Lens Sequences

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Abstract

We study a family of sequences produced by a non-homogeneous linear recurrence formula derived from the geometry of circles inscribed in lenses. The sequences lead to mysterious "underground" sequences underlying them. The results are consequences of the theorem on four disks in general configuration, a generalization of the celebrated Descartes theorem on circles.

1 Introduction

Advantages of the theorem on disks (or circles) in general configuration [3, 4] (the Descartes formula for four tangent disks [2, 8, 6] being a special case) are seen in the results presented in this paper. We introduce new families of integral sequences that result from a geometric construction described below. They have interesting algebraic properties, starting with a three-step non-homogeneous recurrence.
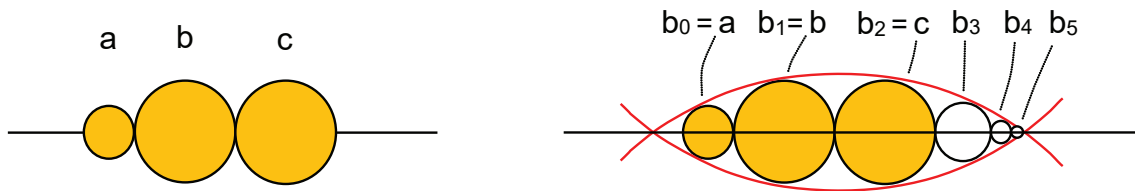

Figure 1: Three circles determine a sequence

Here is the construction illustrated in Figure 1: Start with three disks/circles of curvatures a, b, and c centered on the same line, so that pairs of consecutive circles are tangent, as in the left side of Figure 1. The three disks determine a pair of congruent circles (that is, of the same radius) that are simultaneously tangent to the original triple (right side of Figure 1). The common region formed by this pair defines a symmetric *lens*. Now, continue to inscribe circles inside the lens, as shown in the figure. The resulting chain of circles defines a bilateral sequence of curvatures $(b_n)_{n \in \mathbb{Z}}$. Sequences obtained this way will be called *lens sequences*.

It is remarkable that the sequences defined this way admit existence of integral cases. Interestingly, they include some known sequences [9], but the context presented here will reveal new properties. Especially intriguing is the fact that such sequences can be factored $b_n = f_n f_{n+1}$. The geometric meaning of the underlying sequence (f_n), called here the *underground sequence*, remains an enigma.

1.0.1 Geometry

We denote disks and their curvatures by the same letter: disk a has curvature a, i.e., radius $1/a$. The disk that extends outside of a circle (complement of the standard disk) is given negative curvature and radius. The disks D_1 and D_2, each of radius R, define the lens as the region $D_1 \cap D_2$. They do not need to be of positive curvatures. Figure 2 shows a selection of possibilities. For instance, if the curvatures D_1 and D_2 are negative, the bounding circles do not need to overlap. In this case, any integral sequence will be periodic. If the bounding circles overlap, there are two regions to consider, depending on the choice of the bounding disks, exterior vs interior.

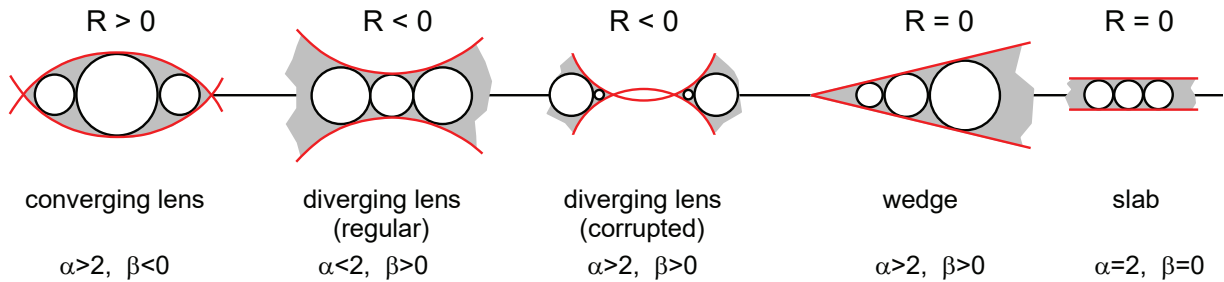| R > 0	R < 0	R < 0	R = 0	R = 0
converging lens	diverging lens (regular)	diverging lens (corrupted)	wedge	slab
α>2, β<0	α<2, β>0	α>2, β>0	α>2, β>0	α=2, β=0

Figure 2: Different types of lens sequences due to geometry of the lens

1.0.2 Terminology

We shall say that a triplet (a, b, c) *generates* the lens sequence, and we will call it a *seed* of the sequence. Notice that *any* three consecutive terms of a lens sequence may serve as a seed. The two disks defining the region of the lens will be called *bounding disks* and their boundaries—*bounding circles*.

Our opening result is this:

Theorem 1. *Let a, b and c be the curvatures of the initial three disks generating a lens sequence, $b \neq 0$. Then the sequence is determined by the following non-homogeneous three-term recurrence formula:*

$$b_n = \alpha\, b_{n-1} - b_{n-2} + \beta\,, \tag{1}$$

where α and β are constants determined by the original triple:

$$\alpha = \frac{ab + bc + ca}{b^2} - 1 \quad and \quad \beta = \frac{b^2 - ac}{b}\,. \tag{2}$$

In particular, if $b_0 = a$ and $b_1 = b$ then $b_2 = c$.

The proof of this theorem is rather involved. It uses a theorem on disks/circles in general configuration [3]. To keep a transparent flow of the exposition, the proof is moved to Section 5 at the end of the paper. Not any triple leads to an integral sequence, for instance $(3, 2, 5)$ does not. But $(4, 3, 6)$ does. One calculates $\alpha = 5$, $\beta = -5$, and the seed extends to

$$\ldots,\ 1134,\ 238,\ 51,\ 12,\ \mathbf{4},\ \mathbf{3},\ \mathbf{6},\ 22,\ 99,\ 468,\ 2236,\ \ldots$$

(The seed will often be printed in bold.) Constants α and β code the geometry of the lens. In particular

$$\cos\varphi = \frac{6 - \alpha}{2 + \alpha}, \qquad L = -\frac{2\sqrt{\alpha^2 - 4}}{\beta}, \qquad R = \frac{\alpha + 2}{\beta}$$

where φ is the angle the big circles intersect (if they do), L is the length of the lens, and R is the radius of the disk defining the lens (see also Fig. 4).

On the other hand, the sequence may be viewed as an algebraic construct, and then the constants α and β define "invariants" of the sequence understood as a process – their values may be determined from any three consecutive terms of the sequence.

Note that the lens sequences may be treated formally as defined by the properties (1) and (2) as an example of self-defined process without any reference to geometry. Although the scheme defines many new interesting sequences, many sequences known in a different context turn out to be also of this sort and gain new, unknown, properties, due to this geometric origin; besides the above recurrence formula, the following three may be added.

1.0.3   Sum from the lens length

The sum of the reciprocals of the sequence is equivalent to the sum of the radii and therefore, when doubled, coincides with the length L of the lens:

$$\text{Sum:}\qquad \sum_{n=-\infty}^{\infty} \frac{1}{b_n} \;=\; L/2 \;=\; \frac{\sqrt{\alpha^2 - 4}}{-\beta}\,. \tag{3}$$

The condition is $\alpha \geq 2$ for the sum to converge. The case $\alpha < 2$ corresponds to non-intersecting bounding circles.

1.0.4 Limit from the angle

The limit of the consecutive terms is determined by the angle under which the bounding circles intersect (in the infinitesimal neighborhood of the point of intersection the bounding circles become straight lines).

$$\text{Limit:} \qquad \lambda \;=\; \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2} \tag{4}$$

As above, the condition $\alpha \geq 2$ has to be satisfied for the limit to exist. The geometric origin lies in:

$$\lambda = \frac{1 + \sin\theta}{1 - \sin\theta} \qquad \text{and} \qquad \alpha = \lambda + \frac{1}{\lambda}, \tag{5}$$

where θ is as shown in Figure 4.

1.0.5 Underground sequences

The most surprising and enigmatic fact is the existence of "underground sequences" (f_n) such that the entries of the lens sequence are products of its two consecutive terms. (f_n) is defined by a two-step homogeneous recurrence and initial values as follows:

$$\text{Underground:} \qquad f_{n+1} = \kappa_n f_n - f_{n-1},$$

with coefficient κ alternating between two values for even and odd n, so that the lens sequence consists of products: Then the lens sequence is

$$b_n = f_n f_{n+1}.$$

The previous example has the following underground sequence

$$
\begin{array}{ccccccccccccc}
b_n: & \ldots & 1134 & 238 & 51 & 12 & \mathbf{4} & \mathbf{3} & \mathbf{6} & 22 & 99 & 468 & 2236 & \ldots \\
f_n: & \ldots & 81 & 17 & 3 & 4 & 1 & 3 & 2 & 11 & 9 & 52 & 43 & \ldots
\end{array}
$$

In the case of the primitive sequence (i.e., if $\gcd(a, b, c) = 1$), the recurrence is

$$f_n \;=\; \begin{cases} k f_{n-1} - f_{n-2}, & \text{if } n \text{ is even;} \\ s f_{n-1} - f_{n-2}, & \text{if } n \text{ is odd.} \end{cases} \tag{6}$$

where the initial values may be taken as $f_0 = \gcd(a, b)$ and $f_1 = \gcd(b, c)$, and the constants are

$$s = \frac{a + b}{(\gcd(a, b))^2}, \qquad k = \frac{b + c}{(\gcd(b, c))^2}.$$

The four integers (f_0, f_1, k, s) defining the underground sequence will be typically presented as

$$^s[f_o, f_1]^k,$$

called in the following the *symbol*. The connection between the constants of the two sequences is expressed by

$$\begin{cases} \alpha & = ks - 2; \\ \beta & = kf_1^2 + sf_0^2 - ks\,f_0 f_1. \end{cases} \tag{7}$$

The converse also holds; any sequence of the form (6) for some integers k, s, and initial values f_0 and f_1 defines a lens sequence by $b_n = f_{n-1}f_n$, satisfying (1) and (2).

Although we find some additional properties of the underground sequences, their geometric meaning of these sequences remains enigmatic.

1.1   Examples from the Apollonian Window

The Apollonian Window is a well-known Apollonian disk packing with all curvatures being integers [10, 7, 5]. Finding lens sequences in this pattern was the initial motivation for the present work. Figure 3 shows three of them, without proof. Here are the corresponding sequences.
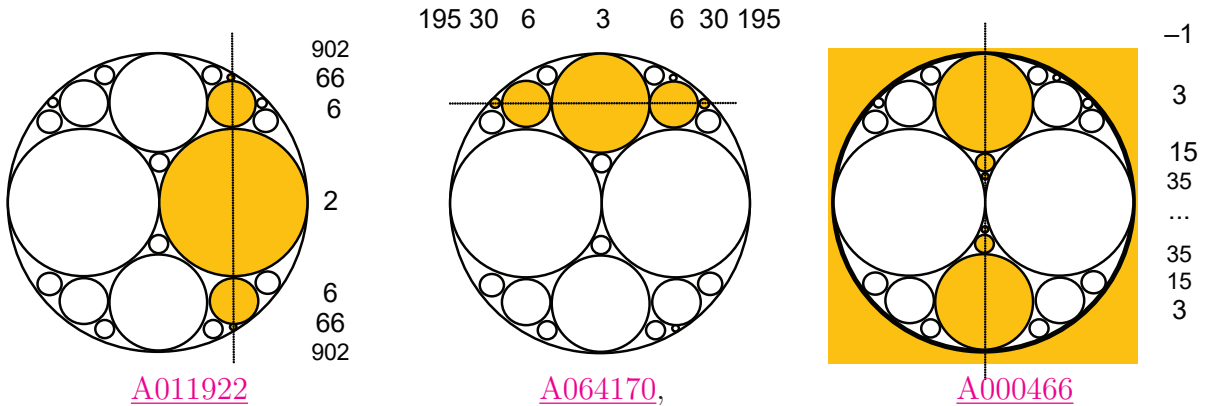

Figure 3: Lens sequences appear in the Apollonian Window. The adjacent numbers are the curvatures of the corresponding disks in the sequence.

