∞, we state that $0 \cdot \infty = 0$. Then (1) holds also for $x = 0$.

We recall the basic properties of the *p-adic order* ν_p.

Proposition 1. *For all* $x, y \in \mathbb{Q}$,

(a) $\nu_p(x) = \infty$ *if and only if* $x = 0$;

(b) $\nu_p(xy) = \nu_p(x) + \nu_p(y)$;

(c) $\nu_p(x + y) \geq \min(\nu_p(x), \nu_p(y))$;

(d) $\nu_p(x + y) = \min(\nu_p(x), \nu_p(y))$ *if* $\nu_p(x) \neq \nu_p(y)$.

Proof. Properties (a) and (b) are trivial. For (c) and (d), see, e.g., [2, Proposition 2.4]. □

Throughout this paper, we let $a \in \mathbb{Q}$, $p, q \in \mathbb{P}$, $p \neq q$, and $\emptyset \neq S \subseteq \mathbb{P}$.

The *arithmetic subderivative* [11, 8, 9] of $x \in \mathbb{Q}$ with respect to S, a.k.a. the *arithmetic type derivative* [4] is

$$D_S(x) = x \sum_{p \in S} \frac{\nu_p(x)}{p}.$$

The *arithmetic partial derivative* [10, 7] of x with respect to p is $D_p(x) = D_{\{p\}}(x)$. The *arithmetic derivative* [12, 3, 13] of x is $D(x) = D_{\mathbb{P}}(x)$. Clearly,

$$D_S(x) = \sum_{p \in S} D_p(x), \quad D(x) = \sum_{p \in \mathbb{P}} D_p(x).$$

The function D_S is very strongly discontinuous at any a [8, Theorem 4] with respect to the ordinary absolute value. But do we succeed better if we use the *p-adic absolute value* of x, defined by

$$|x|_p = \frac{1}{p^{\nu_p(x)}}?$$

(In particular, $|0|_p = 1/\infty = 0$.)

We recall the basic properties of $|\cdot|_p$.

Proposition 2. *For all* $x, y \in \mathbb{Q}$,

(a) $|x|_p = 0$ *if and only if* $x = 0$;

(b) $|xy|_p = |x|_p |y|_p$;

(c) $|x+y|_p \leq \max(|x|_p, |y|_p)$;

(d) $|x+y|_p = \max(|x|_p, |y|_p)$ *if* $|x|_p \neq |y|_p$.

Proof. This proposition is equivalent to Proposition 1. □

Let us write

$$x = x\frac{1}{p^{\nu_p(x)}}p^{\nu_p(x)} = x|x|_p p^{\nu_p(x)} = \mu_p(x)p^{\nu_p(x)},$$

where

$$\mu_p(x) = x|x|_p = \frac{x}{p^{\nu_p(x)}}. \tag{2}$$

Proposition 3. *For all* $x, y \in \mathbb{Q}$,

(a) $\nu_p(\mu_p(x)) = 0$ *if* $x \neq 0$, $\quad \nu_p(\mu_p(0)) = \nu_p(0) = \infty$;

(b) $|\mu_p(x)|_p = 1$ *if* $x \neq 0$, $\quad |\mu_p(0)|_p = 0$;

(c) $\mu_p(xy) = \mu_p(x)\mu_p(y)$;

(d) $D_p(\mu_p(x)) = 0$.

Proof. Trivial. □

The *p-adic distance* $|x-y|_p$ is smaller the larger $\nu_p(x-y)$ is.

We say that a function $\mathbb{Q} \to \mathbb{Q}$ is *p-adically continuous*, in short *p-continuous*, if it is continuous with respect to $|\cdot|_p$. So we ask: Is D_S p-continuous at some a? We study this question by considering sequences (x_i) of rational numbers. If $|x_i - a|_p \to 0$, equivalently $\nu_p(x_i - a) \to \infty$, then (x_i) *converges p-adically*, in short *p-converges*, to a. Let $x_i \to_p a$ denote this convergence.

Sections 2–3 are introductory. We present in Section 2 a "light version" of Dirichlet's theorem on arithmetic progressions. We study p-convergence in Section 3.

Sections 4–7 contain our main results. We prove in Section 4 that D_S is p-continuous at $a = 0$ if S is finite or $p \notin S$. We also prove that D_p is p-continuous also at $a \neq 0$. On the other hand, we show in Section 5 that D_q can be (and conjecture that it always is) p-discontinuous at $a \neq 0$. In Section 6, we extend the results of Section 5 to D_S when S is finite. Although Section 5 is only a special case of Section 6, we find it instructive to present it separately. We complete our paper with the conclusion in Section 7.

2 "Poor man's theorem on arithmetic progressions"

Throughout this section, $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$. As suggested by Graham et al. [5], we let $a \perp b$ denote that $\gcd(a, b) = 1$. See also [6, 4].

We recall Dirichlet's theorem on arithmetic progressions.

Theorem 4. *If $b > 0$, then the set*

$$T = \{a + nb : n \in \mathbb{Z}_+\} \tag{3}$$

contains infinitely many primes.

