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Symmetrized Poly-Bernoulli Numbers and Combinatorics

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Abstract

Poly-Bernoulli numbers are one of the generalizations of the classical Bernoulli numbers. Since a negative indexed poly-Bernoulli number is an integer, it is an interesting problem to study this number from a combinatorial viewpoint. In this short article, we give a new combinatorial relation between symmetrized poly-Bernoulli numbers and Dumont-Foata polynomials.

1 Introduction

A *poly-Bernoulli polynomial* $B_m^{(\ell)}(x)$ of index $\ell \in \mathbb{Z}$ is defined by the generating series

$$e^{-xt}\frac{\mathrm{Li}_\ell(1-e^{-t})}{1-e^{-t}} = \sum_{m=0}^{\infty} B_m^{(\ell)}(x)\frac{t^m}{m!},$$

where $\mathrm{Li}_\ell(z)$ is the polylogarithm function given by

$$\mathrm{Li}_\ell(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^\ell} \quad (|z| < 1).$$

The polynomial $B_m^{(1)}(x)$ coincides with the classical Bernoulli polynomial $B_m(1-x) = (-1)^m B_m(x)$ since $\mathrm{Li}_1(z) = -\log(1-z)$ holds. Following Kaneko [7], the special value $B_m^{(\ell)} := B_m^{(\ell)}(0)$ is called a poly-Bernoulli number of index ℓ.

The aim of this study is giving a combinatorial perspective to the special values of $B_m^{(\ell)}(x)$ at integers $k \in \mathbb{Z}$. We assume that the index $\ell \leq 0$. In this case the values $B_m^{(\ell)}(k)$ are always integers. One of the first such investigations was Brewbaker's study [4]. He noticed the coincidence of two numbers, the poly-Bernoulli number $B_m^{(\ell)}(0)$ and the number of 01 lonesum matrices of size $m \times |\ell|$. Recently, Bényi and Hajnal [3] also established more combinatorial relations in this direction.

In this article, we take a step in another direction similar to Kaneko, Sakurai, and Tsumura [9]. To describe this more precisely, let G_n be the *Genocchi number* A110501 defined by $G_n = 2(2^{n+2}-1)|B_{n+2}|$, where $B_m = B_m^{(1)}(1)$ is the classical Bernoulli number. In [9, Theorem 4.2] the authors showed

$$\sum_{\ell=0}^{n} (-1)^\ell B_{n-\ell}^{(-\ell-1)}(1) = (-1)^{n/2} G_n \tag{1}$$

for any $n \geq 0$. As they mentioned, this equation is an analogue of the result of Arakawa and Kaneko [2],

$$\sum_{\ell=0}^{n} (-1)^\ell B_{n-\ell}^{(-\ell)}(0) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0. \end{cases} \tag{2}$$

In addition, Sakurai asked in her master's thesis whether we can generalize these equations for any positive integers $x = k$, and give some combinatorial meaning to them. Our main result provides an answer to these two questions in terms of the Dumont-Foata polynomial as follows.

Theorem 1. *Let $G_n(x,y,z)$ be the n-th Dumont-Foata polynomial defined in (3), and $\mathscr{B}_m^{(-\ell)}(k)$ the symmetrized poly-Bernoulli number defined in (5). Then we have*

$$\sum_{\ell=0}^{n} (-1)^\ell \mathscr{B}_{n-\ell}^{(-\ell)}(k) = k! \cdot (-1)^{n/2} G_n(1,1,k)$$

for any non-negative integers $n, k \geq 0$. In particular, both sides equal zero for odd n.

This theorem recovers the equations (1) and (2) since $\mathscr{B}_m^{(-\ell)}(0) = B_m^{(-\ell)}(0), \mathscr{B}_m^{(-\ell)}(1) = B_m^{(-\ell-1)}(1)$, and $G_n(1,1,1) = G_n$ hold as we see later.