Example 2 (Vesica Piscis). Starting with $(a, b, c) = (3, 1, 3)$ we get the recurrence formula

$$\boxed{b_n = 14b_{n-1} - b_{n-2} - 8\,,}$$

which produces

$$\dots \mathbf{3,\ 1,\ 3},\ 33,\ 451,\ 6273,\ 87363,\ 1216801,\ 16947843,\ 236052993,\ \dots,$$

The right-tail of the sequence is known as A011922.

$$\lambda = \lim_{n \to \infty} \frac{b_{n+1}}{b_n} = 7 + 4\sqrt{3} = (2 + \sqrt{3})^2$$

5

$$\Sigma_{i=-\infty}^{\infty} 1/b_i = \ldots + 1 + 1/3 + 1/33 + 1/451 + \ldots = \sqrt{3}$$

$$(f_n) = 1, 1, 3, 11, 41, \ldots \qquad f_{n+1} = 4f_n - f_{n-1}, \quad \text{symbol: } {}^4[1,1]^4$$

The sequence, when doubled, coincides with the curvatures in the chain of circles in the Apollonian Window as shown in Figure 3, left.

Example 3 (Golden Vesica)**.** Seed $(a, b, c) = (1, 2, 10)$ implies $\alpha = 7$ and $\beta = -3$. The sequence is generated by

$$\boxed{b_n = 7b_{n-1} - b_{n-2} - 3}$$

is

$$\ldots, \mathbf{1}, \mathbf{2}, \mathbf{1}0, 65, 442, 3026, 20737, 142130, 974170, \ldots$$

The right tail if this sequence is listed as Sloane's A064170. The geometry of the lens relates to the golden proportion, hence the proposed name.

$$\lambda = \lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \frac{7 + 3\sqrt{5}}{2} = \left(\frac{1 + \sqrt{5}}{2}\right)^4 = \varphi^4$$

$$L = \sqrt{5} \implies \Sigma_{i=1}^{\infty} 1/b_i = 1 + 1/2 + 1/10 + 1/65 + \ldots = (1 + \sqrt{5})/2 = \varphi$$

$$(f_n) = \ldots, 1, 1, 2, 5, 13, 34, \ldots \qquad f_{n+1} = 3f_n - f_{n-1}, \quad \text{symbol: } {}^3[1,1]^3$$

where φ stands for the golden ratio. The underground sequence are odd terms in the Fibonacci numbers: When tripled, the sequence appears in the Apollonian Window along the horizontal line including three disks $(6, 3, 6)$, see Figure 3. One may observe that the edge (left or right) of the central disk of curvature (3) defines the golden cut of the lens.

Example 4 (A000466)**.** Triplet $(-1, 3, 15)$ produces recurrence

$$\boxed{b_n = 2b_{n-1} - b_{n-2} + 8\,.}$$

$$\ldots 99, 63, 35, 15, 3, -\mathbf{1}, \mathbf{3}, \mathbf{1}5, 35, 63, 99, 143, \ldots$$

The sequence $(3, 15, 35, \ldots)$ is known as A000466 and is usually defined by $b_n = 4n^2 - 1$.

$$\lambda = \lim_{n \to \infty} \frac{b_{n+1}}{b_n} = 1$$

$$\Sigma_{n=1}^{\infty} 1/b_n = \sum_n 1/(4n^2 - 1) = 1/3 + 1/15 + 1/35 + \ldots = 1/2$$

$$(f_n) = 1, 3, 5, 7, 9, \ldots \qquad f_{n+2} = 2f_n - f_{n-1}.$$

This sequence too, appears in the Apollonian Window, as the central vertical chain of circles, Figure 3. Note that the negative term (-1) has a clear geometric meaning: it is the curvature of the greatest, external, disk of the Apollonian Window.

Example 5. A lens sequence does not necessarily need to be symmetric. For instance the triple $(2, 1, 3)$ produces the following recursion and bilateral sequence (not in the Apollonian Window):

$$\boxed{b_n = 10b_{n-1} - b_{n-2} - 5\,.} \qquad \lambda = 5 + 2\sqrt{6} \qquad L = \frac{4\sqrt{6}}{5}$$

$$\ldots, 12972, 1311, 133, 14, \mathbf{2}, \mathbf{1}, \mathbf{3}, 24, 232, 2291, 22673, 224434, \ldots$$

The radius of the bounding circles is $12/5$ and their centers are $8\sqrt{3}/5$ apart. The underground sequence is

$$(f_n) \;=\; \ldots, 19, 7, 2, 1, 1, 3, 8, 29, 79, \ldots . \qquad {}^{3}[1, 1]^{4}$$

Other integral examples include triangle numbers and one-sided power sequence (a^n). But the lens sequences include new sequences and new relations for known sequences.

2   Basic algebraic properties of lens sequences

Theorem 1 in the previous section suggests the following definition:

Definition 6. A *formal sequence* extended from a triplet (a, b, c), called a *seed*, is defined by the following non-homogeneous three-term recurrence formula:

$$b_n = \alpha\, b_{n-1} - b_{n-2} + \beta\,, \tag{8}$$

where α and β are constants determined by the original triple:

$$\alpha = \frac{ab + bc + ca}{b^2} - 1 \quad \text{and} \quad \beta = \frac{b^2 - ac}{b}\,. \tag{9}$$

and $b_0 = a$, $b_1 = b$, and consequently $b_2 = c$.

The values of the constants α and β do not depend on the particular choice of the triplet of consecutive terms as the seed. A random choice of three integers (a, b, c) will not in general lead to an integral sequence. For now, note the following:

Proposition 7. *If* $b^2 | (ab + bc + ca)$ *and* $b | ac$ *for some* $a, b, c \in \mathbb{N}$, *then a lens sequence extended from* (a, b, c) *consists of integers.*

A sequence is *primitive* if there is no common divisor of all entries other than ± 1.

7

Proposition 8. *An integer lens sequence is primitive if the common divisor of three consecutive entries is 1.*

Proof. If $\gcd(b_k, b_{k+1}, b_{k+2}) = n$ holds for some k, then it holds for all $k \in \mathbb{Z}$. Indeed, if n divides each of (b_k, b_{k+1}, b_{k+2}), then it divides β of the recurrence formula. Hence it divides the terms b_{k+2} and b_{k-1} neighboring the triple. By induction, n divides every term of the sequence. $\square$

The next proposition concerns the basic properties of lens sequences related to geometry. It is also proved in the last section:

Proposition 9. *Let (b_n) be a lens sequence. Then the following hold:*

(i) (Length and the sum) If $\alpha \geq 2$ then the sum of the reciprocals converges and equals:

$$\sum_{n=-\infty}^{\infty} \frac{1}{b_n} = \frac{L}{2} = \frac{\sqrt{\alpha^2 - 4}}{-\beta} . \tag{10}$$

(ii) (Limit and the angle) If $\alpha > 2$ then the limit of the ratios of consecutive terms exists and equals:

$$\lambda := \lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2} = \frac{\sqrt{\alpha + 2} + \sqrt{\alpha - 2}}{2} . \tag{11}$$

Proof. (i) From the geometry of lenses, cf. (39c). See also Figure 4. (ii) Divide the recurrence formula by b_{n-1} to get

$$b_n/b_{n-1} = \alpha - b_{n-2}/b_{n-1} + \beta/b_{n-1} .$$

For large values of n, if the sequence is divergent, the last term becomes irrelevant and the equation becomes $\lambda = \alpha - 1/\lambda$, or simply

$$\lambda^2 - \alpha\lambda + 1 = 0 , \tag{12}$$

with the solution as above. Figure 4 provides the geometric insight, which also relates λ to the lens angle via similar triangles. $\square$

The value of λ given in (11) will be called the *characteristic constant* of the sequence. The ring over rational numbers generated by $\sqrt{\alpha^2 - 4}$ plays an important role in other properties of lens sequences, as we shall soon see. Note that the sequence constant α may be expressed in terms of the characteristic constant in a graceful way:

$$\alpha = \lambda + \frac{1}{\lambda} , \qquad \sqrt{\alpha^2 - 4} = \lambda - \frac{1}{\lambda}$$

Remark 10 (On geometry). Moreover, if $\alpha > -2$, then the sequence may be interpreted in terms of a chain of circles inscribed in a lens made by two disks each of curvature $R^{-1} = -\beta/(\alpha + 2)$ separated by distance $\delta = 4R/\sqrt{\alpha + 2}$. Formal integer lens sequences exist also for $\alpha < -2$ (see Table 4.4 for examples), but in such a case the geometric interpretation is unclear as the distance between the lens circles becomes imaginary.

$$\frac{R-r}{R+r} = \sin\theta$$

$$\frac{\lambda-1}{\lambda+1} = \sin\theta$$

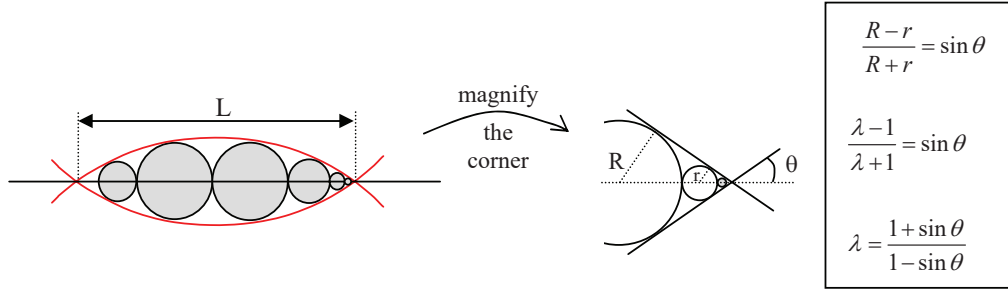$$\lambda = \frac{1+\sin\theta}{1-\sin\theta}$$

Figure 4: Geometric meaning of $\sum_n 1/b_n$ and λ (see Proposition 9)

2.1 Alternative generating formulae

There are many rearrangements of the defining recurrence formula of the lens sequences. Here are some that might be used. For more see the summary.

Proposition 11. *The sequence (b_n) satisfies a homogeneous 4-term linear recurrence formula with only α as the constant, and another quadratic formula with only β as the constant:*

$$\begin{aligned}(a) \qquad & b_n = (\alpha+1)b_{n-1} - (\alpha+1)b_{n-2} + b_{n-3}\,. \\ (b) \qquad & b_{n+1}b_{n-1} + b_n b_{n-2} = \alpha b_n b_{n-1}\end{aligned} \tag{13}$$

Proposition 12. *Constant α has an alternative form involving any four consecutive entries of a lens sequence:*

$$\alpha = \frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}}\,. \tag{14}$$

Proof. Start with the formula for β and express it as follows:

$$\beta = \frac{b_n^2 - b_{n+1}b_{n-1}}{b_n} = b_n - \frac{b_{n+1}b_{n-1}}{b_n} \quad \Rightarrow \quad b_n - \beta = \frac{b_{n+1}b_{n-1}}{b_n}\,.$$

Use the recurrence formula to modify the left-hand side of the last equation,

$$\alpha\,b_{n+1} - b_{n+2} = \frac{b_{n+1}b_{n-1}}{b_n}\,.$$

Now extract α to get (14). $\qquad\square$

Proposition 13 (Compatibility condition)**.** *Two consecutive circles a and b in a lens sequence satisfy the following condition:*

$$a^2 + b^2 = \alpha ab + \beta(a+b)\,. \tag{15}$$

Proof. Eliminate c from the expressions for α and β in (9), and simplify. $\qquad\square$

9

In general, a sequence satisfying a three-step non-homogeneous recurrence will not be necessarily a lens sequence. However:

Proposition 14. *If a sequence (b_n) satisfies these two conditions:*

$$
\begin{array}{lll}
(i) & b_n = \alpha b_{n-1} - b_{n-2} + \beta & \textit{[recurrence formula]} \\
(ii) & a^2 + b^2 = \alpha\, ab + \beta(a + b) & \textit{[compatibility relation]}
\end{array}
$$

where (ii) holds for some $(a, b) = (b_n, b_{n+1})$, then (b_n) is a lens sequence.