Proof. See, e.g., [1, Theorem 7.9]. □

Corollary 5. *If $b \neq 0$, then the set (3) contains infinitely many primes or their additive inverses.*

Proof. Trivial. □

Theorem 4 is advanced, while our paper is elementary. We do not need the full force of this theorem, and we want to use only elementary methods. Therefore we apply, instead of Theorem 4, the following "poor man's theorem on arithmetic progressions." It is elementary but strong enough for us. (Remember that $\emptyset \neq S \subseteq \mathbb{P}$ throughout.)

Theorem 6. *If S is finite and $b \neq 0$, then the set (3) contains infinitely many numbers that are not divisible by any element of S.*

Proof. If $a = 0$, then $b = \pm 1$, since otherwise $a \not\perp b$, a contradiction. Therefore $T = \mathbb{Z}_+$ or $T = \mathbb{Z}_-$, and the claim is trivially true.

Now assume that $a \neq 0$. Let

$$S = \{p_1, \ldots, p_h, q_1, \ldots, q_k\}, \quad p_1, \ldots, p_h \nmid a, \quad q_1, \ldots, q_k \mid a.$$

(Either the p_i or the q_i can be missing. Clearly, $p_i \neq q_j$ for all i, j.) We show that the numbers $a + nb$ apply when n goes through the set

$$N = \{mp_1 \cdots p_h : m \in \mathbb{Z}_+,\ q_1, \ldots, q_k \nmid m\}.$$

Write

$$c = p_1 \cdots p_h.$$

(If the p_i are missing, then the "empty product" $c = 1$.)

Let $x \in T$ with $n \in N$, that is,

$$x = a + mcb, \quad q_1, \ldots, q_k \nmid m. \tag{4}$$

Each $p_i \mid c$ but $p_i \nmid a$, so $p_i \nmid x$. Each $q_i \mid a$ but $q_i \nmid mcb$. (Clearly, $q_i \nmid m, c$. If $q_i \mid b$, then $a \not\perp b$, a contradiction.) Therefore also $q_i \nmid x$. Consequently, $s \nmid x$ for all $s \in S$. Because there are infinitely many numbers (4), the claim follows. □

3 Convergence

Continuity is usually proved by the "$\varepsilon - \delta$ technique", while discontinuity is often proved using suitable sequences. For consistency, we use sequences also in proving continuity. To that end, we need a characterization of p-convergence.

Proposition 7. *Let (x_i) be a sequence of rational numbers. If $a \neq 0$, then the following conditions are equivalent.*

(a) $x_i \to_p a$;

(b) $\mu_p(x_i) \to_p \mu_p(a)$ *and there is* $i_0 \in \mathbb{Z}_+$ *such that* $\nu_p(x_{i_0}) = \nu_p(x_{i_0+1}) = \cdots = \nu_p(a)$.

If $a = 0$, then (b) $\Rightarrow$ (a) *but not conversely.*

Proof.
Case 1: $a \neq 0$.
(a) $\Rightarrow$ (b): If i_0 does not exist, then (x_i) has a subsequence (x_{i_k}) whose each term satisfies $\nu_p(x_{i_k}) \neq \nu_p(a)$. Consequently,

$$\nu_p(x_{i_k} - a) \overset{\text{Prop.1(d)}}{=} \min(\nu_p(x_{i_k}), \nu_p(a)) \leq \nu_p(a) \overset{a\neq 0}{<} \infty, \quad k \geq 1.$$

Hence $\nu_p(x_{i_k} - a) \not\to \infty$, implying $x_i \not\to_p a$, a contradiction. Therefore i_0 exists, i.e.,

$$x_i = \mu_p(x_i)p^{\nu_p(a)}, \quad i \geq i_0.$$

Now, for $i \geq i_0$, we have

$$x_i - a = (\mu_p(x_i) - \mu_p(a))p^{\nu_p(a)}, \tag{5}$$

and further

$$\nu_p(\mu_p(x_i) - \mu_p(a)) + \nu_p(a) \overset{(5),\text{Prop. 1(b)}}{=} \nu_p(x_i - a) \overset{\text{(a)}}{\to} \infty,$$

verifying $\mu_p(x_i) \to_p \mu_p(a)$.
(b) $\Rightarrow$ (a): Since

$$x_i - a \overset{\text{(b)}}{=} \mu_p(x_i)p^{\nu_p(a)} - \mu_p(a)p^{\nu_p(a)} = (\mu_p(x_i) - \mu_p(a))p^{\nu_p(a)}, \quad i \geq i_0,$$

we have

$$\nu_p(x_i - a) \overset{\text{Prop. 1(b)}}{=} \nu_p(\mu_p(x_i) - \mu_p(a)) + \nu_p(a) \overset{\text{(b)}}{\to} \infty,$$

verifying (a).
Case 2: $a = 0$.
(b) $\Rightarrow$ (a). Since $\nu_p(x_{i_0}) = \nu_p(x_{i_0+1}) = \cdots = \nu_p(0) = \infty$, it follows that $x_{i_0} = x_{i_0+1} = \cdots = 0$. Therefore $x_i \to_p 0$.
(a) $\not\Rightarrow$ (b). If $x_i = p^i$, then $x_i \to_p 0$, but $\mu_p(x_i) = 1 \to_p 1 \neq 0 = \mu_p(0)$. □