Remark 2. I feel that there are a lot of possibilities of establishing similar identities as this theorem. Recent work of Bényi and Hajnal [3, Section 6] pointed out that the sequence of diagonal sums

$$\sum_{\ell=0}^{n} B_{n-\ell}^{(-\ell-1)}(1) \quad (n \geq 0)$$

appears as A136127 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [11]. Furthermore, there are many types of generalizations of poly-Bernoulli numbers as referred in [5], and analogues such as poly-Euler numbers [10], and poly-cosecant numbers [8].

2 Definitions

2.1 Dumont-Foata polynomials

We review the work of Dumont and Foata [6] here. For a positive even integer $n \in 2\mathbb{Z}$, we consider the surjective map $p : \{1, 2, \ldots, n\} \to \{2, 4, \ldots, n\}$ with $p(x) \geq x$ for each $x \in \{1, 2, \ldots, n\}$. This map is called a (surjective) *pistol* of size n, and corresponds to the following diagram. Here we draw an example for $n = 6$.

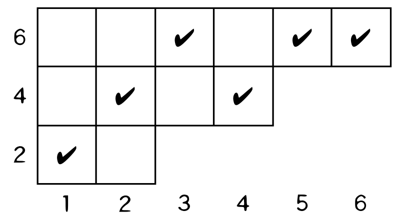

Figure 1: $p(1) = 2, p(2) = p(4) = 4, p(3) = p(5) = p(6) = 6$

Let $\mathcal{P}_n$ be the set of all pistols of size n. For each pistol $p \in \mathcal{P}_n$, we define three quantities called bulging, fixed, and maximal points. First, the number $x \in \{1, 2, \ldots, n\}$ is a *bulging point* of $p \in \mathcal{P}_n$ if $p(y) < p(x)$ for any $0 < y < x$. We let $b(p)$ denote the number of bluging points of p. In the diagram, $b(p)$ corresponds to the number of steps of the minimal stair covering all check marks. For the above example p, the points $x = 1, 2, 3$ are bulging points, so that $b(p) = 3$.

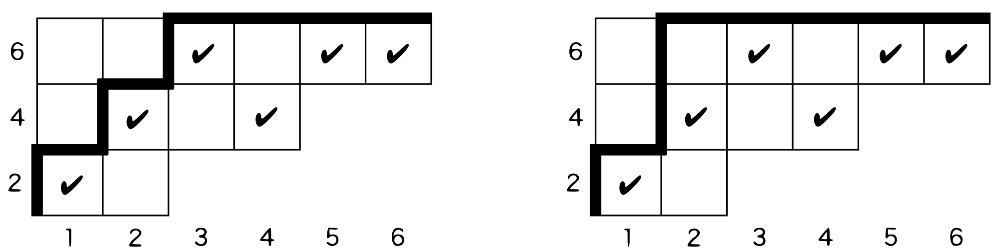

Figure 2: minimal stair (left), not minimal stair (right)

Next, the point $x \in \{1, 2, \ldots, n\}$ is called a *fixed point* of $p \in \mathcal{P}_n$ if $p(x) = x$. Finally, the point $x \in \{1, 2, \ldots, n-1\}$ is a *maximal point* of $p \in \mathcal{P}_n$ if $p(x) = n$. We let $f(p)$ and $m(p)$ denote the numbers of fixed points and maximal points of $p \in \mathcal{P}_n$, respectively. For the above example, we have $f(p) = 2$ and $m(p) = 2$. Dumont and Foata [6, Théorème 1a, 2] established the following interesting theorem.

Theorem 3. [6] *Let $n \in 2\mathbb{Z}_{>0}$ be a positive even integer. The polynomial defined by*

$$G_n(x,y,z) := \sum_{p\in\mathcal{P}_n} x^{b(p)}y^{f(p)}z^{m(p)} \tag{3}$$

is a symmetric polynomial in three variables, and satisfies $G_n(1,1,1) = G_n$.

