Journal of Integer Sequences, Vol. 23 (2020),
Article 20.11.4

Minimum Coprime Graph Labelings

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Abstract

A coprime labeling of a graph G is a labeling of the vertices of G with distinct integers from 1 to k such that adjacent vertices have coprime labels. The minimum coprime number of G is the least k for which such a labeling exists. In this paper, we determine the minimum coprime number for a few well-studied classes of graphs, including the coronas of complete graphs with empty graphs and the joins of two paths. In particular, we resolve a conjecture of Seoud, El Sonbaty, and Mahran and two conjectures of Asplund and Fox. We also provide an asymptotic bound for the minimum coprime number of the Erdős-Rényi random graph.

1 Introduction

1.1 Background

Let G be a simple graph with $n = |V(G)|$ vertices. A *coprime labeling* of G is an injection $f\colon V(G) \to \{1, 2, \ldots, k\}$, for some integer $k \geq n$, such that if $(u, v) \in E(G)$ then $\gcd(f(u), f(v)) = 1$. The *minimum coprime number* $\mathfrak{pr}(G)$ is the least k for which such a labeling exists; a coprime labeling of G using only integers up to $\mathfrak{pr}(G)$ is called a *minimum coprime labeling* of G. If $\mathfrak{pr}(G) = n$, then a minimum coprime labeling of G is called a *prime labeling* and G is a *prime graph*.

The notion of prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy, and Howalla [32]. It is conceptually related to the *coprime graph of integers*, the graph with vertex set $\mathbb{Z}$ that contains the edge (m, n) if and only if $\gcd(m, n) = 1$. The induced subgraph $G(A)$ with vertex set $A \subset \{1, \dots, N\}$ is called the *coprime graph of A* and was first studied by Erdős [8], who posed the famous problem of finding the largest set $A \subset \{1, \dots, N\}$ such that $K_k \not\subset G(A)$. Newman's coprime mapping conjecture, which was proven by Pomerance and Selfridge [22], involves the existence of perfect matchings in $G(A)$. Various other properties of the coprime graph of integers have been studied by Ahlswede and Khachatrian [1, 2, 3], Erdős [9], Erdős and A. Sárközy [10], Erdős and G. N. Sárközy [13], Erdős, A. Sárközy, and Szémeredi [11, 12], and G. N. Sárközy [26]. Further discussion of the coprime graph of integers may be found in [21].

The problem of finding a minimum coprime labeling of a graph H is equivalent to showing that H is a subgraph of the coprime graph $G(\{1, 2, \dots, \mathfrak{pr}(H)\})$. Much work has been done to prove that various classes of graphs are prime; we refer the reader to [15] for a full catalog of results. In particular, it is known that all paths, cycles, helms, fans, flowers, books, and wheels of even order are prime [15, 27, 29].

The primality of trees has been especially well studied. Entringer and Tout conjectured around 1980 that every tree is prime. While this conjecture remains open, it is now known that many classes of trees are prime, including paths, stars, caterpillars, spiders, and complete binary trees [15]. Fu and Huang [14] proved in 1994 that trees with 15 or fewer vertices are prime, and Pikhurko [20] extended this result to trees with 50 or fewer vertices in 2007. Salmasian [25] showed that any tree T with $n \geq 50$ vertices satisfies $\mathfrak{pr}(T) \leq 4n$. Pikhurko [20] improved upon this by showing that the Entringer-Tout conjecture holds asymptotically, i.e., for any $c > 0$, there is an N such that for any tree T of order $n > N$, $\mathfrak{pr}(T) < (1 + c)n$. In 2011, Haxell, Pikhurko, and Taraz [17] proved the Entringer-Tout conjecture for trees of sufficiently large order.

In this paper, we focus on the minimum coprime numbers of a few well-studied classes of graphs. The first class we consider is formed by taking the coronas of complete graphs and empty graphs. The *corona* of a graph G with a graph H, denoted $G \odot H$, is obtained by combining one copy of G with $|V(G)|$ copies of H by attaching the i^{th} vertex in G to every vertex in the i^{th} copy of H. Tout, Dabboucy, and Howalla [32] showed that the *crown graphs* $C_n \odot \overline{K}_m$ are prime for all positive integers n and m. The graphs $K_n \odot \overline{K}_m$, which are spanning supergraphs of the crowns $C_n \odot \overline{K}_m$, have also been studied in this context. Youssef and Elsakhawi [33] showed that $K_n \odot K_1$ is prime for $n \leq 7$, and $K_n \odot \overline{K}_2$ is prime for $n \leq 16$. Seoud, El Sonbaty, and Mahran [28] then observed that $K_n \odot \overline{K}_m$ is not prime if $n > \pi(n(m+1)+1)$. (Throughout this paper, we denote $\pi(x)$ to be the number of primes less than or equal to x, and p_n to be the n^{th} prime number.) They also conjectured the converse,

Conjecture 1 ([28, Conjecture 3.9])**.** The graph $K_n \odot \overline{K}_m$ is prime if $n \leq \pi(n(m+1)) + 1$.