Proposition 8. *A function $F : \mathbb{Q} \to \mathbb{Q}$ is p-continuous at a if and only if any sequence (x_i) of rational numbers satisfying $x_i \to_p a$ satisfies $F(x_i) \to_p F(a)$.*

Proof. Proceed as in proving the corresponding property of the ordinary continuity. □

There are three formally different ways to consider p-convergence. First, use ν_p everywhere. Second, use $|\cdot|_p$ everywhere. Third, use either ν_p or $|\cdot|_p$, depending on the situation. We follow the first way.

4 The cases of D_S, $a = 0$, and D_p, a arbitrary

We begin with a lemma that may be interesting on its own.

Lemma 9. *Let S be finite and $y \in \mathbb{Q}$. Assume that*

$$\{q \in \mathbb{P} \mid \nu_q(y) \neq 0\} \subseteq S.$$

Factorize

$$y = \prod_{q \in S} q^{\nu_q(y)} = u(y)w(y),$$

where

$$u(y) = \prod_{q \in S} q^{\nu_q(y)-1}, \quad w(y) = \prod_{q \in S} q.$$

Then

$$D_S(y) = u(y)v(y),$$

where

$$v(y) = \sum_{q \in S} \nu_q(y) \prod_{r \in S \setminus \{q\}} r.$$

Proof. We have

$$\begin{aligned} D_S(y) &= \sum_{q \in S} D_S(q^{\nu_q(y)}) \prod_{r \in S \setminus \{q\}} r^{\nu_r(y)} = \sum_{q \in S} \nu_q(y) q^{\nu_q(y)-1} \prod_{r \in S \setminus \{q\}} r^{\nu_r(y)} \\ &= \sum_{q \in S} \nu_q(y) q^{\nu_q(y)-1} \prod_{r \in S \setminus \{q\}} r^{\nu_r(y)-1} r = \sum_{q \in S} \nu_q(y) \Big(\prod_{r \in S} r^{\nu_r(y)-1} \Big) \Big(\prod_{r \in S \setminus \{q\}} r \Big), \end{aligned}$$

verifying the claim. □

Theorem 10. *If S is finite or $p \notin S$, then D_S is p-continuous at the origin.*

Proof. Let

$$x_i \to_p 0. \tag{6}$$

We show that $D_S(x_i) \to_p 0 = D_S(0)$.

If (x_i) has only a finite number of nonzero terms, then the claim is trivially true. So, we assume that there are infinitely many $x_i \neq 0$. Because zeros do not cause any problem in the proof, we can omit them and thus assume that each $x_i \neq 0$.

Write

$$x_i \overset{(1)}{=} (\operatorname{sgn} x_i) \prod_{q \in \mathbb{P}} q^{\nu_q(x_i)} = (\operatorname{sgn} x_i) \Big(\prod_{q \in S} q^{\nu_q(x_i)} \Big) \Big(\prod_{q \in \mathbb{P} \setminus S} q^{\nu_q(x_i)} \Big) = (\operatorname{sgn} x_i) y_i z_i,$$

where

$$y_i = \prod_{q \in S} q^{\nu_q(x_i)}, \quad z_i = \prod_{q \in \mathbb{P} \setminus S} q^{\nu_q(x_i)}.$$

(If $S = \mathbb{P}$, then the "empty product" $z_i = 1$.) Then

$$D_S(x_i) = (\operatorname{sgn} x_i) z_i D_S(y_i). \tag{7}$$

First, let us assume that S is finite. By Lemma 9,

$$D_S(y_i) = u(y_i) v(y_i). \tag{8}$$

Since $v(y_i) \in \mathbb{Z}$, it follows that

$$\nu_p(v(y_i)) \geq 0. \tag{9}$$

If $p \notin S$, then

$$\nu_p(D_S(x_i)) \overset{(7),(8)}{=} \nu_p(z_i u(y_i) v(y_i)) \overset{\text{Prop. 1(b)}}{=} \nu_p(z_i) + 0 + \nu_p(v(y_i)) \overset{(9)}{\geq} \nu_p(z_i) = \nu_p(x_i) \overset{(6)}{\to} \infty.$$

If $p \in S$, then

$$\begin{aligned} \nu_p(D_S(x_i)) &\overset{(7),(8)}{=} \nu_p(z_i u(y_i) v(y_i)) \overset{\text{Prop. 1(b)}}{=} 0 + \nu_p(u(y_i)) + \nu_p(v(y_i)) \overset{(9)}{\geq} \nu_p(u(y_i)) \\ &= \nu_p(x_i) - 1 \overset{(6)}{\to} \infty. \end{aligned}$$

Thus, $D_S(x_i) \to_p 0$ in each case.