In addition, $G_0(x,y,z) = 1$ and $G_n(x,y,z) = 0$ for a positive odd integer $n \in \mathbb{Z}$. The polynomial $G_n(x,y,z)$ is called the n-th *Dumont-Foata polynomial.* Furthermore, Dumont and Foata showed that the polynomial for $n > 0$ has the form $G_n(x,y,z) = xyzF_n(x,y,z)$, and the polynomial $F_n(x,y,z)$ satisfies the recurrence relation

$$F_n(x,y,z) = (x+z)(y+z)F_{n-2}(x,y,z+1) - z^2F_{n-2}(x,y,z)$$

with initial values $F_1(x,y,z) = 0$ and $F_2(x,y,z) = 1$. This implies that the polynomial $G_n(z) := G_n(1,1,z)$ called the *Gandhi polynomial* A036970 satisfies

$$G_{n+2}(z) = z(z+1)G_n(z+1) - z^2G_n(z) \tag{4}$$

with $G_0(z) = 1, G_1(z) = 0$.

For instance, there exist three pistols of size 4. The pistols have $(b(p), f(p), m(p)) = (2,2,1), (2,1,2)$ and $(1,2,2)$, so that the Dumont-Foata polynomial is given by

$$G_4(x,y,z) = x^2y^2z + x^2yz^2 + xy^2z^2 = xyz(xy+yz+zx).$$

Indeed, $G_4(1,1,1) = 3$ coincides the 4-th Genocchi number given by $G_4 = 2(2^6-1)|B_6| = 3$.

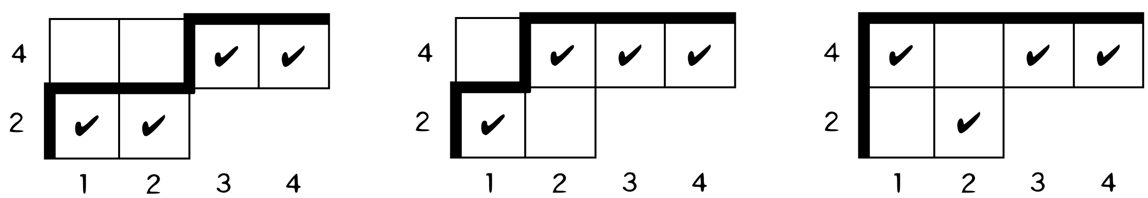

Figure 3: All elements of $\mathcal{P}_4$

2.2 Symmetrized poly-Bernoulli numbers

Table 1. includes the first few values of poly-Bernoulli numbers $\{B_m^{(-\ell)}(0)\}$ and $\{B_m^{(-\ell)}(1)\}$ with $m, \ell \geq 0$.

$\ell\backslash m$	0	1	2	3	4
0	1	1	1	1	1
1	1	2	4	8	16
2	1	4	14	46	146
3	1	8	46	230	1066
4	1	16	146	1066	6902

$\ell\backslash m$	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	3	7	15	31
3	1	7	31	115	391
4	1	15	115	675	3451

Table 1: $B_m^{(-\ell)}(0)$ A099594, $B_m^{(-\ell)}(1)$ A136126

We can see the symmetric property of these numbers at a glance. On the other hand, for $k \geq 2$ it seems unlikely that such a simple symmetric property can be given. In order to reproduce the symmetric properties for any $k \geq 2$, Kaneko-Sakurai-Tsumura [9] considered combinations of $B_m^{(-\ell)}(k)$. To make this precise, let $m, \ell, k \geq 0$ be non-negative integers. We now define the *symmetrized poly-Bernoulli number* $\mathscr{B}_m^{(-\ell)}(k)$ by

$$\mathscr{B}_m^{(-\ell)}(k) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} B_m^{(-\ell-j)}(k), \tag{5}$$

where $\begin{bmatrix} k \\ j \end{bmatrix}$ is the unsigned Stirling number of the first kind A130534 defined in [1, Definition 2.5]. This number satisfies the symmetry property

$$\mathscr{B}_m^{(-\ell)}(k) = \mathscr{B}_\ell^{(-m)}(k)$$

for any $m, \ell, k \geq 0$. Note that

$$\mathscr{B}_m^{(-\ell)}(0) = B_m^{(-\ell)}(0), \quad \mathscr{B}_m^{(-\ell)}(1) = B_m^{(-\ell-1)}(1).$$