They computed all values of n satisfying this condition for all $m \leq 20$. Most recently, Asplund and Fox [4] computed the minimum coprime numbers of $K_n \odot K_1$ and $K_n \odot \overline{K}_2$, showing that $\mathfrak{pr}(K_n \odot K_1) = p_{n-1}$ if $n > 7$ and $\mathfrak{pr}(K_n \odot \overline{K}_2) = p_{n-1}$ if $n > 16$. They conjecture that their results extend whenever n is sufficiently large relative to m.

Conjecture 2 ([4, Conjecture 1])**.** For all positive integers m, there exists an $M > m$ such that for all $n > M$, we have $\mathfrak{pr}(K_n \odot \overline{K}_m) = p_{n-1}$.

We prove the following statement, which resolves Conjectures 1 and 2 affirmatively.

Theorem 3. *For all positive integers m and n,*

$$\mathfrak{pr}(K_n \odot \overline{K}_m) = \max(mn + n, p_{n-1}).$$

The next class of graphs we consider is constructed via the join operation. The *join* of two disjoint graphs G and H, denoted $G + H$, consists of the graph union $G \cup H$ with edges added to connect each vertex in G to each vertex in H. We focus on the joins of two paths, which are well-studied. It was shown in [27, 28] that $P_n + K_1 = P_n + P_1$ is prime, $P_n + \overline{K}_2$ is prime if and only if $n \geq 3$ is odd, and $P_n + \overline{K}_m$ is not prime for $m \geq 3$. Asplund and Fox [4] computed the minimum coprime numbers of $P_m + P_n$ for various m and n. They showed the following theorem.

Theorem 4 ([4, Theorems 17,18,19])**.** *For $m \geq 4$ even and $n = 2$, or $m \geq n$ and $n = 3$ or 4, the minimum coprime number of $P_m + P_n$ is given by*

$$\mathfrak{pr}(P_m + P_n) = \begin{cases} m + 2n - 2, & \textit{if } m \textit{ is odd;} \\ m + 2n - 1, & \textit{if } m \textit{ is even.} \end{cases} \tag{1}$$

They proceeded to show that (1) holds for $2 \leq n \leq 10$ if $m > 118$. They conjecture that this result extends to all n as long as m is sufficiently large.

Conjecture 5 ([4, Conjecture 2])**.** For any positive integer N, there exists a positive integer M such that for all $m > M$ and $2 \leq n \leq N$, the minimum coprime number of $P_m + P_n$ is given by

$$\mathfrak{pr}(P_m + P_n) = \begin{cases} m + 2n - 2, & \text{if } m \text{ is odd;} \\ m + 2n - 1, & \text{if } m \text{ is even.} \end{cases}$$

The k^{th} *Ramanujan prime* R_k is the least integer for which $\pi(x) - \pi(\frac{x}{2}) \geq k$ holds for all $x \geq R_k$. Ramanujan primes were introduced in [23] as a generalization of Bertrand's postulate; they are published as sequence A104272 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [30]. We resolve Conjecture 5 affirmatively, showing the following.

Theorem 6. *For any positive integer N, if $M \geq R_{N-1} - 2N + 1$, then for all $m \geq M$ and $2 \leq n \leq N$, the minimum coprime number of $P_m + P_n$ is*

$$\mathfrak{pr}(P_m + P_n) = \begin{cases} m + 2n - 2, & \textit{if } m \textit{ is odd;} \\ m + 2n - 1, & \textit{if } m \textit{ is even.} \end{cases}$$

Our results on $P_m + P_n$ automatically yield upper bounds on the minimum coprime numbers of the complete bipartite graphs $K_{m,n}$, and our constructions also generalize to the join of two cycles and the join of a path and a cycle, enabling us to compute $\mathfrak{pr}(C_m + C_n)$, $\mathfrak{pr}(C_m + P_n)$, and $\mathfrak{pr}(P_m + C_n)$ for sufficiently large m. Our results on these classes of graphs are given in Theorems 17 and 21, and Corollaries 22 and 23.

We conclude this paper by providing an asymptotic bound for the minimum coprime number of a random subgraph, a topic which, to the best of our knowledge, has not been previously studied. Given a graph G and $p \in (0,1)$, we obtain a probability distribution G_p called a *random subgraph* by taking subgraphs of G with each edge appearing independently with probability p. When $G = K_n$, this is called the *Erdős-Rényi random graph*, denoted $G(n,p)$. We compute the following asymptotic bounds for the minimum coprime number of the Erdős-Rényi random graph.