Second, assume that S is infinite. Because $w(y)$ and $v(y)$ contain a divergent infinite product, applying Lemma 9 needs some preparation. Define

$$S_i = \{ q \in S \mid \nu_q(x_i) \neq 0 \}.$$

If $D_S(x_i) \neq 0$ only for finitely many terms, then the claim is trivially true. So, we assume that there are infinitely many such terms. We can omit all x_i satisfying $D_S(x_i) = 0$, because they do not violate the convergence. Then each $S_i \neq \emptyset$.

Now

$$D_S(x_i) = D_{S_i}(x_i) \overset{(7)}{=} (\operatorname{sgn} x_i) z_i D_{S_i}(y_i). \tag{10}$$

By Lemma 9,

$$D_{S_i}(y_i) = u_i(y_i) v_i(y_i), \tag{11}$$

where

$$u_i(y_i) = \prod_{q \in S_i} q^{\nu_q(y_i)-1}, \quad v_i(y_i) = \sum_{q \in S_i} \nu_q(y_i) \prod_{r \in S_i \setminus \{q\}} r.$$

If $p \notin S$, then

$$\nu_p(D_S(x_i)) \overset{(10),(11)}{=} \nu_p(z_i u_i(y_i) v_i(y_i)) \overset{\text{Prop. 1(b)}}{=} \nu_p(z_i) + 0 + \nu_p(v_i(y_i)) \overset{(9)}{\geq} \nu_p(z_i) = \nu_p(x_i) \overset{(6)}{\to} \infty.$$

Consequently, $D_S(x_i) \to_p 0$. We discuss the case of $p \in S$ at the end of this section. □

Theorem 11. *The function D_p is p-continuous everywhere.*

Proof. If $a = 0$, then apply Theorem 10. If $a \neq 0$, then let $x_i \to_p a$, and let i_0 be as in Proposition 7. For $i \geq i_0$,

$$\begin{aligned} D_p(x_i) - D_p(a) &= D_p(\mu_p(x_i) p^{\nu_p(a)}) - D_p(\mu_p(a) p^{\nu_p(a)}) \\ &\overset{\text{Prop. 3(d)}}{=} \nu_p(a) p^{\nu_p(a)-1} \mu_p(x_i) - \nu_p(a) p^{\nu_p(a)-1} \mu_p(a) \\ &= \nu_p(a) p^{\nu_p(a)-1} (\mu_p(x_i) - \mu_p(a)) = c(\mu_p(x_i) - \mu_p(a)), \quad c = \nu_p(a) p^{\nu_p(a)-1}. \end{aligned} \tag{12}$$

If $\nu_p(a) = 0$, then

$$D_p(x_i) \overset{(12)}{=} D_p(a), \quad i \geq i_0.$$

If $\nu_p(a) \neq 0$, then

$$\nu_p(D_p(x_i) - D_p(a)) \overset{(12),\text{Prop. 1(b)}}{=} \nu_p(c) + \nu_p(\mu_p(x_i) - \mu_p(a)) \overset{\text{Prop. 7}}{\to} \infty.$$

Therefore $D_p(x_i) \to_p D_p(a)$ in each case. □

Can we extend the proof of Theorem 10 to the case where S is infinite and $p \in S$? If $p \in S_i$, then

$$\begin{aligned} \nu_p(D_S(x_i)) &\overset{(10),(11)}{=} \nu_p(z_i u_i(y_i) v_i(y_i)) \overset{\text{Prop. 1(b)}}{=} 0 + \nu_p(u_i(y_i)) + \nu_p(v_i(y_i)) \overset{(9)}{\geq} \nu_p(u_i(y_i)) \\ &= \nu_p(x_i) \overset{(6)}{\to} \infty, \end{aligned}$$

implying the convergence.

If

$$p \in S \setminus S_i, \tag{13}$$

then

$$\nu_p(D_S(x_i)) \overset{(10),(11)}{=} \nu_p(z_i u_i(y_i) v_i(y_i)) \overset{\text{Prop. 1(b)}}{=} 0 + 0 + \nu_p(v_i(y_i)) = \nu_p(x_i).$$

If (13) holds only for finitely many indices i, then the corresponding x_i do not effect on the convergence, and therefore they do not bother us. If there are infinitely many such indices, then the question of convergence remains open.

We thus conclude that $D_S(x_i) \to_p 0$ also if, for any sequence (x_i) of nonzero numbers with $x_i \to_p 0$, only finitely many terms satisfy (13). However, we find this assumption useless, because its validity cannot be tested.

5 The case of D_q, $a \neq 0$

In this and the next section, we show that D_q is under certain assumptions discontinuous outside the origin. These sections are quite technical and require the use of rather heavy notation. In order to increase readability, we consider the special case $S = \{q\}$ separately.

Theorem 12. *Let*

$$a \neq 0. \tag{14}$$

If

$$D_q(a) = 0, \tag{15}$$

then D_q is p-discontinuous at a.

Proof. Let

$$x_i = a + \frac{p^i}{q}. \tag{16}$$

Then

$$\nu_p(x_i - a) = i \to \infty, \tag{17}$$

implying $x_i \to_p a$. Since

$$\nu_q(a) \overset{(15)}{=} 0, \quad \nu_q(\frac{p^i}{q}) = -1,$$

we have $\nu_q(x_i) \overset{(16),\text{Prop. 1(d)}}{=} -1$.