$\ell \backslash m$	0	1	2	3	4
0	1	-1	1	-1	1
1	1	0	0	0	0
2	1	2	2	2	2
3	1	6	18	42	90
4	1	14	86	374	1382

$\ell \backslash m$	0	1	2	3	4
0	2	2	2	2	2
1	2	8	20	44	92
2	2	20	104	416	1472
3	2	44	416	2744	15032
4	2	92	1472	15032	120632

Table 2: $B_m^{(-\ell)}(2)$ and $\mathscr{B}_m^{(-\ell)}(2)$

Moreover, the authors showed the following explicit formula for $\mathscr{B}_m^{(-\ell)}(k)$.

$$\mathscr{B}_m^{(-\ell)}(k) = \sum_{j=0}^{\min(m,\ell)} j!(k+j)! \left\{ \begin{matrix} m+1 \\ j+1 \end{matrix} \right\} \left\{ \begin{matrix} \ell+1 \\ j+1 \end{matrix} \right\}, \tag{6}$$

where $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}$ is the Stirling number of the second kind A008277 defined in [1, Definition 2.2]. We prove our main theorem using this formula in the next section.

3 Proof

To prove Theorem 1, it suffices to show that the function

$$\tilde{G}_n(k) := \frac{(-1)^{n/2}}{k!} \sum_{\ell=0}^{n} (-1)^\ell \mathscr{B}_{n-\ell}^{(-\ell)}(k)$$

satisfies the recurrence relation (4) for any integer $k \geq 0$. First, we can easily see that $\tilde{G}_0(k) = 1$ and $\tilde{G}_1(k) = 0$, which are the initial cases. Moreover, for any odd integer n, $\tilde{G}_n(k) = 0$ follows from the symmetric property of $\mathscr{B}_m^{(-\ell)}(k)$. For an even integer $n \geq 2$, by the formula (6) we have

$$
\begin{aligned}
&(-1)^{n/2}k!\Big(k(k+1)\tilde{G}_n(k+1) - k^2\tilde{G}_n(k) - \tilde{G}_{n+2}(k)\Big) \\
&\quad = k\sum_{j=0}^{n/2} j!(k+j+1)!\sum_{\ell=j}^{n-j}(-1)^{\ell}\left\{ n-\ell+1 \atop j+1 \right\}\left\{ \ell+1 \atop j+1 \right\} && (7)\\
&\qquad - k^2\sum_{j=0}^{n/2} j!(k+j)!\sum_{\ell=j}^{n-j}(-1)^{\ell}\left\{ n-\ell+1 \atop j+1 \right\}\left\{ \ell+1 \atop j+1 \right\} && (8)\\
&\qquad + \sum_{j=0}^{n/2+1} j!(k+j)!\sum_{\ell=j}^{n+2-j}(-1)^{\ell}\left\{ n-\ell+3 \atop j+1 \right\}\left\{ \ell+1 \atop j+1 \right\}. && (9)
\end{aligned}
$$

Since $\left\{ k \atop 1 \right\} = 1$ holds for any $k \geq 1$, we can split the third line (9) according as $j = 0$ or not, which equals

$$
k! + \sum_{j=0}^{n/2}(j+1)!(k+j+1)!\sum_{\ell=j+1}^{n+1-j}(-1)^{\ell}\left\{ n+3-\ell \atop j+2 \right\}\left\{ \ell+1 \atop j+2 \right\}.
$$

Let

$$
a_{n,j} := \sum_{\ell=j}^{n-j}(-1)^{\ell}\left\{ n-\ell+1 \atop j+1 \right\}\left\{ \ell+1 \atop j+1 \right\}. \tag{10}
$$

Then the total of (7), (8), and (9) equals

$$
k! + \sum_{j=0}^{n/2} j!(k+j)!\Big(k(j+1)a_{n,j} + (j+1)(k+j+1)a_{n+2,j+1}\Big). \tag{11}
$$