Proof. A simple elimination of β (respectively, α) reproduces the formula (9) for α (respectively, β). $\qquad\square$

2.2 Sequence as a process: invariants

The formulas for the coefficients α and β in the recurrence formula (8) may be represented diagrammatically as shown in Figure 5, which exhibits the coefficients' algebraic "structure" (the mnemonic role aside). The dots on the line represent the consecutive terms of the sequence. The arcs represent products of the joined terms, and the position above/below the line position indicates their appearance in the numerator/denominator of the formula. Dotted lines indicate the negative sign.
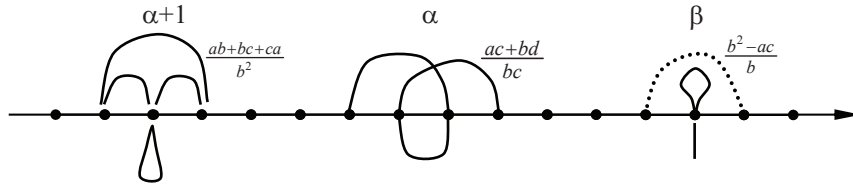

Figure 5: Diagrammatic representation of the lens sequence invariants

Since the formulas do not depend on the particular choice of the three seed circles, one may position the diagram at any place in the line/sequence. In this sense, α and β represent *invariants* of the sequence with respect to translation along the sequence. But this also means that each of them gives rise to a new non-linear recurrence formula!

2.3 Integrality

The question is how to choose seeds (a, b, c) in order to obtain lens sequences that are *integral*.

To ensure that α and β are integers, it suffices to choose (a, b, c) so that $b|ac$ and $b^2|ab + bc + ca$, or, equivalently, that $b^2|(a + b)(b + c)$. Thus triples of the form $(a, 1, c)$ always generate integer sequences for any $a, c \in \mathbb{N}$. For now, let us review the following families of integer lens sequences:

Proposition 15. *If $b^2|(b+c)$ then for any $a \in \mathbb{Z}$ the triple (a,b,c) is a seed of an integral sequence.*

Proof. Write $k = (b+c)/b^2$. This implies that $c = b^2 - kb$. The seed

$$(a, b, b^2 - kb)$$

produces the constants

$$
\begin{aligned}
\alpha &= (a+b)k - 2 \\
\beta &= (a+b) - ab\,k\,,
\end{aligned}
\tag{16}
$$

both being evidently integers. $\qquad\square$

Hence a simple method of producing integer sequence arises: choose arbitrarily a pair of integers (a, b) and some integer k. Then a seed (a, b, c) with

$$c = b(bk - 1)\,,\tag{17}$$

will generate an integer sequence. Indeed, calculate the recurrence constants α and β from (17) to get the integer values. (Only one of a or b can be negative if the sequence is to represent an actual geometric disk chain.)

Not all lens sequences are of this form. Here is an example:

$$\ldots,\ 175,\ 21,\ 6,\ 10,\ 65, 187,\ \ldots \qquad b_{n+1} = 10b_n - b_{n-1} - 29$$

It is easy to check that for no two consecutive terms the condition of Proposition 15 is satisfied.

We will arrive at a general rule that produces all integer lens sequences and an improved version of the integrality criterion in the last section.

2.4   Binet-like formula

Proposition 16. *If $\alpha \neq 2$, then the lens sequence generated from a seed (a, b, c) has the following Binet-like formula*

$$b_n = w\lambda^n + \bar{w}\bar{\lambda}^n + \gamma \tag{18}$$

where

$$\lambda = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2} \qquad \bar{\lambda} = \frac{\alpha - \sqrt{\alpha^2 - 4}}{2}.$$

and where

$$w = \frac{a - 2b + c}{2(\alpha - 2)} + \frac{c - a}{2(\alpha^2 - 4)}\sqrt{\alpha^2 - 4}, \qquad \gamma = \frac{-\beta}{\alpha - 2}$$

and $\bar{w}$ and $\bar{\lambda}$ denote conjugates of w and λ in $\mathbb{Q}(\sqrt{\alpha^2-4})$, respectively. In particular, $(a, b, c) = (b_{-1}, b_0, b_1)$.

Proof. Define a new sequence whose entries are shifted by a constant, namely $a_n = b_n + \beta/(\alpha - 2)$. The sequence (a_n) satisfies a homogeneous three-term recurrence formula $a_n = \alpha\, a_{n-1} - a_{n-2}$, which resolves to (18) by the standard procedure. $\qquad\square$

Denoting "jumps" around the central element b_0 by $\Delta_+ = b_1 - b_0$ and by $\Delta_- = b_0 - b_{-1}$, we get a more suggestive form of term w in the formula (18):

$$
\begin{aligned}
w &= \frac{\Delta_+ - \Delta_-}{2(\alpha - 2)} + \frac{\Delta_+ + \Delta_-}{2(\alpha^2 - 4)}\sqrt{\alpha^2 - 4} \\[2mm]
&= \frac{\Delta_+ - \Delta_-}{2(\lambda^{1/2} - \lambda^{-1/2})^2} + \frac{\Delta_+ + \Delta_-}{2\,|\lambda - 1/\lambda|}\,,
\end{aligned}
$$

where the second form is obtained by expressing α in terms of λ. Re-indexing the sequence (b_n) will change the value of w.

The above suggest Chebyshev polynomials as a way to "explain" the lens sequences. This path however does not seem to provide any deeper insight.

Remark 17. For the standard Binet formula in the context of Fibonacci numbers, see e.g. [1], p. 202.

Example 18. Vesica Piscis [A011922] (see Example 2): $(\ldots, \mathbf{3}, \mathbf{1}, \mathbf{3}, 33, 451, 6273, \ldots)$, $b_n = 14b_{n-1} - b_{n-2} - 8$, implies

$$
b_n = \frac{4 + \lambda^n + \bar{\lambda}^n}{6} \qquad \text{where} \qquad \lambda = 7 + 4\sqrt{3} = (2 + \sqrt{3})^2\,.
$$

The formula is centered around $b_0 = 1$.

Example 19. Golden Vesica [A064170] (see Example 3): $(\ldots \mathbf{2}, \mathbf{1}, \mathbf{2}, 10, 35, 442, \ldots)$, $b_n = 7b_{n-1} - b_{n-2} - 3$, implies

$$
b_n = \frac{3 + \lambda^n + \bar{\lambda}^n}{5} \qquad \text{where} \qquad \lambda = \frac{7 + 3\sqrt{5}}{2} = \varphi^4\,,
$$

with the the fourth power of the golden ratio as the characteristic constant of this sequence.

Example 20. To account for non-symmetric cases, consider the sequence extended from $(6, 2, 3)$:

$$
\ldots, 2346, 299, 39, \mathbf{6}, \mathbf{2}, \mathbf{3}, 15, 110, 858, 6747, \ldots
$$

The recurrence is $b_n = 8b_{n-1} - b_{n-2} - 7$ and the Binet-like formula is

$$
b_n = \frac{70 + (25 - 3\sqrt{15})\lambda^n + (25 + 3\sqrt{15})\bar{\lambda}^n}{60} \qquad \text{where} \qquad \lambda = 4 + \sqrt{15}\,.
$$

3 Underground sequences

The main mystery in the structure of lens sequences and one of the most remarkable properties is this: the entries of an integer lens sequence are products of consecutive pairs of a certain "underlying" integer sequence. Here is an example of a sequence of *Vesica Piscis* (A011922, see Example 2 in Section 1):

$$b_n: \quad \ldots \; \underset{-1}{3} \quad \underset{0}{1} \quad \underset{1}{3} \quad \underset{2}{33} \quad 451 \quad 6273 \quad 87363 \quad 1216801 \; \ldots$$

$$\Uparrow \text{multiply}$$

$$f_n: \qquad \underset{-1}{1} \quad \underset{0}{1} \quad \underset{1}{3} \quad \underset{2}{11} \quad 41 \quad 153 \quad 571 \quad 2131$$

Thus the lens sequence may be represented as $b_n = f_{n-1}f_n$, for some integer sequence (f_n). The sequence (f_n) will be called in this context the *underground* sequence of sequence (b_n) (Table 1 provides examples). We present the formalism of this amazing and unexpected property.

Let us start with a general fact.

Proposition 21. *Every factorization $\{f_n\}$ of lens sequence in a sense that $b_n = f_{n-1}f_n$, satisfies 3-term recurrence formula*

$$f_{n+2} + f_{n-2} = \alpha f_n \tag{19}$$

Proof. Starting with the expression for α given in Proposition 12, we have

$$\alpha = \frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}} = \frac{f_{n-2}f_{n-1}}{f_{n-1}f_n} + \frac{f_{n+1}f_{n+2}}{f_n f_{n+1}} = \frac{f_{n-2}}{f_n} + \frac{f_{n+2}}{f_n} = \frac{f_{n-2} + f_{n+2}}{f_n}.$$

$\square$

This property is true for *any* —not necessarily integer— factorization of (b_n). Such factorization is easy to produce, e.g., set $f_0 = 1$, $f_1 = b_1$, $f_2 = b_2/b_1$, $f_3 = b_3 b_1/b_0$, $f_4 = b_4 b_2 b_0/b_3 b_1$, etc. However:

Theorem 22 (Factorization theorem). *Every integer lens sequence (b_n) may be factored into an integer sequence (f_n) so that $b_n = f_{n-1}f_n$. If the lens sequence is primitive, the factorization is (up to a sign) unique. Moreover, in such a case $|f_n| = \gcd(b_n, b_{n+1})$.*

Proof. Assume that (b_n) is a primitive lens sequence. Consider three consecutive terms (a, b, c) of the lens sequence and define

$$f_0 = \frac{a}{\gcd(a,b)}, \quad f_1 = \gcd(a,b), \quad f_2 = \frac{b}{f_1} = \frac{b}{\gcd(a,b)}, \quad f_3 = \frac{c}{f_2} = \frac{\gcd(a,b)\,c}{b}$$

13

Clearly, $a = f_0 f_1$, $b = f_1 f_2$, and $c = f_2 f_3$. We need to show that these four terms are integers. Terms f_0, f_1, and f_2 are integers by definition. As to the last term, use the formula $\beta = \frac{b^2 - ac}{b}$:

$$\beta \in \mathbb{Z} \quad \Rightarrow \quad \frac{ac}{b} \in \mathbb{Z} \quad \Rightarrow \quad \frac{\gcd(a,b)\, c}{b} \in \mathbb{Z}$$

hence f_3 is an integer. Thus $f_0, f_1, f_2, f_3 \in \mathbb{Z}$ and the integrality of the whole sequence (f_n) follows immediately from (19).