Consequently,

$$D_q(x_i) = \frac{\nu_q(x_i)}{q} x_i = -\frac{x_i}{q}, \tag{18}$$

and further

$$\nu_p(D_q(x_i) - D_q(a)) \overset{(15)}{=} \nu_p(D_q(x_i)) \overset{(18)}{=} \nu_p(\frac{x_i}{q}) = \nu_p(x_i).$$

We show that $\nu_p(x_i) \not\to \infty$; then $D_q(x_i) \not\to_p D_q(a)$, verifying the claim. If

$$\nu_p(x_i) \to \infty, \tag{19}$$

then

$$\nu_p(a) = \nu_p(x_i - (x_i - a)) \overset{\text{Prop. 1(c)}}{\geq} \min(\nu_p(x_i), \nu_p(x_i - a)) \overset{(17),(19)}{\to} \infty.$$

Hence $\nu_p(a) = \infty$, i.e., $a = 0$, contradicting (14). □

Theorem 13. *Let*

$$a = \frac{a_1}{a_2}, \quad 0 \neq a_1 \in \mathbb{Z},\ a_2 \in \mathbb{Z}_+, \quad a_1 \perp a_2. \tag{20}$$

If

$$D_q(a) \neq 0 \tag{21}$$

and

$$q \nmid a_2, \tag{22}$$

then D_q is p-discontinuous at a.

Proof. Let $i \in \mathbb{Z}_+$. Since $\mu_p(a_1) \overset{(20),\text{Prop. 3(a)}}{\perp} \mu_p(a_2)p^i$ and $\mu_p(a_2) \overset{(20)}{>} 0$, there are, by Theorem 6 ($S = \{p, q\}$, $a = \mu_p(a_1)$, $b = \mu_p(a_2)p^i$) positive integers $r_{i1} < r_{i2} < \cdots$ satisfying

$$r_{ik} = \mu_p(a_1) + n_{ik}\mu_p(a_2)p^i, \quad n_{ik} \in \mathbb{Z}_+, \quad p, q \nmid r_{ik}, \quad k = 1, 2, \dots. \tag{23}$$

Consequently,

$$\mu_p(a) + n_{ik}p^i \overset{\text{Prop. 3(c)}}{=} \frac{\mu_p(a_1)}{\mu_p(a_2)} + n_{ik}p^i = \frac{\mu_p(a_1) + n_{ik}\mu_p(a_2)p^i}{\mu_p(a_2)} \overset{(23)}{=} \frac{r_{ik}}{\mu_p(a_2)}.$$

Choose $r_{1k_1} < r_{2k_2} < \cdots$ and write $n_1 = n_{1k_1}, n_2 = n_{2k_2}, \dots,\ r_1 = r_{1k_1}, r_2 = r_{2k_2}, \dots$. Define the sequence (y_i) by

$$y_i = \mu_p(a) + n_i p^i = \frac{r_i}{\mu_p(a_2)}. \tag{24}$$

Then

$$\nu_p(y_i) \overset{(24),\text{Prop. 1(b)}}{=} \nu_p(r_i) - \nu_p(\mu_p(a_2)) \overset{(23),\text{Prop. 3(a)}}{=} 0 \tag{25}$$

and

$$\nu_p(y_i - \mu_p(a)) \overset{(24)}{=} \nu_p(n_i p^i) \overset{\text{Prop. 1(b)}}{=} \nu_p(n_i) + i \overset{n_i \in \mathbb{Z}_+}{\geq} i \to \infty. \tag{26}$$

For all $i \in \mathbb{Z}_+$, we have

$$\begin{aligned} D_q(y_i) &\overset{(24)}{=} \frac{\mu_p(a_2)D_q(r_i) - r_i D_q(\mu_p(a_2))}{\mu_p(a_2)^2} \\ &\overset{q \nmid r_i}{=} -\frac{r_i D_q(\mu_p(a_2))}{\mu_p(a_2)^2} \overset{(2)}{=} -\Big(r_i D_q\big(\frac{a_2}{p^{\nu_p(a_2)}}\big)\Big) \Big/ \Big(\frac{a_2}{p^{\nu_p(a_2)}}\Big)^2 \\ &= -\Big(\frac{r_i}{p^{\nu_p(a_2)}} D_q(a_2)\Big) \Big/ \Big(\frac{a_2}{p^{\nu_p(a_2)}}\Big)^2 = -\frac{r_i p^{\nu_p(a_2)} D_q(a_2)}{a_2^2} \overset{(22)}{=} 0. \end{aligned} \tag{27}$$