Theorem 7. *We have*

$$\mathfrak{pr}(G(n,p)) \sim n \log n$$

almost surely, i.e., with probability tending to 1 *as* $n \to \infty$*.*

As a corollary, we observe that

Corollary 8. *For all* p*,* $G(n,p)$ *is almost surely not prime.*

1.2 Outline

In Section 2, we state several straightforward results on the minimum coprime number of an arbitrary graph which we will later use in our proofs. In Section 3, we discuss the graphs $K_n \odot \overline{K}_m$, proving Theorem 3. In Section 4, we focus on the graphs $P_m + P_n$, proving Theorem 6. We also consider the complete bipartite graphs $K_{m,n}$ and the graphs $C_m + C_n$ and $C_m + P_n$. In Section 5, we discuss the minimum coprime number of the Erdős-Rényi random graph, proving Theorem 7. We conclude by posing a number of open questions in coprime graph labeling in Section 6.

2 Preliminaries

In this section we state several propositions relating the minimum coprime number to other graph properties. Recall that if G and H are graphs with $V(G) = V(H)$ and $E(G) \subseteq E(H)$, we call H a *spanning supergraph* of G and G a *spanning subgraph* of H. The *independence number* $\alpha(G)$ of G is the size of the largest set of vertices S in $V(G)$ such that no two vertices in S are adjacent to one another.

Proposition 9. *Let* G *and* H *be graphs such that* H *is a spanning supergraph of* G*. Then* $\mathfrak{pr}(G) \leq \mathfrak{pr}(H)$*.*

Proof. Let $f : V(H) \to \{1, \ldots, \mathfrak{pr}(H)\}$ be a minimum coprime labeling of H. As $V(G) = V(H)$, f induces a coprime labeling of G. Hence $\mathfrak{pr}(G) \leq \mathfrak{pr}(H)$. □

The following corollary of Proposition 9 has been noted in numerous places in the literature.

Corollary 10. *Any spanning subgraph of a prime graph is prime. Any spanning supergraph of a nonprime graph is nonprime.*

Proposition 11. *For any graph G, $\mathfrak{pr}(G) \geq 2(|V(G)| - \alpha(G)) - 1$.*

Proof. Under any coprime labeling of G, the vertices with even labels must form an independent set. Hence at most $\alpha(G)$ even integers may be used to label G. The lower bound follows from noting that at least $|V(G)| - \alpha(G)$ odd integers must be used as labels. □

The following corollary of Proposition 11 has been stated elsewhere in the literature, for instance in [14, 29].

Corollary 12. *If G is a prime graph, then $\alpha(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.*

3 Minimum coprime numbers of $K_n \odot \overline{K}_m$

In this section we prove Theorem 3.

Proof of Theorem 3. Set $N = \max(p_{n-1}, mn+n)$. It is apparent that $\mathfrak{pr}(K_n \odot \overline{K}_m) \geq mn+n$ as $|V(K_n \odot \overline{K}_m)| = mn + n$. By Proposition 9, we have $\mathfrak{pr}(K_n \odot \overline{K}_m) \geq \mathfrak{pr}(K_n)$. It is well known that $\mathfrak{pr}(K_n) = p_{n-1}$ (see [4], for example). Therefore $\mathfrak{pr}(K_n \odot \overline{K}_m) \geq N$.

It now suffices to show that it is possible to construct a coprime labeling of $K_n \odot \overline{K}_m$ using only labels up to N. Denote $u_1, \ldots, u_n$ to be the vertices of K_n and $v_{i,1}, \ldots, v_{i,m}$ to be the vertices of the i^{th} copy of $\overline{K}_m$. Label u_1 with 1 and the remaining u_i with p_{i-1}, so that we have used the first $n-1$ primes as labels. We will label $v_{i,1}, \ldots, v_{i,m}$ with integers not exceeding $mn + n$, excluding $1, p_1, \ldots, p_{n-1}$.

After labeling the vertices of K_n as indicated, there are at least mn integers less than or equal to N which have not yet been used as labels. Let L_1 denote the list of all such integers. We label the vertices $v_{i+1,k}$ for $1 \leq i \leq n-1$ as follows: for each i, select the first m integers in L_i which are coprime to p_i (the label of u_{i+1}). Use these integers to label $v_{i+1,1}, \ldots, v_{i+1,m}$, and remove them from L_i to form a new list L_{i+1}. We may use any integers in L_n to label $v_{1,1}, \ldots, v_{1,m}$, as they are all coprime to 1, the label of u_1.

We claim that at each step of this process there exist m integers in L_i which are coprime to p_i, so that this process in fact yields a coprime labeling for $K_n \odot \overline{K}_m$ using only integers up to N. Observe that L_i contains at least $|L_1| - m(i-1) = m(n-i+1)$ elements. Suppose for the sake of contradiction that there are at most $m-1$ elements in L_i that are coprime to p_i; then the remaining $m(n-i)+1$ elements must all be multiples of p_i.