As to uniqueness of factorization of a primitive lens sequence, assume *a contrario* that two integer quadruples, (f_0, f_1, f_2, f_3) and (g_0, g_1, g_2, g_3), are the initial terms of two different factorizations of (b_n). Then $g_0/f_0 = p/q$ for some coprimes $p, q \in \mathbb{Z}$. At least one of p and q is not 1; assume that it is $q \neq 1$. Since $g_i g_{i+1} = f_i f_{i+1}$, we must have

$$(g_0, g_1, g_2, g_3) = \left(\frac{p}{q} f_0, \ \frac{q}{p} f_1, \ \frac{p}{q} f_2, \ \frac{q}{p} f_3 \right) \ \in \mathbb{Z}^4 .$$

Thus $q|f_0$ and $q|f_2$ (because $\gcd(p,q) = 1$). But this means that $q \mid a$ (since $a = f_0 f_1$), $q \mid b$ (since $b = f_1 f_2$), and $q \mid c$ (since $c = f_2 f_3$), against the assumption of primitivity of the lens sequence. $\qquad \square$

The 3-term recurrence (19) for the underground sequence involves only α. Another interesting non-linear 4-term recurrence involves only β:

Proposition 23. *Every underground sequence (f_i) of a lens sequence (b_i) satisfies the following quadratic recurrence formula:*

$$\det \begin{bmatrix} f_n & f_{n+1} \\ f_{n+2} & f_{n+3} \end{bmatrix} \equiv f_{n+3} f_n - f_{n+1} f_{n+2} = -\beta. \tag{20}$$

Proof. For any n we have

$$\begin{aligned}
f_n f_{n+3} - f_{n+1} f_{n+2} &= \frac{f_n f_{n+1} f_{n+2} f_{n+3}}{f_{n+1} f_{n+2}} - f_{n+1} f_{n+2} \\
&= \frac{b_{n+1} b_{n+3}}{b_{n+2}} - b_{n+2} = \frac{b_{n+1} b_{n+3} - b_{n+2}^2}{b_{n+2}} = -\beta .
\end{aligned}$$

$\qquad \square$

Note that not every initial quadruple (f_1, f_2, f_3, f_4) leads via recurrence (19) to an underground sequence of a lens sequence. When do they? First, we notice that the underground sequences of lens sequences have an interesting anatomy. It turns out that they are determined by three-term linear recurrences with a "variable constant". Here is the main theorem for the underground sequences:

Theorem 24 (Underground sequence structure). *(i) Let $k, s \in \mathbb{Z}$. Define a sequence (f_n) by*

$$f_n = \begin{cases} k f_{n-1} - f_{n-2}, & \text{if } n \text{ is even;} \\ s f_{n-1} - f_{n-2}, & \text{if } n \text{ is odd.} \end{cases} \tag{21}$$

with some arbitrary initial terms $f_0, f_1 \in \mathbb{Z}$. Define $b_n = f_{n-1} f_n$. Then (b_n) is a lens sequence

$$b_n = \alpha \, b_{n-1} - b_{n-2} + \beta \tag{22}$$

with

$$\begin{cases} \alpha & = ks - 2; \\ \beta & = k f_1^2 + s f_0^2 - ks \, f_0 f_1 \, . \end{cases} \tag{23}$$

(ii) Every lens sequence is of such type. In particular, for a primitive lens sequence with a seed $(b_{-1}, b_0, b_1) = (a, b, c)$, the underground sequence is defined as is in (21), where

$$f_0 = \gcd(a, b), \qquad f_1 = \gcd(b, c),$$
$$s = \frac{a + b}{f_0^2}, \qquad k = \frac{b + c}{f_1^2}.$$

Proof. We start with part (i). Let (f_n) be a sequence defined by (21). First, we shall show that the following expression

$$\Delta_n = \det \begin{bmatrix} f_n & f_{n+1} \\ f_{n+2} & f_{n+3} \end{bmatrix} \tag{24}$$

is an invariant of sequence (f_n), that is it does not depend on n. Indeed, let p denote k or s, depending on whether n is even or odd (it will not matter!). Then

$$\Delta_n = \det \begin{bmatrix} f_n & f_{n+1} \\ f_{n+2} & f_{n+3} \end{bmatrix} = \det \begin{bmatrix} f_n & p f_n - f_{n-1} \\ f_{n+2} & p f_{n+2} - f_{n+1} \end{bmatrix}$$
$$= \det \begin{bmatrix} f_n & -f_{n-1} \\ f_{n+2} & -f_{n+1} \end{bmatrix} = \det \begin{bmatrix} f_{n-1} & f_n \\ f_{n+1} & f_{n+2} \end{bmatrix} = \Delta_{n-1} \, .$$

Thus, by induction, the value of Δ_n does not depend on n and may be denoted by Δ. Hence it may be brought down to our initial terms of the sequence and then, after another substitutions (21), shown to be

$$\Delta = k f_1^2 + s f_0^2 - ks \, f_0 f_1 \, .$$

As to the recurrence formula, calculate the sum $b_{n-1} + b_{n+1}$ (assume even n; the odd case

goes analogously):

$$
\begin{aligned}
b_{n+1} + b_{n-1} &= f_n f_{n+1} + f_{n-2} f_{n-1} = (k f_{n-1} - f_{n-2})(s f_n - f_{n-1}) + f_{n-2} f_{n-1} \\
&= ks\, f_{n-1} f_n - k f_{n-1}^2 - s f_n f_{n-2} + f_{n-1} f_{n-2} + f_{n-1} f_{n-2} \\
&= ks\, f_{n-1} f_n - f_{n-1}(k f_{n-1} - f_{n-2}) - f_{n-2}(s f_n - f_{n-1}) \\
&= ks\, f_{n-1} f_n - f_{n-1} f_n - f_{n-2} f_{n+1} \\
&= ks\, f_{n-1} f_n - 2 f_{n-1} f_n + f_{n-1} f_n - f_{n-2} f_{n+1} \\
&= (ks - 2)\, f_{n-1} f_n + (f_{n-1} f_n - f_{n-2} f_{n+1}) \\
&= (ks - 2)\, b_n - \Delta \, .
\end{aligned}
\tag{25}
$$

Now, rename $ks - 2 = \alpha$, and $\Delta = -\beta$ (see Prop. 23). Then (25) becomes an non-homogeneous three-term recurrence formula

$$
b_{n+1} + b_{n-1} = \alpha\, b_n + \beta \, .
$$

Finally, we need to show that (b_n) is actually a lens sequence. As for α, we need simply to show

$$
\alpha = \frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}}
$$

(see Proposition 12). Consider the right hand side:

$$
\frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}} = \frac{f_{n-2} f_{n-1}}{f_{n-1} f_n} + \frac{f_{n+1} f_{n+2}}{f_n f_{n+1}} = \frac{f_{n-2}}{f_n} + \frac{f_{n+2}}{f_n} = \frac{f_{n-2} + f_{n+2}}{f_n} = \alpha \, ,
$$

where the last step is true because of (19). Indeed, assume that n is even:

$$
\begin{aligned}
\frac{f_{n-2} + f_{n+2}}{f_n} &= \frac{f_{n-2} + k f_{n+1} - f_n}{f_n} \\
&= \frac{f_{n-2} + k(s f_n - f_{n-1}) - f_n}{f_n} \\
&= \frac{ks f_n - k f_{n-1} + f_{n-2} - f_n}{f_n} \\
&= \frac{ks f_n - f_n - f_n}{f_n} \\
&= ks - 2 = \alpha \, .
\end{aligned}
$$

As to the formula for β in terms of a seed, follow similar calculations as in the proof of Proposition 23:

$$
\beta = -\Delta = -\det \begin{bmatrix} f_0 & f_1 \\ f_2 & f_3 \end{bmatrix} = f_1 f_2 - f_0 f_3 = \frac{(f_1 f_2)^2 - f_0 f_1 f_2 f_3}{f_1 f_2} = \frac{b_2^2 - b_1 b_3}{b_2} \, ,
$$

This ends the proof of (i). The proof of (ii) follows easily. Let (a, b, c) be a primitive seed of an integer sequence. Define a quadruple of numbers (f_0, f_1, f_2, f_3) by setting $f_1 = \gcd(a, b)$, $f_2 = \gcd(b, c)$, and $f_0 = a / f_1$, and $f_3 = c / f_2$. Then $k = (f_0 + f_2)/f_1$ and $s = (f_1 + f_3)/f_2$. $\quad\square$

Remark 25. Note that Δ may be viewed as a quadratic form given by the matrix

$$G = \begin{bmatrix} k & ks \\ 0 & s \end{bmatrix}$$

evaluated on the vector $\mathbf{v} = [f_0, f_1]^T$, i.e., $\Delta = \mathbf{v}^T G \mathbf{v}$. In particular, vectors $[f_{2n}, f_{2n+1}]^T \in \mathbb{R}^2$ all stay on a quadric defined by G. Exchange k and s for vectors $[f_{2n+1}, f_{2n}]^T$.

The above implies a simple criterion on whether a triple of integers is a good candidate for an integer lens sequence:

Theorem 26 (Integrality Criterion). *A triple $(a, b, c) \in \mathbb{Z}^3$ is a seed of an integer lens sequence iff*

$$(i) \ \gcd(a, b) \cdot \gcd(b, c) = b \cdot \gcd(a, b, c)$$
$$(ii) \ (\gcd(a, b))^2 \ \text{divides} \ (a + b) \cdot \gcd(a, b, c)$$
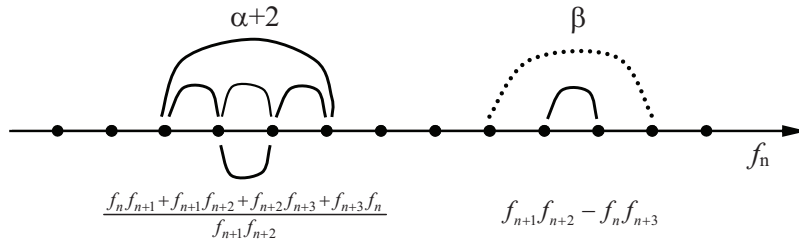$$(iii) \ (\gcd(b, c))^2 \ \text{divides} \ (b + c) \cdot \gcd(a, b, c)$$

Figure 6: Diagrammatic representation of the lens sequence invariants, calculated from the underground sequence (cf. Fig. 5). Dotted arc indicates negative sign.

The above result leads to another property of lens sequences:

Corollary 27. *The sum of any pair of consecutive terms of a lens sequence is a multiple of a square, namely:*
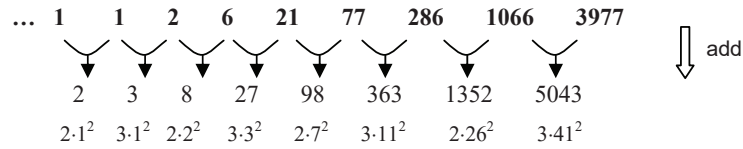
$$b_n + b_{n+1} = \begin{cases} k \cdot f_n^2, & \text{if } n \text{ is even;} \\ s \cdot f_n^2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Elementary: $b_n + b_{n+1} = f_n f_{n-1} + f_{n+1} f_n = f_n(f_{n+1} + f_{n-1}) = f_n p f_n$, where p stands for k or s, depending on the parity of n. $\qquad\square$

For example, the sums of two consecutive entries of A011922 are perfect squares:

17

The sequence [A101265](#) gives the following:

$$\ldots \quad 1 \quad 1 \quad 2 \quad 6 \quad 21 \quad 77 \quad 286 \quad 1066 \quad 3977$$

(with downward "add" arrows combining adjacent terms)

$$2 \quad 3 \quad 8 \quad 27 \quad 98 \quad 363 \quad 1352 \quad 5043$$

$$2 \cdot 1^2 \quad 3 \cdot 1^2 \quad 2 \cdot 2^2 \quad 3 \cdot 3^2 \quad 2 \cdot 7^2 \quad 3 \cdot 11^2 \quad 2 \cdot 26^2 \quad 3 \cdot 41^2$$

3.1 Generating lens sequences

Let us return to the question of generating integer lens sequences. The existence of underground sequences allows one to label all lens sequences.

Definition 28. A *symbol* of a lens sequence is the quadruple

$$^s(p, q)^k$$

which defines the underground sequence (f_i) with $f_0 = p$, $f_1 = q$, and with constants s and k as in (21), and therefore defines the corresponding lens sequence (b_i), namely, $b_i = f_{i-1} f_i$. More directly, symbol $^s(p, q)^k$ defines a lens sequence via its seed $(a, b, c) = ((sp - q)p, pq, q(kq - p))$.

Every integer quadruple $^s(p, q)^k$ leads to an integer lens sequence. And vice versa, given a seed of a primitive sequence (a, b, c), we easily reproduce the symbol:

$$p = \gcd(a, b), \qquad k = \frac{b + c}{q^2},$$

$$q = \gcd(b, c), \qquad s = \frac{a + b}{p^2}.$$

Proposition 29. *The lens sequence generated by $^s(p, q)^k$ is primitive if and only if*

$$\gcd(p, q) = \gcd(p, k) = \gcd(s, q) = 1. \tag{26}$$

Proof. Write the "central" four terms of the underground sequences and the corresponding lens sequence:

$$(f_i): \quad \ldots, f_{-1} = (sp - q), f_0 = p, f_1 = q, f_2 = (kq - p), \ldots$$
$$(b_i): \quad \ldots, a = (sp - q)p, b = pq, c = q(kq - p), \ldots$$

For (b_i) to be primitive we must have $\gcd(a, b, c) = 1$, which implies (26). $\qquad \square$

Remark 30. To use this generator of sequences as a unique *label* system for lens sequences, one would have to remove the ambiguity of the choice of the initial terms. We may demand that, say, $p = f_0$ has the smallest absolute value among (f_i) and that $|f_{-1}| > f_0 \leqslant f_1$.