Once the sum is 0, the proof completes. By using the generating function given in [1, Proposition 2.6 (8)], we have

$$
\frac{t^j}{(1-t)(1-2t)\cdots(1-(j+1)t)} = \sum_{\ell\geq j}\left\{ \ell+1 \atop j+1 \right\}t^{\ell} = \sum_{\ell\leq n-j}\left\{ n-\ell+1 \atop j+1 \right\}t^{n-\ell}
$$

for any non-negative integers $n, j \in \mathbb{Z}_{\geq 0}$. Multiplying these two expressions, we obtain

$$
\frac{s^j t^j}{(1-s)(1-t)\cdots(1-(j+1)s)(1-(j+1)t)} = \sum_{\ell\geq j}\sum_{k\leq n-j}\left\{ \ell+1 \atop j+1 \right\}\left\{ n-k+1 \atop j+1 \right\}s^{\ell}t^{n-k}.
$$

By specializing at $s=-x, t=x$,

$$\frac{(-1)^j x^{2j}}{(1-x^2)\cdots(1-(j+1)^2x^2)} = \sum_{\ell\geq j}\sum_{k\leq n-j}(-1)^\ell \left\{ {\ell+1 \atop j+1} \right\} \left\{ {n-k+1 \atop j+1} \right\} x^{n+\ell-k}. \qquad (12)$$

Thus, we see that the number $a_{n,j}$ defined in (10) appears as the n-th coefficient of (12). By the expression of the left-hand side of (12), we easily see that $a_{n,j}=0$ when n is an odd integer or $2j>n$. Further, we get the initial values $a_{2j,j}=(-1)^j, a_{n,0}=1$ for even n, and the recurrence relation

$$a_{n+2,j}=(j+1)^2a_{n,j}-a_{n,j-1}.$$

Applying this to the equation (11), we get

$$\begin{aligned} k! + \sum_{j=0}^{n/2} j!(k+j)!\Big(k(j+1)a_{n,j}+(j+1)(k+j+1)((j+2)^2a_{n,j+1}-a_{n,j})\Big) \\ = k! + \sum_{j=0}^{n/2}(j+2)(j+2)!(k+j+1)!a_{n,j+1} - \sum_{j=0}^{n/2}(j+1)(j+1)!(k+j)!a_{n,j}. \end{aligned}$$

Since $a_{n,n/2+1}=0$ and $a_{n,0}=1$, this equals 0, which concludes the proof of Theorem 1.

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References

[1] T. Arakawa, T. Ibukiyama, and M. Kaneko, *Bernoulli Numbers and Zeta Functions*, with an appendix by Don Zagier, Springer, 2014.

[2] T. Arakawa and M. Kaneko, On poly-Bernoulli numbers, *Comment. Math. Univ. Sanct. Pauli* **48** (1999), 159–167.

[3] B. Bényi and P. Hajnal, Combinatorial properties of poly-Bernoulli relatives, *Integers* **17** (2017), #A31.

[4] C. Brewbaker, A combinatorial interpretation of the poly-Bernoulli numbers and two Fermat analogues, *Integers* **8** (2008), #A02.

[5] C. B. Corcino, R. B. Corcino, T. Komatsu, and H. Jolany, On multi poly-Bernoulli polynomials, *J. Inequal. Spec. Funct.* **10** (2019), 21–34.

[6] D. Dumont and D. Foata, Une propriété de symétrie des nombres de Genocchi, *Bull. Soc. Math. France* **104** (1976), 433–451.

[7] M. Kaneko, Poly-Bernoulli numbers, *J. Théor. Nombres Bordeaux* **9** (1997), 199–206.

[8] M. Kaneko, M. Pallewatta, and H. Tsumura, On polycosecant numbers, *J. Integer Sequences* **23** (2020) Article 20.6.4.

[9] M. Kaneko, F. Sakurai, and H. Tsumura, On a duality formula for certain sums of values of poly-Bernoulli polynomials and its application, *J. de Théorie des Nombres de Bordeaux* **30** (2018), 203–218.

[10] Y. Ohno and Y. Sasaki, On poly-Euler numbers, *J. Aust. Math. Soc.* **103** (2017), 126–144.

[11] N. J. A. Sloane et al., The on-line encyclopedia of integer sequences, 2020. Available at `https://oeis.org`.

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