We consider the following cases. Note that by [32, 33, 4], the statement in the theorem is known for all $n \leq 3$ or $m \leq 2$, so we may assume that $n \geq 4$ and $m \geq 3$.

- If $p_i > n$, we have

$$p_i - \frac{ip_i}{n} + \frac{p_i}{mn} \geq p_i - \frac{(n-1)p_i}{n} + \frac{p_i}{mn} = \frac{p_i}{n} + \frac{p_i}{mn} > 1 + \frac{1}{m}.$$

- If $p_i \leq n$ and $i \geq 3$, we have

$$p_i - \frac{ip_i}{n} + \frac{p_i}{mn} \geq p_i - i + \frac{p_i}{mn} > 2 > 1 + \frac{1}{m},$$

as $p_i - i \geq 2$ for all $i \geq 3$.

- If $p_i \leq n$ and $i = 2$, we have

$$p_i - \frac{ip_i}{n} + \frac{p_i}{mn} = 3 - \frac{6}{n} + \frac{3}{mn} > \frac{3}{2} > 1 + \frac{1}{m}.$$

Thus, in each case, the largest integer in L_i is at least

$$p_i(mn - mi + 1) = mn\left(p_i - \frac{ip_i}{n} + \frac{p_i}{mn}\right) > mn\left(1 + \frac{1}{m}\right) = mn + n,$$

which is a contradiction. □

4 Minimum coprime numbers of $P_m + P_n$

In this section we prove Theorem 6 and discuss the minimum coprime numbers of a number of graphs related to $P_m + P_n$. We will use the following theorem and lemma.

Theorem 13 ([19], Brun-Titchmarsh theorem)**.** *Let $\pi(x; k, a)$ denote the number of primes at most x that are equivalent to a modulo k. Then*

$$\pi(x + y; k, a) - \pi(x; k, a) \leq \frac{2y}{\varphi(k)\log(y/k)}$$

whenever $y \geq k$, where φ is the Euler totient function.

Lemma 14. *For all $x \geq 1$, there is a prime $p \in (x, 2x]$ such that $p \not\equiv 1, 10 \pmod{11}$.*

Proof. By [24, Corollary 3 to Theorem 2], we have

$$\pi(2x) - \pi(x) > \frac{3}{5}\frac{x}{\log x}$$

whenever $x \geq 20.5$. Combining this with Theorem 13 shows that the total number of primes in $(x, 2x]$ which are not congruent to 1 or 10 modulo 11 is at least

$$\frac{3x}{5\log x} - \frac{2x}{5(\log x - \log 11)}.$$

When $x > 1331$, the above expression is positive, and we have the desired result. For $x \leq 1331$, we may manually verify the statement in the lemma. □

Proof of Theorem 6. Set $L = 2\left\lceil \frac{m-1}{2} \right\rceil + 2n - 1$. The lower bound $\mathfrak{pr}(P_m + P_n) \geq L$ follows from Proposition 11, observing that $\alpha(P_m+P_n) = \left\lceil \frac{m}{2} \right\rceil$. Thus, to show that $\mathfrak{pr}(P_m+P_n) = L$, it suffices to construct a coprime labeling using only labels up to L.

By Theorem 4, we know that such a labeling exists for $n \leq 4$. Assume that $n \geq 5$ and hence $L \geq R_4 = 29$. Since $L \geq R_{n-1}$, we may label the vertices of P_n with 1 and $n-1$ primes between $\left\lceil \frac{L}{2} \right\rceil$ and L, which we denote $q_1, \ldots, q_{n-1}$ in increasing order. By Lemma 14, we may assume that $q_\ell \not\equiv 1, 10 \pmod{11}$ for some $1 \leq \ell \leq n-1$. Moreover, by our assumptions on m and n, each of these primes is at least 17. As $1, q_1, \ldots, q_{n-1}$ are each coprime to all of the other integers up to L, any coprime labeling of P_m using the remaining integers will yield a minimum coprime labeling for $P_m + P_n$.

If $q_1 > m+1$ we are done, as we may simply label the vertices of P_m with $2, \ldots, m+1$ in order. Otherwise, let S_1 be the ordered list of integers up to L, excluding $\{1, q_1, \ldots, q_{n-1}\}$. We will inductively construct sets S_i so that $|S_i| = L - n - i + 1$ and every element less than q_i contained in S_i is coprime to its neighbors in S_i that are also less than q_i. It is clear that S_1 satisfies the conditions above. For $1 \leq i \leq n-2$, we construct S_{i+1} from S_i as follows.