3.1.1 Remark on diagrammatic use of symbols

The first of the following two diagrams means that 13 is obtained as $13 = 5 \times 3 - 2$. The second represents equation $2 = 5 \times 3 - 2$:

Now, the symbol $^1(2, 3)^5$ may be graphically developed into an underground sequence (f_i), and consequently into a lens sequence (b_i), in the following way:
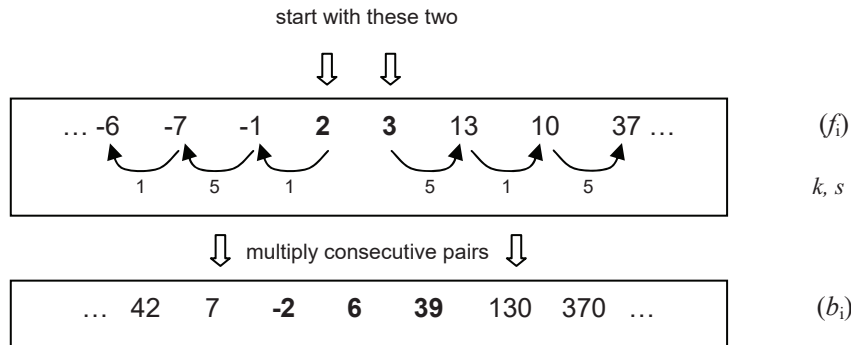
Note that the four central terms of (f_i) suffice to generate the three central terms of (b_i), which yield the constants α and β. The recurrence formula for sequence (f_i) is bilateral and may be represented diagrammatically as shown below:

19
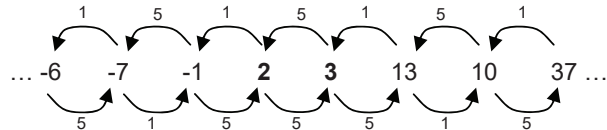
Example 1 (Vesica Piscis): $\mathbf{b} = $ <u>A011922</u> $\mathbf{f} = $ <u>A001835</u> $= $ <u>A079935</u>
$\{b_i\} = (..., 1, 3, 33, 451, 6273, 87363, \dots),$ $b_{n+1} = 14b_n - b_{n-1} - 8$
$\{f_i\} = (..., 1, 3, 11, 41, 153, 571, \dots),$ $^4(1,1)^4$ (number of domino packings in a $(3 \times 2n)$ rectangle)

Example 2 (Golden Vesica): $\mathbf{b} = $ <u>A064170</u> $\mathbf{f} = $ <u>A001519</u>
$\{b_i\} = (..., 1, 2, 10, 65, 442, 30, 26, 20737, \dots)$ $b_{n+1} = 7b_n - b_{n-1} - 3$
$\{f_i\} = (..., 1, 2, 5, 13, 34, 89, 233, \dots),$ $^3(1,1)^3$ (odd Fibonacci numbers)

Example 3: $\mathbf{b} = $ <u>A000466</u> $\mathbf{f} = $ <u>A005408</u>
$\{b_i\} = (..., -1, 3, 15, 35, 63, 99, 143, 195, 255, \dots)$ $b_{n+1} = 2b_n - b_{n-1} + 8$
$\{f_i\} = (... - 1, 1, 3, 5, 7, 9, 11, 13, 15, \dots),$ $^2(-1,1)^2$ (odd numbers)

Example 4: $\mathbf{b} = $ <u>A081078</u> $\mathbf{f} = $ <u>A002878</u>
$\{b_i\} = (..., -1, 4, 44, 319, 2204, \dots)$ $b_{n+1} = 7b_n - b_{n-1} + 15$
$\{f_i\} = (... - 1, 1, 4, 11, 29, 76, 199, 521, 1364 \dots),$ $^3(-1,1)^3$ (odd Lucas numbers)

Example 5: $\mathbf{b} = $ <u>A005247</u> $\mathbf{f} = $ <u>A005247</u>
$\{b_i\} = (..., 2, 2, 3, 6, 14, 35, 90, 234, 611, 1598, \dots)$ $b_{n+1} = 3b_n - b_{n-1} - 1$
$\{f_i\} = (..., 2, 1, 3, 2, 7, 5, 18, 13, 47, 34, \dots),,$ $^1(2,1)^5$
(even Lucas numbers interlaced with odd Fibonacci numbers)

Example 6: $\mathbf{b} = $ <u>A027941</u> $\mathbf{f} = $ <u>A005013</u>
$\{b_i\} = (..., 0, 1, 4, 12, 33, 88, 232, 609, \dots)$ $b_{n+1} = 3b_n - b_{n-1} + 1$
$\{f_i\} = (..., 0, 1, 1, 4, 3, 11, 8, 29, 21, 76, \dots),,$ $^1(1,1)^5$
(odd Lucas numbers interlaced with even Fibonacci numbers)

Example 7: $\mathbf{b} = $ <u>A000217</u> $\mathbf{f} = $ <u>A026741</u>
$\{b_i\} = (..., 0, 1, 3, 6, 10, 15, 21, 28 \dots)$ $b_{n+1} = 2b_n - b_{n-1} + 1$
$\{f_i\} = (..., 0, 1, 1, 3, 2, 5, 3, 7, 4, 9, \dots),$ $^4(0,1)^1$
(the sequence of natural numbers interlaced with odd natural numbers)

Table 1: Examples of underground sequences $\mathbf{f}$ for some integer lens sequences $\mathbf{b}$.

3.2   Towards the meaning of the underground sequence

Finally, let us consider yet another recurrence formula for lens sequences and its surprising context. Let us start with this:

Proposition 31. *Lens sequences obey the following nonlinear 4-step recurrence formula:*

$$b_{n+2} = \frac{(b_{n+1} - \beta)(b_n - \beta)}{b_{n-1}} . \tag{27}$$

Proof. Rewrite the definition of β in the form $ac = b(b - \beta)$. Since the three consecutive terms (a, b, c) may start with any entry of the sequence, let us write its two instances as follows:

$$\begin{cases} b_{n+1}b_{n-1} = b_n(b_n - \beta) \\ b_{n+2}b_n = b_{n+1}(b_{n+1} - \beta) \end{cases}$$

Multiply side-wise to get $b_{n+1}b_{n-1}b_{n+2}b_n = (b_n - \beta)(b_{n+1} - \beta)b_n b_{n+1}$. Canceling the repeated terms results in

$$b_{n-1}b_{n+2} = (b_n - \beta)(b_{n+1} - \beta),$$

which is equivalent to (27). □

Corollary 32. *A lens sequence satisfies the following identity:*

$$\det \begin{bmatrix} b_{n+1} - \beta & b_{n+2} \\ b_{n-1} & b_n - \beta \end{bmatrix} = 0\,.$$

This leads to yet another implication. The above determinant looks like a characteristic equation with β playing the role of the eigenvalue. Note that it does not depend on n. What are the corresponding eigenvectors?

Theorem 33. *If (f_n) is the underground sequence of a lens sequence (b_n), then the following "eigen-equation" holds:*

$$\begin{bmatrix} b_{n+1} & b_{n+2} \\ b_{n-1} & b_n \end{bmatrix} \begin{bmatrix} f_{n+1} \\ -f_{n-1} \end{bmatrix} = \beta \begin{bmatrix} f_{n+1} \\ -f_{n-1} \end{bmatrix}. \tag{28}$$

Proof. Write b's in terms of f's, evaluate the left hand side, and use the formula (20) to factor out β in each vector entry. □

It is an intriguing result, yet even more puzzling is that the second eigenvector of this matrix is also composed of the terms of (f_n):

$$\begin{bmatrix} b_{n+1} & b_{n+2} \\ b_{n-1} & b_n \end{bmatrix} \begin{bmatrix} f_{n+2} \\ f_{n-2} \end{bmatrix} = h_n \begin{bmatrix} f_{n+2} \\ f_{n-2} \end{bmatrix}, \tag{29}$$

except that the eigenvalues change with n, namely $h_n = \kappa_n f_{n-1} f_{n+1}$ were κ_n is k or s for n even or odd, respectively. The spectral decomposition readily follows.

The full meaning of the underground sequences remains to be understood.

4 Examples of lens sequences and their underground sequences

The following tables show a handful of examples of lens sequences. The first shows bilateral sequences that are not symmetric. The entries (1.a) and (1.b) are reflections of each other. The next table shows symmetric sequences. Some of them, when curtailed to only one wing, are known from other contexts. Here they are juxtaposed, revealing their affinity.

Finally, examples of periodic sequences are shown, and then formal sequences that do not correspond to a simple geometry of lens.

	seed	constants	sequence	symbol
1a.	$(3,1,2)$	$\alpha = 10,\ \beta = -5$	$\ldots, 24, 3, 1, 2, 14, 133, 1311, 12972, 128404, 1271063,\ldots$	$^3(1,2)^4$
b.	$(2,1,3)$		$\ldots, 14, 2, 1, 3, 24, 232, 2291, 22673, 224434, 2221662,\ldots$	$^4(1,3)^3$
2.	$(5,3,6)$	$\alpha = 6,\ \beta = -7$	$\ldots, 108, 20, 5, 3, 6, 26, 143, 825, 4800, 27968,\ldots$	$^8(1,3)^1$
3.	$(3,-1,4)$	$\alpha = 4,\ \beta = 11$	$\ldots, 403, 104, 24, 3, -1, 4, 28, 119, 459, 1728,\ldots$	$^3(1,4)^2$
4.	$(15,12,20)$	$\alpha = 4,\ \beta = -13$	$\ldots, 400, 112, 35, 15, 12, 20, 55, 187, 680, 2520,\ldots$	$^2(4,5)^3$
5.	$(21,6,10)$	$\alpha = 10,\ \beta = -29$	$\ldots, 16796, 1700, 175, 21, 6, 10, 65, 611, 6016, \ldots$	$^4(2,5)^3$
6.	$(1,2,4)$	$\alpha = 5/2,\ \beta = 0$	$\ldots, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots$	$^3(1,2)^{2/3}$

Table 2: Examples of non-symmetric sequences. Example 6 (A000079) is integer in one direction only.