Also define

$$x_i = y_i p^{\nu_p(a)}; \tag{28}$$

then

$$D_q(x_i) = p^{\nu_p(a)} D_q(y_i) \overset{(27)}{=} 0. \tag{29}$$

Since $\mu_p(x_i) \overset{(25),(28)}{=} y_i \overset{(26)}{\to_p} \mu_p(a)$, it follows that $x_i \overset{\text{Prop. 7}}{\to_p} a$. On the other hand, since

$$D_q(x_i) - D_q(a) \overset{(29)}{=} -D_q(a) \overset{(21)}{\neq} 0,$$

we have

$$\nu_p(D_q(x_i) - D_q(a)) = \nu_p(D_q(a)) \overset{(21)}{<} \infty,$$

verifying $D_q(x_i) \not\to_p D_q(a)$. □

Corollary 14. *If $0 \neq a \in \mathbb{Z}$, then D_q is p-discontinuous at a.*

Proof. Apply Theorem 12 if $q \nmid a$, and Theorem 13 if $q \mid a$. □

If $D_q(a) \neq 0$ and $q \mid a_2$ (where a_2 is as in (20)), then our question remains open. We conjecture that discontinuity holds also in this case.

Conjecture 15. The function D_q is p-discontinuous outside the origin.

6 The case of D_S, S finite, $a \neq 0$

We extend Theorems 12 and 13, Corollary 14, and Conjecture 15. The presentation branches according to properties of $D_S(a)$ and S, summarized in Section 7.

Theorem 16. *Let S be finite, $p \notin S$, and $a \neq 0$. If*

$$D_S(a) = 0, \tag{30}$$

then D_S is p-discontinuous at a.

Proof. Let $S = \{q_1, \ldots, q_m\}$, take $\gamma \in \mathbb{Z}_+$ satisfying

$$\gamma > -\nu_{q_1}(a), \ldots, -\nu_{q_m}(a), \tag{31}$$

and define

$$x_i = a + \frac{p^i}{(q_1 \cdots q_m)^\gamma}. \tag{32}$$

Let $i \in \mathbb{Z}_+$ and $j \in \{1, \ldots, m\}$. Then

$$\nu_p(x_i - a) \overset{(32)}{=} i \to \infty, \quad x_i \to_p a, \quad \nu_{q_j}(x_i - a) \overset{(32)}{=} -\gamma \overset{(31)}{<} \nu_{q_j}(a),$$
$$\nu_{q_j}(x_i) = \nu_{q_j}((x_i - a) + a) \overset{\text{Prop. 1(d)}}{=} -\gamma, \quad D_{q_j}(x_i) = \frac{\nu_{q_j}(x_i)}{q_j} x_i = -\frac{\gamma x_i}{q_j}. \tag{33}$$

As in the proof of Theorem 12, we see that

$$\nu_p(x_i) \not\to \infty. \tag{34}$$

Now

$$D_S(x_i) \overset{(33)}{=} -\Big(\frac{1}{q_1} + \cdots + \frac{1}{q_m}\Big)\gamma x_i = -\frac{e_{m-1}(q_1, \ldots, q_m)\gamma x_i}{q_1 \cdots q_m}, \tag{35}$$

where e_{m-1} denotes the $(m-1)$'th elementary symmetric function. Therefore

$$\nu_p(D_S(x_i) - D_S(a)) \overset{(30)}{=} \nu_p(D_S(x_i)) \overset{(35),\text{Prop. 1(b)}}{=} \nu_p(e_{m-1}(q_1, \ldots, q_m)) + \nu_p(\gamma) + \nu_p(x_i).$$

Since $\nu_p(e_{m-1}(q_1, \ldots, q_m))$ and $\nu_p(\gamma)$ are (finite) constants and $\nu_p(x_i) \not\to \infty$, we have

$$\nu_p(D_S(x_i) - D_S(a)) \not\to \infty,$$

i.e., $D_S(x_i) \not\to_p D_S(a)$. □

Theorem 17. *Let $S \neq \{p\}$ be finite, $p \in S$, and $a \neq 0$. If*

$$D_S(a) = 0, \tag{36}$$

then D_S is p-discontinuous at a.

Proof. Let

$$S = \{q_1, \ldots, q_m, p\}, \quad S_0 = \{q_1, \ldots, q_m\}, \tag{37}$$

and let (x_i) be as in (32). For $i > \nu_p(a)$,

$$\nu_p(x_i - a) \overset{(32)}{=} i > \nu_p(a) \tag{38}$$

and further

$$\nu_p(x_i) = \nu_p((x_i - a) + a) \overset{(38),\text{Prop. 1(d)}}{=} \nu_p(a);$$

hence

$$D_p(x_i) = \frac{\nu_p(a)}{p} x_i. \tag{39}$$

Now

$$D_S(x_i) = D_{S_0}(x_i) + D_p(x_i) \overset{(35),(39)}{=} \frac{e_{m-1}(q_1, \ldots, q_m)\gamma x_i}{q_1 \cdots q_m} + \frac{\nu_p(a)}{p} x_i = c x_i, \tag{40}$$

where

$$c = \frac{e_{m-1}(q_1, \ldots, q_m)\gamma}{q_1 \cdots q_m} + \frac{\nu_p(a)}{p}.$$

We can choose $\gamma \in \mathbb{Z}_+$ so that, in addition to (31), the inequality $c \neq 0$ holds. Then

$$\nu_p(c) < \infty. \tag{41}$$

Since

$$\nu_p(D_S(x_i) - D_S(a)) \overset{(36)}{=} \nu_p(D_S(x_i)) \overset{(40),\text{Prop. 1(b)}}{=} \nu_p(c) + \nu_p(x_i) \overset{(34),(41)}{\not\to} \infty,$$

$D_S(x_i) \not\to_p D_S(a)$ follows. □

Theorem 18. *Let a be as in (20), let S be finite, and $p \notin S$. If*

$$D_S(a) \neq 0 \tag{42}$$

and

$$\nu_p(a_2 D_S(a)) \neq \nu_p(a D_S(a_2)), \tag{43}$$

then D_S is p-discontinuous at a.