- If $q_{i+1} > q_i + 2$ and $q_i > q_{i-1} + 2$ (if $i > 1$), then S_i contains the sequence $q_i - 2, q_i - 1, q_i + 1, q_i + 2$, where $q_i - 2$ and $q_i + 2$ are odd and composite. Observe that 3 divides at most one of $q_i - 2$ and $q_i + 2$. If $3 \nmid q_i + 2$, we can set $S_{i+1} = S_i \setminus \{q_i + 1\}$, as $\gcd(q_i - 2, q_i - 1) = \gcd(q_i - 1, q_i + 2) = 1$. Otherwise, $3 \nmid q_i - 2$, so we can set $S_{i+1} = S_i \setminus \{q_i - 1\}$, as $\gcd(q_i - 2, q_i + 1) = \gcd(q_i + 1, q_i + 2) = 1$.

- If $q_{i+1} = q_i + 2$, then it suffices to set $S_{i+1} = S_i \setminus \{q_i + 1\}$.

- If $q_{i+1} > q_i + 2$, $i > 1$, and $q_i = q_{i-1} + 2$, then S_i contains the sequence $q_i - 4, q_i - 3, q_i + 1, q_i + 2$, where $q_i - 4$ and $q_i + 2$ are odd and composite. Observe that 5 divides at most one of $q_i - 4$ and $q_i + 2$. If $5 \nmid q_i + 2$, we can set $S_{i+1} = S_i \setminus \{q_i + 1\}$, as $\gcd(q_i - 4, q_i - 3) = \gcd(q_i - 3, q_i + 2) = 1$ necessarily. Otherwise, $5 \nmid q_i - 4$, so we can set $S_{i+1} = S_i \setminus \{q_i - 3\}$, as $\gcd(q_i - 4, q_i + 1) = \gcd(q_i + 1, q_i + 2) = 1$.

We now construct a final ordered list $S_n \subseteq S_{n-1}$ such that every element in S_n is coprime to its neighbors. If $q_{n-1} = L$ we are done, as we may set $S_n = S_{n-1}$. Otherwise, since L is always odd, we have $q_{n-1} + 2 \leq L$, so S_{n-1} contains the sequence $q_{n-1} - k - 1$, $q_{n-1} - k$, $q_{n-1} + 1$, $q_{n-1} + 2$, where $k = 1$ or 3 depending on whether $q_{n-1} - 2$ is composite or prime respectively. As in the cases above, it is always possible to obtain S_n by removing one of $q_{n-1} - k$ and $q_{n-1} + 1$.

If $q_{n-1} = L$, then we have $|S_n| = |S_{n-1}| = L - 2n + 2 \geq m$, and we obtain a minimum coprime labeling for $P_m + P_n$ by labeling the vertices of P_m with the elements of S_n in order. Otherwise, we have $|S_n| = L - 2n + 1$. If m is even, $L - 2n + 1 = m$ and we again obtain a minimum coprime labeling for $P_m + P_n$. If m is odd, $L - 2n + 1 = m - 1$, and we are short of one label. We resolve this by recalling that there exists some $q_\ell \not\equiv 1, 10 \pmod{11}$. To construct S_n, we have deleted one of $q_i - 1$, $q_i + 1$ for each q_i; hence there is some $x = q_\ell \pm 1$, $11 \nmid x$ that does not appear in S_n. As $q_1 > 23$, we may label the vertices of P_m with the sequence

$$x, 11, 12, 5, 4, 3, 8, 7, 6, 13, 10, 9, 14,$$

followed by all of the elements $x \in S_n$ such that $15 \leq x \leq L$, followed by 2. This yields a minimum coprime labeling for $P_m + P_n$. □

The condition $M \geq R_{N-1} - 2N + 1$ in Theorem 6 is sufficient but certainly not necessary; by Theorem 4, it is known, for instance, that if $N = 3$ or 4, it suffices to set $M \geq N$. The following theorem extends this result to $N = 5$.

Theorem 15. *For $m \geq 5$, the minimum coprime number of $P_m + P_5$ is*

$$\mathfrak{pr}(P_m + P_5) = \begin{cases} m + 8, & \text{if } m \text{ is odd;} \\ m + 9, & \text{if } m \text{ is even.} \end{cases}$$

Proof. We prove the theorem for each case as follows.

- If $m \geq 20$, we are done by Theorem 6 as $R_4 = 29$.
- If $m = 5$, we may label the vertices of the first path with the sequence $1, 3, 5, 9, 13$, and the vertices of the second path with $2, 7, 4, 11, 8$.
- If $6 \leq m \leq 10$, we may label the vertices of P_5 with the sequence $3, 5, 9, 1, 15$, and the vertices of P_m with the first m integers in the sequence

$$2, 7, 4, 11, 8, 13, 14, 17, 16, 19.$$

- If $11 \leq m \leq 17$, we may label the vertices of P_5 with the sequence $1, 11, 13, 17, 19$, and the vertices of P_m with the first m integers in the sequence