	central elements	constants	OEIS nr	sequence	symbol
Central sequences					
1.	$(2,1,2)$	$\alpha = 7,\ \beta = -3$	A064170	$1, 2, 10, 65, 442, 3026, 20737, 142130,\ldots$	$^3(1,1)^3$
2.	$(3,1,3)$	$\alpha = 14,\ \beta = -8$	A011922	$1, 3, 33, 451, 6273, 87363, 1216801,\ldots$	$^4(1,1)^4$
3.	$(4,1,4)$	$\alpha = 23,\ \beta = -15$	—	$1, 4, 76, 1729, 39676, 910804, 20908801,\ldots$	$^5(1,1)^5$
4.	$(5,\ 1,\ 5)$	$\alpha = 34,\ \beta = -24$	—	$1, 5, 145, 4901, 166465, 5654885, 192099601,\ldots$	$^6(1,1)^6$
5.	$(3,-1,3)$	$\alpha = 2,\ \beta = 8$	A000466	$-1, 3, 15, 35, 63, 99, 143, 195, 255, 323, 399,\ldots$	$^2(1,3)^2$
6.	$(4,-1,4)$	$\alpha = 7,\ \beta = 15$	A081078	$-1, 4, 44, 319, 2204, 15124, 103679, 710644,\ldots$	$^3(1,4)^3$
7.	$(5,-1,5)$	$\alpha = 14,\ \beta = 24$	—	$-1, 5, 95, 1349, 18815, 262085, 3650399$	$^4(1,5)^4$
8.	$(6,-1,6)$	$\alpha = 23,\ \beta = 35$	—	$-1, 6, 174, 4031, 92574, 2125206, 48787199,\ldots$	$^5(1,6)^5$
Bicentral sequences					
9.	$(2,1,1,2)$	$\alpha = 4,\ \beta = -1$	A101265	$1, 2, 6, 21, 77, 286, 1066, 3977, 14841,\ldots$	$^2(1,1)^3$
10.	$(3,1,1,3)$	$\alpha = 6,\ \beta = -2$	A011900	$1, 3, 15, 85, 493, 2871, 16731, 97513, 568345,\ldots$	$^2(1,1)^4$
11.	$(4,1,1,4)$	$\alpha = 8,\ \beta = -3$	—	$1, 4, 28, 217, 1705, 13420, 105652, 831793,\ldots$	$^2(1,1)^5$
12.	$(5,1,1,5)$	$\alpha = 10,\ \beta = -4$	A054318	$1, 5, 45, 441, 4361, 43165, 427285, 4229681,\ldots$	$^2(1,1)^6$
13.	$(3,2,2,3)$	$\alpha = 3,\ \beta = -1$	A032908	$2, 3, 6, 14, 35, 90, 234, 611, 1598, 4182, 10947,\ldots$	$^5(1,2)^1$
14.	$(5,2,2,5)$	$\alpha=5,\ \beta = -3$	—	$2, 5, 20, 92, 437, 2090, 10010, 47957, 229772,\ldots$	$^7(1,2)^1$
15.	$(0,0,1,3)$	$\alpha = 2,\ \beta = 1$	A000217	$0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66,\ldots$	$^1(1,1)^4$
16.	$(0,0,1,4)$	$\alpha = 3,\ \beta = 1$	A027941	$0, 1, 4, 12, 33, 88, 232, 609, 1596, 4180,\ldots$	$^1(1,1)^5$
17.	$(0,0,1,5)$	$\alpha = 4,\ \beta = 1$	A061278	$0, 1, 5, 20, 76, 285, 1065, 3976, 14840, 55385,\ldots$	$^1(1,1)^6$

Table 3: Examples of symmetric lens sequences

22

	seed	constants	sequence	symbol
1.	$(2, -1, 2)$	$\alpha = -1, \ \beta = 3$	$2, 2, -1, 2, 2, -1, 2, 2, -1, 2, 2, -1, \ldots$	$^1(1,2)^1$
2.	$(3, -1, 2)$	$\alpha = 0, \ \beta = 5$	$2, 6, 3, -1, 2, 6, 3, -1, 2, 6, 3, -1, \ldots$	$^1(1,2)^2$
3.	$(14, -6, 15)$	$\alpha = 0, \ \beta = 29$	$14, -6, 15, 35, 14, -6, 15, 10, 35, \ldots$	$^2(5,7)^1$
4.	$(1, 1, 0)$	$\alpha = 0, \ \beta = 1$	$1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, \ldots$	$^1(1,1)^2$
5.	$(4, -1, 2)$	$\alpha = 1, \ \beta = 7$	$2, 10, 15, 12, 4, -1, 2, 10, 15, 12, 4, -1, \ldots$	$^1(1,2)^3$
6.	$(10, -6, 33)$	$\alpha = 1, \ \beta = 49$	$33, 88, 104, 65, 10, -6, 33, 88, 104, 65, 10, -6, \ldots$	$^3(5,2)^1$

Table 4: Examples of periodic sequences. Example 4 is known as [A021913](#).
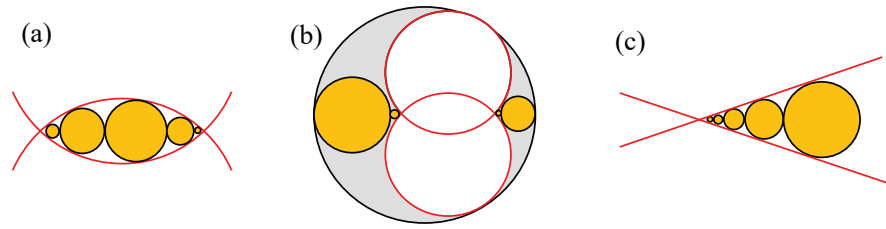

Figure 7: Geometric representations for examples from Table 2: (a) Examples 1–2 (b) Example 3, (c) Example 4.
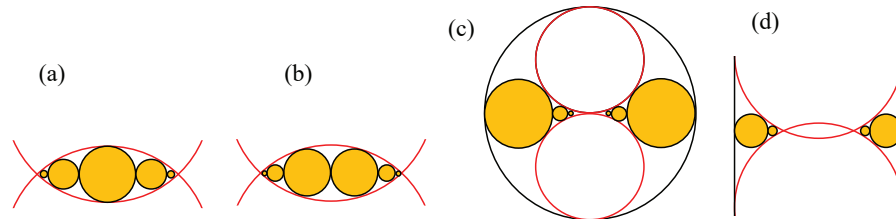

Figure 8: Geometric representations (only general shape) for examples from Table 3: (a) Examples 1–4, (b) Examples 9–14, (c) Examples 5–8, (d) Examples 15–17.
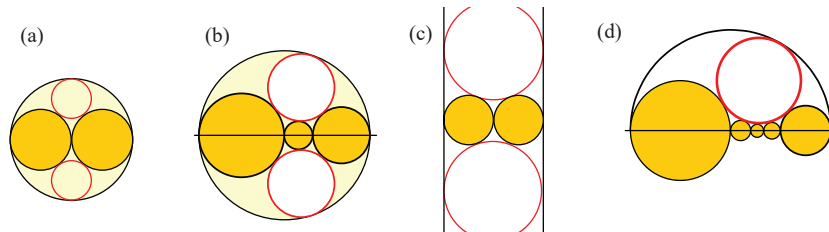

Figure 9: Periodic sequences, geometric representations for examples from Table 4: (a) Example 1, (b) Example 2, (c) Example 3, (d) Example 4.

	constants	OEIS	sequence	symbol
1.	$\alpha = -3,\ \beta = 1$	A001654	$0, 1, -2, 6, -15, 40, -104, 273, -714, 1870, -4895, 12816, \ldots$	$^1(1,1)^{-1}$
2.	$\alpha = -3,\ \beta = 5$	A075269	$2, -3, 12, -28, 77, -198, 522, -1363, 3572, -9348, \ldots$	$^{-1}(1,2)^1$
3.	$\alpha = -4,\ \beta = 3$	—	$1, -2, 10, -35, 133, -494, 1846, -6887, 25705, -95930, \ldots$	$^2(1,1)^{-1}$
4.	$\alpha = -4,\ \beta = -1$	A109437	$0, -1, 3, -12, 44, -165, 615, -2296, 8568, -31977, \ldots$	$^2(1,3)^{-1}$
5.	$\alpha = -5,\ \beta = 1$	A099025	$0, 1, -4, 20, -95, 456, -2184, 10465, -50140, 240236, \ldots$	$^{-3}(1,1)^1$
6.	$\alpha = -6,\ \beta = -4$	A084159	$1, -3, 21, -119, 697, -4059, 23661, -137903, 803761, \ldots$	$^{-2}(1,1)^{-2}$
6.	$\alpha = -6,\ \beta = 1$	A084158	$0, 1, -5, 30, -174, 1015, -5915, 34476, -200940, \ldots$	$^{-4}(1,1)^1$

Table 5: Examples of formal lens sequences

4.1 Classification of symmetric integral lens chains

Now consider lens sequences with mirror symmetry.

Proposition 34. *A sequence is called central if it contains a triple of the form (a, b, a). Only $b = \pm 1$ leads to primitive integer sequences.*

Proof. For β to be an integer, b must divide β, thus $b|ac$. Since $\gcd(a, b, c) = b$, which must be ± 1 for the sequence to be primitive. $\qquad\square$

Proposition 35. *A lens sequence is called bicentral, if it contains a quadruplet of the form (a, b, b, a). Only if b is chosen from $\{0, \pm 1, \pm 2\}$, does a primitive integer sequence result. Note that for the case $b = 0$, the seed needs to be chosen in the form $(0, a, c)$.*

Proof. Analogous to the previous case, except using α rather than β. $\qquad\square$

Such sequences will be called symmetric. The list below shows examples of symmetric lens sequences for small values of the initial terms. Only the right tails are displayed.

There are five types of symmetric lens sequence that may be realized as curvatures of disks. Below, we summarize the general formulas for each of them. Recall that $L =$ length of the lens, $R =$ radius of the lens circles, $\delta =$ their relative distance, and $\lambda =$ characteristic constant. Due to symmetry, the sums of the reciprocals are curtailed to one (right) tail of the sequence.

---

1. Seed: $[n, 1, n]$.   Symbol: $^{n+1}(1,1)^{n+1}$.   Recurrence: $\alpha = (n+1)^2 - 2,\ \beta = 1 - n^2$

     Geometry: $R = \frac{n+1}{n-1}$, $L = 2\frac{n+3}{n-1}$, $\delta = \frac{4}{n+1}$.   (Inner chain)

     Characteristic constant: $\lambda = \left(\frac{n+1+\sqrt{(n+3)(n-1)}}{2} \right)^2 = \frac{n^2 + 2n - 1 + (n+1)\sqrt{(n+3)(n-1)}}{2}$

     Binet: $b_k = \frac{\lambda^k + \bar{\lambda}^k + n + 1}{n+3}$

     Sum of reciprocals: $\sum_{k=0}^{\infty} 1/b_k = 1 + \frac{1}{n} + \ldots = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{n+3}{n-1}}$

---

2. Seed: $[n, 1, 1, n]$.   Symbol: $^2(1, 1)^{n+1}$.   Recurrence: $\alpha = 2n$, $\beta = 1 - n$.

    Geometry: $R = 2\frac{n+1}{n-1}$, $L = 2\sqrt{\frac{n+1}{n-1}}$, $\delta = \frac{4}{\sqrt{2(n+1)}}$.   (Inner chain)

    Characteristic constant: $\lambda = \left(\frac{\sqrt{2n+2}+\sqrt{2n-2}}{2}\right)^2 = \frac{n+\sqrt{n^2-1}}{2}$

    Binet: $b_k = \frac{1}{4}\left(1 + \sqrt{\frac{n-1}{n+1}}\right)\lambda^k + \frac{1}{4}\left(1 - \sqrt{\frac{n-1}{n+1}}\right)\lambda^{-k} + \frac{1}{2}$

    Sum of reciprocals: $\sum\limits_{k=1}^{\infty} 1/b_k = 1 + \frac{1}{n} + \ldots = \sqrt{\frac{n+1}{n-1}}$

---

3. Seed: $[n, 2, 2, n]$.   Symbol: $^1(2, 1)^{n+2}$.   Recurrence: $\alpha = n$, $\beta = 2 - n$.

    Geometry: $R = \frac{n+2}{n-2}$, $L = 2\sqrt{R} = 2\frac{n+2}{n-2}$, $\delta = \frac{4}{\sqrt{n+2}}$.   (Inner chain)

    Characteristic constant: $\lambda = \left(\frac{\sqrt{n+2}+\sqrt{(n-2)}}{2}\right)^2 = \frac{n+\sqrt{n^2-4}}{2}$

    Binet: $b_k = \frac{1}{2}\left(1 + \sqrt{\frac{n-2}{n+2}}\right)\lambda^k + \frac{1}{2}\left(1 - \sqrt{\frac{n-2}{n+2}}\right)\lambda^{-k} + 1$

    Sum of reciprocals: $\sum\limits_{k=1}^{\infty} 1/b_k = \frac{1}{2} + \frac{1}{n} + \ldots = \frac{1}{2}\sqrt{\frac{n+2}{n-2}}$

---

4. Seed: $[n, -1, n]$.   Symbol: $^{n-1}(1, 1)^{n+1}$.   Recurrence: $\alpha = (n-1)^2 - 2$, $\beta = n^2 - 1$.

    Geometry: $R = -\frac{n-1}{n+1}$, $L = 2\frac{n-3}{n+1}$, $\delta = \frac{4}{n-1}$.   (Outer chain)

    Characteristic constant: $\lambda = \left(\frac{n-1+\sqrt{(n-3)(n+1)}}{2}\right)^2 = \frac{n^2-2n-1+(n-1)\sqrt{(n-3)(n+1)}}{2}$