Proof. Let $S = \{q_1, \dots, q_m\}$, $i \in \mathbb{Z}_+$, and $j \in \{1, \dots, m\}$. Proceeding as in the proof of Theorem 13, we have

$$r_{ik} = \mu_p(a_1) + n_{ik}\mu_p(a_2)p^i, \quad p, q_1, \dots, q_m \nmid r_{ik}, \quad k = 1, 2, \dots,$$

$$y_i = \mu_p(a) + n_i p^i = \frac{r_i}{\mu_p(a_2)}, \quad p, q_1, \dots, q_m \nmid r_i, \tag{44}$$

$$D_{q_j}(y_i) = -\frac{r_i p^{\nu_p(a_2)} D_{q_j}(a_2)}{a_2^2}, \tag{45}$$

$$x_i = y_i p^{\nu_p(a)} \to_p a, \tag{46}$$

and

$$D_{q_j}(x_i) \overset{(45),(46)}{=} -p^{\nu_p(a)} \frac{r_i p^{\nu_p(a_2)} D_{q_j}(a_2)}{a_2^2} \overset{(20),\text{Prop. }1\text{(b)}}{=} -\frac{r_i p^{\nu_p(a_1)} D_{q_j}(a_2)}{a_2^2} = -c r_i D_{q_j}(a_2), \tag{47}$$

where

$$c = \frac{p^{\nu_p(a_1)}}{a_2^2}.$$

Consequently,

$$D_S(x_i) = \sum_{j=1}^{m} D_{q_j}(x_i) \overset{(47)}{=} -c r_i \sum_{j=1}^{m} D_{q_j}(a_2) = -c r_i D_S(a_2), \tag{48}$$

and further

$$\begin{aligned}
\nu_p(D_S(x_i)) &\overset{(48)}{=} \nu_p(c r_i D_S(a_2)) \overset{\text{Prop. }1\text{(b)}}{=} \nu_p(a_1) - 2\nu_p(a_2) + \nu_p(D_S(a_2)) \\
&\overset{(20),\text{Prop. }1\text{(b)}}{=} \nu_p(a) - \nu_p(a_2) + \nu_p(D_S(a_2)) \qquad (49) \\
&\overset{(43),\text{Prop. }1\text{(b)}}{\neq} \nu_p(a_2) - \nu_p(a_2) + \nu_p(D_S(a)) \\
&= \nu_p(D_S(a)). \qquad (50)
\end{aligned}$$

Let

$$u(x_i) = D_S(x_i) - D_S(a).$$

Then

$$\nu_p(u(x_i)) \overset{(50),\text{Prop. 1(d)}}{=} \min(\nu_p(D_S(x_i)), \nu_p(D_S(a)))$$

$$\overset{(48),\text{Prop. 1(b)}}{=} \min(\nu_p(cD_S(a_2)), \nu_p(D_S(a))) \tag{51}$$

$$\leq \nu_p(D_S(a)) \overset{(42)}{<} \infty, \tag{52}$$

and $D_S(x_i) \not\to_p D_S(a)$ follows. □

Theorem 19. *Let a be as in (20), let $S \neq \{p\}$ be finite and $p \in S$. Assume that $D_S(a) \neq 0$. If*

$$D_{S\setminus\{p\}}(a) = 0, \tag{53}$$

then D_S is p-continuous at a. If

$$D_{S\setminus\{p\}}(a) \neq 0 \tag{54}$$

and

$$\nu_p(a_2 D_{S\setminus\{p\}}(a)) \neq \nu_p(a D_{S\setminus\{p\}}(a_2)), \tag{55}$$

then D_S is p-discontinuous at a.

Proof. If (53) holds, then

$$D_S(a) = D_{S\setminus\{p\}}(a) + D_p(a) = D_p(a),$$

and the claim follows from Theorem 11.