$$2, 3, 4, 5, 6, 7, 8, 9, 14, 15, 16, 21, 10, 23, 12, 25, 24.$$

- If $m = 18$ or 19, we may label the vertices of P_5 with the sequence $1, 13, 17, 19, 23$, and the vertices of P_m with the first m integers in the sequence

$$2, 3, 4, 5, 6, 7, 26, 9, 10, 11, 14, 15, 16, 21, 22, 25, 8, 27, 20.$$

□

Our work on the join of two paths also offers insight on the complete bipartite graphs. Fu and Huang [14] proved that, for $m \leq n$, $K_{m,n}$ is prime if and only if $m \leq \pi(m+n) - \pi(\frac{m+n}{2}) + 1$. Seoud, Diab, and Elsakhawi [27] showed that $K_{2,n}$ is prime for all n and that $K_{3,n}$ is prime unless $n = 3, 7$. Berliner et al. [5] provide all values of n for $m \leq 13$ for which $K_{m,n}$ is prime and note that $K_{m,n}$ is prime for all $n \geq R_{m-1} - m$, as implied by [14]. They also ask about the behavior of $\mathfrak{pr}(K_{m,n})$ when $n < R_{m-1} - m$. By Proposition 9, Theorem 6 immediately implies the following bound on $\mathfrak{pr}(K_{m,n})$, which provides a partial answer to this question.

Corollary 16. *If* $n \geq R_{m-1} - 2m + 1$, *then*

$$\mathfrak{pr}(K_{m,n}) \leq 2\left\lceil \frac{n-1}{2} \right\rceil + 2m - 1.$$

In fact, a slight modification of the first part of the proof of Theorem 6 enables us to show that, more generally,

Theorem 17. *If* $m \leq n \leq R_{m-1} - m$, *then* $\mathfrak{pr}(K_{m,n}) \leq R_{m-1}$.

Proof. We label the vertices of $\overline{K}_m$ with 1 and the $m-1$ largest primes up to R_{m-1}, which are, in particular, at least $\left\lceil \frac{R_{m-1}}{2} \right\rceil$. Hence they are each coprime to all of the other integers up to R_{m-1}. We may use any n of the remaining integers to label the vertices of $\overline{K}_n$. □

However, we note that for $N \geq 6$, it is in general not sufficient to take $M \geq N$.

Remark 18. There exist positive integers $m \geq n$ such that

$$\mathfrak{pr}(P_m + P_n) > 2\left\lceil \frac{m-1}{2} \right\rceil + 2n - 1.$$

We now present a number of examples and counterexamples for Remark 18 that illustrate the complexity of the behavior of $\mathfrak{pr}(P_m + P_n)$ for $n \leq m \leq R_{n-1} - 2n$.

Example 19 ($n = 6$).

- For $6 \leq m \leq 9$, we have

$$\mathfrak{pr}(P_m + P_6) = \begin{cases} m + 10, & \text{if } m \text{ is odd;} \\ m + 11, & \text{if } m \text{ is even.} \end{cases}$$

  This follows from labeling the vertices of P_6 with the sequence $3, 5, 9, 1, 15, 11$ and the vertices of P_m with the first m integers in the sequence

$$2, 7, 4, 17, 8, 13, 14, 19, 16.$$

- For $m = 10$ or 11, this equality does not hold. We prove this by contradiction: if $\mathfrak{pr}(P_{10}+P_6) = \mathfrak{pr}(P_{11}+P_6) = 21$, then there exist minimum coprime labelings of $P_{10}+P_6$ and $P_{11}+P_6$ using 5 or 6 even integers respectively and each of the odd integers up to 21. The vertices of P_6 must be labeled with the set of integers $S = \{3, 5, 7, 9, 15, 21\}$, as no subset of S is pairwise coprime to its complement in S. Thus all of the integers in S must be used as labels for the same path, and there are too many odds in S to be used as labels in P_{10} or P_{11}. However, because S contains a total of 6 integers, 4 of which are multiples of 3, it is not possible to arrange the elements of S in a sequence such that adjacent elements are coprime. Therefore $\mathfrak{pr}(P_m+P_6) > 21$ for $m = 10$ or 11. The same reasoning shows that the minimum coprime numbers of $P_{10}+P_6$ and $P_{11}+P_6$ exceeds 22. In fact, we may construct labelings to show that $\mathfrak{pr}(P_{10}+P_6) = \mathfrak{pr}(P_{11}+P_6) = 23$.

Example 20 ($n = 7$)**.**

- For $m = 7$, we have $\mathfrak{pr}(P_7 + P_7) = 19$. This follows from labeling the vertices of one copy of P_7 with the sequence $3, 5, 9, 1, 15, 11, 13$, and the vertices of the second copy with the sequence $2, 7, 4, 17, 8, 19, 16$.