    Binet: $b_k = \frac{\lambda^k + \bar{\lambda}^k - (n-1)}{n-3}$

    Sum of reciprocals: $\sum\limits_{k=1}^{\infty} 1/b_k = \frac{1}{n} + \ldots = \frac{4}{n+1}$

---

5. Seed: $[0, 1, n]$.   Symbol: $^1(1, 1)^{n+1}$.   Recurrence: $\alpha = n - 1$, $\beta = 1$.

    Geometry: $R = -(n+1)$, $L = 2\sqrt{(n+1)(n-3)}$, $\delta = \frac{4}{\sqrt{n+1}}$. (Outer chain)

    Characteristic constant: $\lambda = \left(\frac{n+1+\sqrt{(n-3)}}{2}\right)^2 = \frac{n-1+\sqrt{(n+1)(n-3)}}{2}$

    Binet: $b_k = \frac{1}{2}\left(\frac{n-2}{n-3} + \frac{n}{\sqrt{(n+1)(n-3)}}\right)\lambda^k + \frac{1}{2}\left(\frac{n-2}{n-3} - \frac{n}{\sqrt{(n+1)(n-3)}}\right)\lambda^{-k} + \frac{1}{n-3}$

    Sum of reciprocals: $\sum\limits_{k=0}^{\infty} 1/b_k = 1 + \frac{1}{n} + \ldots = \frac{n+1-\sqrt{(n+1)(n-3)}}{2}$

4.2 Periodic lens chains

Another interesting special case concerns the sequences of periodic disk chains. They appear when the lens circles are separated, like in Figure 9. Understood as the disks of negative curvature, they define a "divergent lens" as the overlapping region. Interestingly, there are only three types of such lenses:
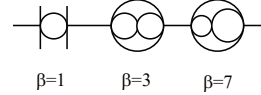
$$
\begin{array}{lll}
\text{A.} & \alpha = -1, & \text{sequences are of order three} \\
\text{B.} & \alpha = 0, & \text{sequences are of order four} \\
\text{C.} & \alpha = 1, & \text{sequences are of order six}
\end{array}
$$

If $\alpha = 2$, the lens circles are tangent and the sequences become infinite. Each type is labeled by integers $m \in \mathbb{N}$. The tables below list the sequences of each type for $m = 0, 1, 2, 3 \ldots$.

$\boxed{\alpha = -1}$ $\quad b_{n+1} = -b_n - b_{n-1} + \beta \qquad \beta = m^2 + m + 1$
$\qquad\qquad (b_i) \; = \; (-m, \; m+1, \; m^2+m, \; -m)$

$\beta = 1$	0	1	0	1	0	1
$\beta = 3$	−1	2	2	−1	2	2
$\beta = 7$	−2	3	6	−2	3	6
$\beta = 13$	−3	4	12	−3	4	12

β=1   β=3   β=7

$\boxed{\alpha = 0}$ $\qquad b_{n+1} = b_{n-1} + \beta \qquad \beta = m^2 + 2m + 2$
$\qquad\qquad (b_i) \; = \; (-m, \; m+2, \; m^2+3m+2, \; m^2+m, \; -m)$

$\beta = 2$	0	2	2	0	0	2	2	0	(⋆)
$\beta = 5$	−1	3	6	2	−1	3	6	2	
$\beta = 10$	−2	4	12	6	−2	4	12	6	(⋆)
$\beta = 17$	−3	5	10	12	−3	5	10	12	

β=1   β=5
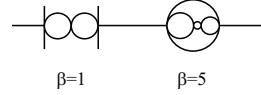
$\boxed{\alpha = 1}$ $\qquad b_{n+1} = b_n - b_{n-1} + \beta \qquad \beta = m^2 + 3m + 3$
$\qquad\qquad (b_i) \; = \; (-m, \; m+3, \; m^2+5m+6, \; 2m^2+7m+6, \; 2m^2+5m+3, \; m^2+m, \; -m)$

$\beta = 3$	0	3	6	6	3	0	0	3	(⋆)
$\beta = 7$	−1	4	12	15	10	2	−1	4	
$\beta = 13$	−2	5	20	28	21	6	−2	5	
$\beta = 21$	−3	6	30	45	36	12	−3	6	(⋆)
$\beta = 31$	−4	7	42	66	55	20	−4	7	

β=1   β=7
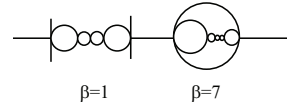
 The cases indicated by (⋆) can be reduced. They concern the cases $2 \mid m$ for $\alpha = 0$, and $3 \mid m$ for $\alpha = 1$.

The corresponding underground sequences are easily determined by factoring the above sequences. They turn to be of order twice longer than the corresponding lens sequences. Here they are:

$$
\begin{array}{ll}
\alpha = -1 & (1, \; m+1, \; m, \; -1) \\
\alpha = 0 & (1, \; m+2, \; m+1, \; m, \; -1) \\
\alpha = 1 & (1, \; m+3, \; m+2, \; 2m+3, \; m+1, \; m, \; -1)
\end{array}
$$

In each case, the last entry (-1) opens the negative version of the cycle. The elegance of their structure can be visualized by points of the pairs of the coefficients at m and 1, connected according to their order in the sequences, see Figure 10.
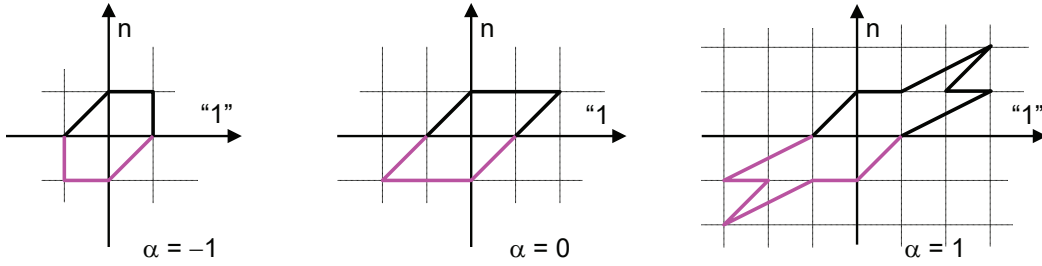

Figure 10: The geometry of the underground sequences of periodic disk chains.

5 Recurrence formula from geometry

In this section we prove Theorem 1 on the recurrence formula of the first section. We shall need a theorem on circle configurations [3], which generalizes that of Descartes' theorem on "kissing circles" [2]. If C_1 and C_2 denote two disks of radii r_1 and r_2 respectively, and d denotes the distance between their centers, then define a product of the disks as

$$\langle C_1, C_2 \rangle = \frac{d^2 - r_1^2 - r_2^2}{2 r_1 r_2} \, , \tag{30}$$

Its values for a few special cases are shown in Figure 11. For any four circles C_i, $i = 1, \ldots, 4$, define a *configuration matrix* f as the matrix with entries

$$f_{ij} = \langle C_i, C_j \rangle \, ,$$

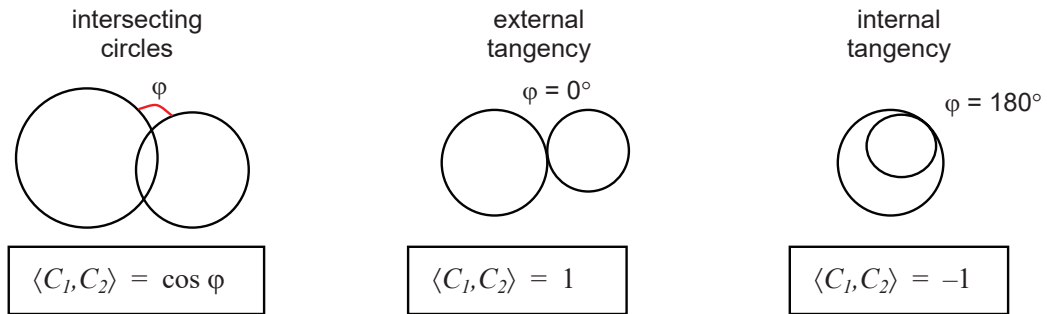where the brackets denote the inner product of circles.

Figure 11: Product of disks/circles: special cases

27

Theorem 36 ([3]). *A configuration of four circles in general position satisfies the following quadratic equation*

$$\boldsymbol{b}^T F \boldsymbol{b} = 0 \tag{31}$$

where $\boldsymbol{b} = [b_1, b_2, b_3, b_4]^T$ is the vector made of the curvatures of the four circles, and where $F = f^{-1}$ is the inverse of the configuration matrix.

Remark 37. Equation (31) is only a fragment of the full matrix formula, which incorporates also the positions of the centers of the circles. For more on this theorem, its proof, and the associated Minkowski geometry of circles, see [3]. For our purposes the version reduced to curvatures (31) is sufficient.

5.0.1   Notation

In the following by a *lens* we mean "symmetric lens"—the intersection of the interiors (exteriors) of two congruent circles, called in this context *lens circles*. A *chain of circles* is a sequence of circles such that every two consecutive circles are tangent.

We are now ready to prove the basic result.

Theorem 38. *A sequence (b_n) of curvatures of a chain of circles inscribed in a lens satisfies a non-homogeneous linear recurrence formula of the form*

$$b_{n+1} = \alpha b_n - b_{n-1} + \beta \tag{32}$$

for some constants α and β, with

$$\alpha = \frac{6 - 2K}{1 + K} = \frac{8}{1 + K} - 2 \qquad and \qquad \beta = -\frac{8A}{1 + K}, \tag{33}$$

where K is the product of the two lens circles and $A = 1/R$ is the curvature of each lens circle.

Proof. Consider two consecutive circles in the chain, of curvatures say x and y. Let the curvatures of the circles that form the lens by A, and their product be denoted by K ($K = \cos \varphi$, if the circles intersect). The configuration matrix f and its inverse are easy to find. In the case of converging lenses we can read it off from Figure 12, left:

$$f = \left[\begin{array}{cc|cc} -1 & K & -1 & -1 \\ K & -1 & -1 & -1 \\ \hline -1 & -1 & -1 & +1 \\ -1 & -1 & +1 & -1 \end{array} \right]$$

28

where the disk indices are ordered as (A, A, x, y). Its inverse F leads via the master equation (31) (after multiplying by a factor of -8) to:

$$\begin{bmatrix} A \\ A \\ x \\ y \end{bmatrix}^T \begin{bmatrix} \frac{4}{K+1} & \frac{-4}{K+1} & 2 & 2 \\ \frac{-4}{K+1} & \frac{4}{K+1} & 2 & 2 \\ 2 & 2 & K+1 & K-3 \\ 2 & 2 & K-3 & K+1 \end{bmatrix} \begin{bmatrix} A \\ A \\ x \\ y \end{bmatrix} = 0,$$

which is equivalent to quadratic equation:

$$(1+K)x^2 + (1+K)y^2 + 2(K-3)xy + 8Ax + 8Ay = 0.$$

One may solve it for y to get two solutions:

$$y_{1,2} = -\frac{4A + (K-3)x \pm 2\sqrt{2(1-K)x^2 - 8Ax + 4A^2}}{1+K} \tag{34}$$

Note that the two solutions y_1 and y_2 correspond to the two possible circles tangent to x: one on the left and one on the right. Adding the two solutions eliminates radicals:

$$y_1 + y_2 = \frac{6 - 2K}{1+K}x - \frac{8A}{1+K}. \tag{35}$$

Since the triple (y_1, x, y_2) forms a sequence in a chain of inscribed circles, we may label these curvatures as $b_{n-1} = y_1$, $b_n = x$, and $b_{n+1} = y_2$, to get

$$b_{n+1} + b_{n-1} = \alpha\, b_n + \beta,$$

which is equivalent to (8). The case of the diverging lens results from similar reasoning, with slightly different initial matrix F. □

Remark 39. Assuming that the lens is not symmetric will not affect the general fact that the inscribed disks define a nonhomogeneous 3-step recurrence.

Corollary 40. *The sequence constants are related:* $\alpha + R\beta = -2$.

Now let us see how three circles determine a sequence.