Now assume that (54) holds. Let S, S_0, and x_i be as in (37) and (46), respectively. Since $\nu_p(x_i) \overset{(44),(46)}{=} \nu_p(a)$, we have

$$D_p(x_i) = \rho x_i, \quad \rho = \frac{\nu_p(a)}{p}.$$

Now

$$D_S(x_i) - D_S(a) = D_{S_0}(x_i) + D_p(x_i) - (D_{S_0}(a) + D_p(a)) = u(x_i) + v(x_i), \tag{56}$$

where

$$u(x_i) = D_{S_0}(x_i) - D_{S_0}(a), \quad v(x_i) = \rho(x_i - a).$$

Because

$$\nu_p(v(x_i)) \overset{\text{Prop. 1(b)}}{=} \nu_p(\rho) + \nu_p(x_i - a) \overset{(46)}{\to} \infty$$

and $\nu_p(u(x_i))$ is bounded by (52), there is $i_0 \in \mathbb{Z}_+$ such that

$$\nu_p(v(x_i)) > \nu_p(u(x_i)) \quad \text{for all} \quad i \geq i_0. \tag{57}$$

Thus, for $i \geq i_0$,

$$\nu_p(D_S(x_i) - D_S(a)) \overset{(56),(57),\text{Prop. 1(d)}}{=} \nu_p(u(x_i)) \overset{(52)}{\not\to} \infty,$$

verifying $D_S(x_i) \not\to_p D_S(a)$. □

Corollary 20. *If $S \neq \{p\}$ is finite and $0 \neq a \in \mathbb{Z}$, then D_S is p-discontinuous at a.*

Proof. Apply Theorem 16 if $p \notin S$ and $D_S(a) = 0$, Theorem 17 if $p \in S$ and $D_S(a) = 0$, Theorem 18 if $p \notin S$ and $D_S(a) \neq 0$, and Theorem 19 if $p \in S$ and $D_S(a) \neq 0$. Note that (43) and (55) are satisfied (the right-hand side is infinite but the left-hand side is finite). □

We conjecture that Theorems 18 and 19 remain true without (43) and (55), respectively.

Conjecture 21. If $S \neq \{p\}$ is finite, $0 \neq a \in \mathbb{Q}$, and $D_{S\setminus\{p\}}(a) \neq 0$, then D_S is p-discontinuous at a.

7 Conclusion

We summarize our results. C denotes p-continuity, D p-discontinuity, and O denotes that the question is open.

1 (Theorem 10). S is finite or $p \notin S$, $a = 0$. C.

2 (the end of Section 5). S infinite, $p \in S$, $a = 0$. C or O.

3 (Theorem 11). $S = \{p\}$, a arbitrary. C.

4 (Theorem 12, a special case of Theorems 16 and 17). $S = \{q\}$, $a \neq 0$, $D_q(a) = 0$. D.

5 (Theorem 13, a special case of Theorems 18 and 19). $S = \{q\}$, $a \neq 0$, $D_q(a) \neq 0$. D.

6 (Theorem 16). S finite, $p \notin S$, $a \neq 0$, $D_S(a) = 0$. D.

7 (Theorem 17). $S(\neq \{p\})$ finite, $p \in S$, $a \neq 0$, $D_S(a) = 0$. D.

8 (Theorem 18). S finite, $p \notin S$, $a \neq 0$, $D_S(a) \neq 0$. D under (43), otherwise O.

9 (Theorem 19). $S(\neq \{p\})$ finite, $p \in S$, $a \neq 0$, $D_S(a) \neq 0$. C under (53), D under (54) and (55), otherwise O.

10. S infinite, $a \neq 0$. O.

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References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[2] A. Baker, An introduction to p-adic numbers and p-adic analysis, lecture notes, University of Glasgow, 2020, http://www.maths.gla.ac.uk/~ajb/course-notes.html.

[3] E. J. Barbeau, Remarks on an arithmetic derivative, *Canad. Math. Bull.* **4** (1961), 117–122.

[4] J. Fan and S. Utev, The Lie bracket and the arithmetic derivative, *J. Integer Sequences* **23** (2020), Article 20.2.5.

[5] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, Addison-Wesley, 2nd ed., 1994.

[6] P. Haukkanen, M. Mattila. J. K. Merikoski, and T. Tossavainen, Perpendicularity in an Abelian group, *Internat. J. Math. Math. Sci.* **2013**, Article 983607.

[7] P. Haukkanen, J. K. Merikoski, and T. Tossavainen, On arithmetic partial differential equations, *J. Integer Sequences* **19** (2016), Article 16.8.6.

[8] P. Haukkanen, J. K. Merikoski, and T. Tossavainen, Arithmetic subderivatives: Discontinuity and continuity, *J. Integer Sequences* **22** (2019), Article 19.7.4.

[9] P. Haukkanen, J. K. Merikoski, and T. Tossavainen, Asymptotics of partial sums of the Dirichlet series of the arithmetic derivative, *Math. Comm.* **25** (2020), 107–115, https://www.mathos.unios.hr/mc/index.php/mc/article/view/3206.

[10] J. Kovič, The arithmetic derivative and antiderivative, *J. Integer Sequences* **15** (2012), Article 12.3.8.

[11] J. K. Merikoski, P. Haukkanen, and T. Tossavainen, Arithmetic subderivatives and Leibniz-additive functions, *Ann. Math. Informat.* **50** (2019), 145–157, http://ami.ektf.hu.

[12] J. Mingot Shelly, Una cuestión de la teoría de los números, *Asociación Española, Granada* (1911), 1–12.

[13] V. Ufnarovski and B. Åhlander, How to differentiate a number, *J. Integer Sequences* **6** (2003), Article 03.3.4.

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