- For $m = 8$ or 10, we have $\mathfrak{pr}(P_m + P_7) = m + 13$. This follows from labeling the vertices of P_7 with the sequence $3, 5, 9, 7, 15, 1, 21$, and the vertices of P_m with the first m integers in the sequence

$$2, 11, 4, 13, 8, 17, 16, 19, 22, 23.$$

- For $m = 9$, the lower bound is not tight. As in Example 19, we prove this by contradiction. If $\mathfrak{pr}(P_9 + P_7) = 21$, then there would exist a minimum coprime labeling of $P_9 + P_7$ using 5 of the even integers and all of the odd integers up to 21. In particular, all the integers in the set $S = \{3, 5, 7, 9, 15, 21\}$ must be used as labels for the vertices of P_7, by the same reasoning as before. However, there are fewer than 5 even integers up to 21 that are coprime to all of the elements in S. We can construct a coprime labeling to show that $\mathfrak{pr}(P_9 + P_7) = 22$. This case demonstrates that the minimum coprime number of the join of certain paths is constrained by the even integers rather than the odds; it is also interesting that the lower bound may be tight for both $m - 1$ and $m + 1$ but fail to be tight for m.

Our methods in Theorem 6 also enable us to prove the following results on the joins of cycles and paths.

Theorem 21. *For any positive integer N, if $M \geq R_{N-1} - 2N + 1$, then for all $m \geq M$ and $n \leq N$, the minimum coprime number of $C_m + C_n$ is*

$$\mathfrak{pr}(C_m + C_n) = \begin{cases} m + 2n, & \text{if } m \text{ is odd;} \\ m + 2n - 1, & \text{if } m \text{ is even.} \end{cases}$$

Proof. Label the vertices of $P_m + P_n$ as in the proof of Theorem 6. Then the labels for P_n are all either 1 or prime, so in particular the endpoints of P_n have coprime labels and we may join them to form C_n without violating the coprime labeling condition. If m is even, then one endpoint of P_m is labeled with 2 and the other is labeled with an odd integer, so we may also join the endpoints of P_m to form C_m without violating the coprime labeling condition, obtaining a coprime labeling of $C_m + C_n$. This labeling is a minimum coprime labeling because we have $\mathfrak{pr}(C_m + C_n) \geq \mathfrak{pr}(P_m + P_n)$ by Proposition 9.

If m is odd, then $\mathfrak{pr}(C_m + C_n) \geq m + 2n$ by Proposition 11, as $\alpha(C_m + C_n) = \lfloor \frac{m}{2} \rfloor$. Recall that the ordered list S_n contains either m or $m - 1$ elements. If $|S_n| = m - 1$, the last element of S_n is $L = m + 2n - 2$. Labeling the vertices of P_m with the sequence S_n with $m + 2n$ appended and joining the endpoints of P_m to form C_m therefore yields a minimum coprime labeling for $C_m + C_n$. If $|S_n| = m$, then $p_{n-1} = L = m + 2n - 2$. We may assume that $n \geq 5$ as the other cases are known by [4]; therefore $p_1 > 17$. We consider the following cases.

- If $m+2n-4$ is composite, the last two elements of S_n are $m+2n-4$ and $m+2n-3$. Labeling the vertices of P_m with the sequence S_n, replacing $m+2n-3$ by $m+2n$, and joining the endpoints of P_m to form C_m therefore yields a minimum coprime labeling for $C_m + C_n$, as we have $\gcd(m + 2n - 4, m + 2n) = \gcd(m + 2n, 2) = 1$.

- If $m + 2n - 4$ is prime, the last two elements of S_n are $m + 2n - 6$ and $m + 2n - 5$. If $3 \nmid m + 2n$, we may replace $m + 2n - 5$ with $m + 2n$ to obtain a minimum coprime labeling for $C_m + C_n$, as above. If $3 \mid m + 2n$, we join the endpoints of P_m to form C_m, replace $m + 2n - 5$ with $m + 2n$, and rearrange the labels surrounding the endpoints to read
$$\ldots, m + 2n - 6, 2, m + 2n, 4, 3, 5, 6, \ldots.$$

□

The proof of Theorem 21 immediately implies the following two corollaries on the join of a cycle and a path.

Corollary 22. *For any positive integer N, if $M \geq R_{N-1} - 2N + 1$, then for all $m \geq M$ and $n \leq N$, the minimum coprime number of $C_m + P_n$ is*

$$\mathfrak{pr}(C_m + P_n) = \begin{cases} m + 2n, & \text{if } m \text{ is odd;} \\ m + 2n - 1, & \text{if } m \text{ is even.} \end{cases}$$

Corollary 23. *For any positive integer N, if $M \geq R_{N-1} - 2N + 1$, then for all $m \geq M$ and $n \leq N$, the minimum coprime number of $P_m + C_n$ is*

$$\mathfrak{pr}(P_m + C_n) = \begin{cases} m + 2n - 2, & \text{if } m \text{ is odd;} \\ m + 2n - 1, & \text{if } m \text{ is even.} \end{cases}$$

5 Minimum coprime number of a random subgraph

In this section, we prove Theorem 7.

Lemma 24. *Denote $n := |V(G)|$. If n is sufficiently large and $\alpha(G) < \sqrt{n}$, then $\mathfrak{pr}(G) \geq p_{n-\alpha(G)\lfloor\sqrt{n}\rfloor}$.*

Proof. Assume that we have a minimum coprime labeling of G. For any prime p, the vertices with labels which are multiples of p must form an independent set. Thus as we let p range over $p_1, \ldots, p_{\lfloor\sqrt{n}\rfloor}$ respectively, we obtain $\lfloor\sqrt{n}\rfloor$ (possibly non-disjoint) independent sets, each of which contains at most $\alpha(G)$ vertices. Hence there are at least $n - \alpha(G)\lfloor\sqrt{n}\rfloor$ vertices whose labels are not multiples of any prime up to $p_{\lfloor\sqrt{n}\rfloor}$. If any of the remaining labels exceeds p_n, then we are done. Otherwise, as the remaining labels only have prime factors larger than $p_{\lfloor\sqrt{n}\rfloor}$ and $p_{\lfloor\sqrt{n}\rfloor}^2 > p_n$ for large n, the remaining labels must all be prime. Using the next $n - \alpha(G)\lfloor\sqrt{n}\rfloor$ primes after $p_{\lfloor\sqrt{n}\rfloor}$ as labels yields the desired result. □

Proof of Theorem 7. By a celebrated result of Bollobás and Erdős [6], we have $\omega(G(n,p)) \sim 2\log_{1/p} n$ almost surely, where $\omega(G)$ denotes the *clique number* of G, i.e., the size of the largest complete subgraph of G. As $\alpha(G(n,p)) = \omega(G(n,1-p))$, by the prime number theorem, $p_{n-\alpha(G)\lfloor\sqrt{n}\rfloor} \sim n\log n$, and using Lemma 24, we have $\mathfrak{pr}(G(n,p)) \sim n\log n$ almost surely as $n \to \infty$.

□

6 Further directions

Here we pose a number of open questions in coprime graph labeling.

Question 25. We observed in Examples 19 and 20 that $\mathfrak{pr}(P_m + P_n)$ exhibits interesting behavior when $n \le m \le R_{n-1} - 2n$. Is there a nice characterization of $\mathfrak{pr}(P_m + P_n)$ in these cases, and in particular, is it possible to predict when $\mathfrak{pr}(P_m + P_n) = 2\left\lceil \frac{m-1}{2} \right\rceil + 2n - 1$?

Question 26. Is it sufficient to take $M \ge N$ in Theorem 6 for sufficiently large N? This is conjectured in [4].

Question 27. Can we improve on the bounds for $\mathfrak{pr}(K_{m,n})$ in Theorem 17? In particular, can we obtain sharper bounds on $\mathfrak{pr}(K_{n,n})$? (The first few values of $\mathfrak{pr}(K_{n,n})$ are given in sequence A213273 in the OEIS [30].)

Question 28. The Cartesian product $P_m \square P_n$ is called a *grid graph*. In particular, if $m = 2$, the graph $P_2 \square P_n$ is called a *ladder*. Dean [7] and Ghorbani and Kamali [16] showed independently that all ladders are prime, resolving a conjecture of Varkey. Other grid graphs have been shown to be prime, including $P_m \square P_n$ if $m \le n$ and n is prime [31], and a few other cases in [18]. It it true that $P_m \square P_n$ is prime for all m and n? This would settle a conjecture in [31].

Question 29. There has been a substantial amount of research conducted on the clique number and independence number of a random subgraph. Would any of these results enable us to obtain lower bounds on $\mathfrak{pr}(G_p)$ for arbitrary G?

Question 30. For arbitrary G, the trivial upper bound $\mathfrak{pr}(G_p) \le \mathfrak{pr}(G)$ is asymptotically tight, as we may observe in the case where G is prime. Is it possible to obtain a better upper bound on $\mathfrak{pr}(G_p)$ for specific classes of G?

7 Acknowledgments

This research was conducted at the University of Minnesota, Duluth REU and was supported by NSF/DMS grant 1659047 and NSA grant H98230-18-1-0010. The author would like to thank Joe Gallian for suggesting this problem, and Aaron Berger and Pat Devlin for numerous helpful conversations.

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2010 *Mathematics Subject Classification*: Primary 05C78; Secondary 05C80, 11A05.
Keywords: coprime labeling, prime labeling, prime graph, Ramanujan prime, random graph.

(Concerned with sequences A104272 and A213273.)

Received July 15 2020; revised versions received October 19 2020; November 29 2020. Published in *Journal of Integer Sequences*, December 2 2020.

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