Theorem 41. *The coefficients of the recurrence (32) are determined by the three curvatures a, b, and c defining the lens, namely:*

$$\alpha = \frac{ab + bc + ca}{b^2} - 1 \quad and \quad \beta = \frac{b^2 - ac}{b}. \tag{36}$$

29

Proof. We apply Theorem 36 in each of the three steps to a different quadruple of circles.
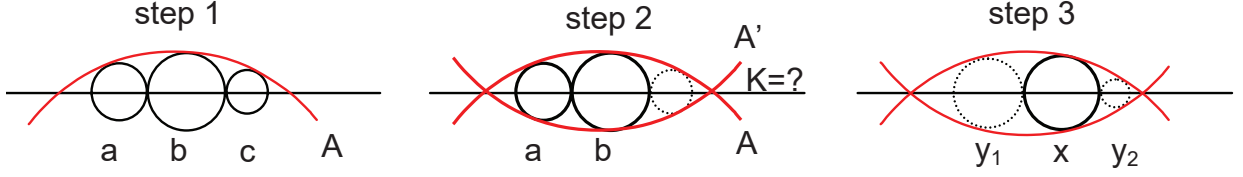

Figure 12: The three steps of the proof of Thmeorem 41

.

5.0.2 Step 1

Consider a configuration of four disks: a triple of three consecutive circles in the chain, a, b, c, and one of the disks forming the lens, $A = 1/R$. The product of two external circles a and c may be easily evaluated from the distance between their centers, $(1/a + 2/b + 1/c)$. We have:

$$\langle a, c \rangle = \frac{\left(\frac{1}{a} + \frac{2}{b} + \frac{1}{c}\right)^2 - \left(\frac{1}{a}\right)^2 - \left(\frac{1}{c}\right)^2}{2 \cdot \frac{1}{a} \cdot \frac{1}{c}} = 2\frac{ab + bc + ca}{b^2} + 1 \, .$$

Let the main fraction of the last expression be denoted by $z = \frac{ab+bc+ca}{b^2}$. Then the configuration matrix and its inverse are

$$f = \left[\begin{array}{ccc|c} -1 & 1 & 2z+1 & -1 \\ 1 & -1 & 1 & -1 \\ 2z+1 & 1 & -1 & -1 \\ \hline -1 & -1 & -1 & -1 \end{array}\right], \quad F = \frac{1}{4}\left[\begin{array}{cccc} -\frac{1}{z+1} & 1 & \frac{1}{z+1} & -1 \\ 1 & -(z+1) & 1 & z-1 \\ \frac{1}{z+1} & 1 & -\frac{1}{z+1} & -1 \\ -1 & z-1 & -1 & -(z+1) \end{array}\right]$$

(the order of entries is: $abcd$). Denoting $\mathbf{v} = [a, b, c, A]^T$ and solving the quadratic equation $\mathbf{v}^T F \mathbf{v} = 0$ for d readily leads to

$$A = \frac{b(ac - b^2)}{ab + bc + ca + b^2} \, . \tag{37}$$

This gives us the curvature A of the two lens bounding circles. Now we need to find their product.

5.0.3 Step 2

Use a quadruple of circles: a, b, and the two circles forming the lens, A and A'. The latter two have the same curvature $A = A'$, the value of which we know from the previous step.

30

The goal is to find the product $K = \langle A, A' \rangle$. The configuration matrix and its inverse are

$$f = \begin{bmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & K \\ -1 & -1 & K & -1 \end{bmatrix} \qquad F = \tfrac{1}{8} \begin{bmatrix} -1-K & 3-K & -2 & -2 \\ 3-K & -1-K & -2 & -2 \\ -2 & -2 & -\frac{4}{K+1} & \frac{4}{K+1} \\ -2 & -2 & \frac{4}{K+1} & -\frac{4}{K+1} \end{bmatrix}$$

(the order of indices agrees with a, b, A, A'). Applying vector $\mathbf{v} = [a, b, A, A]^T$ to the quadratic equation $\mathbf{v}^T F \mathbf{v} = 0$ gives

$$K = \frac{8b^2}{(a+b)(b+c)} - 1 \tag{38}$$

5.0.4   Step 3

Now we can either simply substitute A and K of (37) and (38) to (33) to get the result, or equivalently build the matrix for the configuration in Figure 12, right, and mimic the proof of Theorem 38.   □

5.1   More on lens geometry

Although we are mainly interested in the algebraic properties of lens sequences, some geometric properties explicate their algebraic behavior. Below, we summarize some basic facts.
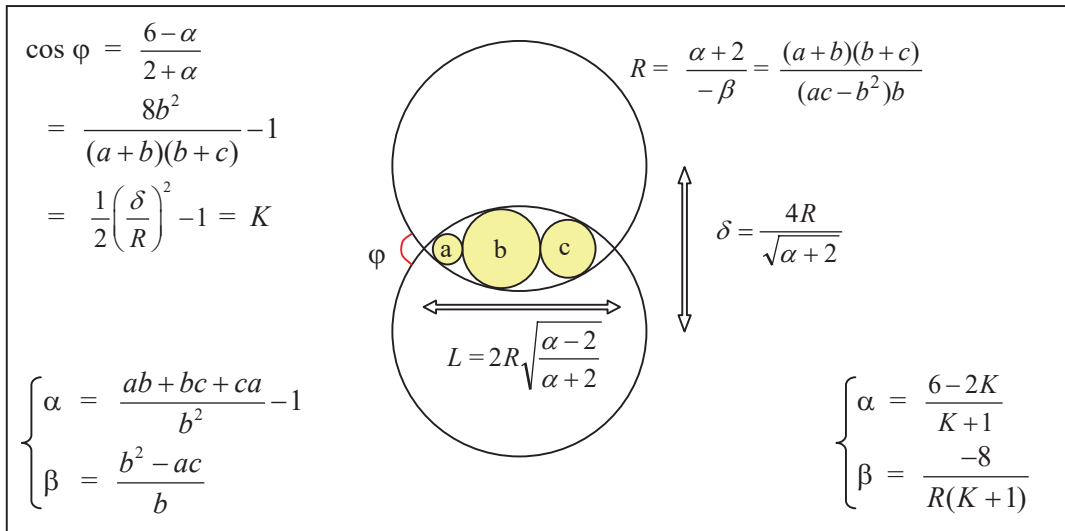
$$\cos \varphi = \frac{6-\alpha}{2+\alpha}$$
$$= \frac{8b^2}{(a+b)(b+c)} - 1$$
$$= \frac{1}{2}\left(\frac{\delta}{R}\right)^2 - 1 = K$$

$$R = \frac{\alpha+2}{-\beta} = \frac{(a+b)(b+c)}{(ac-b^2)b}$$

$$\delta = \frac{4R}{\sqrt{\alpha+2}}$$

$$L = 2R\sqrt{\frac{\alpha-2}{\alpha+2}}$$

$$\begin{cases} \alpha = \dfrac{ab+bc+ca}{b^2} - 1 \\ \beta = \dfrac{b^2-ac}{b} \end{cases}$$

$$\begin{cases} \alpha = \dfrac{6-2K}{K+1} \\ \beta = \dfrac{-8}{R(K+1)} \end{cases}$$

Figure 13: Sequence constants and geometry of a lens

31

Proposition 42. *The radius R of the lens circles is determined by three circles and may be expressed in terms of the sequence constants α and β:*

$$R = \frac{\alpha + 2}{-\beta} = \frac{(a+b)(b+c)}{(ac - b^2)b} \ . \tag{39a}$$

The inner product of the lens circles is

$$K = \frac{6 - \alpha}{2 + \alpha} = \frac{8b^2}{(a+b)(b+c)} - 1 = \frac{1}{2}\left(\frac{\delta}{R}\right)^2 - 1 = \cos\varphi \, , \tag{39b}$$

where the last equation is valid if the circles intersect. The length L of the lens, if defined, is

$$L = 2R\sqrt{\frac{\alpha - 2}{\alpha + 2}} = -\frac{2\sqrt{\alpha^2 - 4}}{\beta} = 2\frac{\sqrt{(a+b)(b+c)[(a+b)(b+c) - 4b^2]}}{(ac - b^2)b} \ . \tag{39c}$$

The separation of the lens bounding circles (distance between their centers) is

$$\delta = \frac{4R}{\sqrt{\alpha + 2}} = -\frac{4\sqrt{\alpha + 2}}{\beta} \quad \text{and} \quad \frac{\delta}{R} = \frac{4}{\sqrt{\alpha + 2}} \ . \tag{39d}$$

Proof. All are direct corollaries of Theorem 38 and simple geometric constructions. □

n	K	α		example	
0	∞	-2	periodic (order 2)	$(1, -1)$	
1	7	-1	periodic (order 3)	$(6, 3, -2)$	disjoint lens
2	3	0	periodic (order 4)	$(3, 6, 2, -1)$	circles
3	$5/3$	1	periodic (order 6)	$(2, 10, 15, 12, 4, -1)$	
4	1	2	Example 1.3	$(-1, 3, 15)$	tangent lens circles
5	$3/4$	3		$(3, 2, 2, 3)$	
6	$1/3$	4		$(2, 1, 1, 2)$	
7	$1/7$	5		$(5, 2, 2, 5)$	
8	0	6	orthogonal	$(3, 1, 1, 3)$	
9	$-1/9$	7	golden Vesica	$(2, 1, 2)$	
10	$-1/5$	8		$(4, 1, 1, 4)$	
11	$-3/11$	9			intersecting lens
12	$-1/3$	10		$(3, 1, 2)$	circles
13	$-5/13$	11			
14	$-3/7$	12		$(6, 1, 1, 6)$	
15	$-7/15$	13		$(4, 1, 2)$	
16	$-1/2$	14	Vesica Piscis	$(3, 1, 3)$	
17	$-9/17$	15			
18	$-5/9$	16		$(8, 1, 1, 8)$	

Table 6: Admissible values of α. If the bounding circle intersect, $K = \cos\varphi$, where φ stands for the intersection angle.

Figure 13 contains these findings for easy reference. Figure 2 in Section 1 categorizes a variety of geometric situations for a lens sequence. In the case of converging lenses, when two circles of radius R intersect at angle φ, the recurrence formula is:

$$b_n = \left(\frac{8}{1 + \cos \varphi} - 2 \right) b_{n-1} - b_{n-2} - \frac{1}{R} \frac{8}{1 + \cos \varphi}.$$

This answers the question of which lenses may lead to integer sequences. Indeed, denote $m = \frac{8}{1+\cos\varphi}$. Then $\alpha = m - 2$, $\beta = -m/R$. For m to be an integer, $m \in \mathbb{N}$, we need $\cos \varphi = 8/m - 1$. Table 6 shows some values.

Note that if $\alpha \leqslant 2$, then only external sequences are possible (corresponding to diverging lenses). Moreover, if $\alpha < 2$, then the integer sequence must be periodic. If $\alpha > 2$, then we can have two families of sequences: inner (inside a converging lens) or outer (outside the lens circles, i.e., inside a corrupted diverging lens (*corrupted*, because of the missing central part)). In the case of the outer sequence we will have exactly one negative entry (the most external circle) or two adjacent "0" entries (two vertical lines).

6 Summary

A lens sequence is an integer sequence (b_i) that satisfies two conditions:

(i) $b_n = \alpha b_{n-1} - b_{n-2} + \beta$ [recurrence formula]

(ii) $a^2 + b^2 = \alpha\, ab + \beta(a + b)$ [compatibility relation]

where α and β are constants and a and b are any two consecutive terms of (b_i). These two conditions assure that the sequence has a geometric realization in terms of the curvatures of a chain of circles inscribed in a symmetric lens (the space of the overlap of the interiors or exteriors of two congruent circles). The sequence constants may be viewed as invariants of a process $i \to b_i$. They may be calculated from a *seed*, i.e., any three consecutive sequence terms (a, b, c):

$$\alpha = \frac{ab + bc + ca}{b^2} - 1 \text{ and } \beta = \frac{b^2 - ac}{b}$$

or, for α, alternatively

$$\alpha = \frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}}.$$

Other, nonlinear, recurrence formulas for the lens sequence include:

[two-step formula] $\qquad 2b_{n+1} = b_n\alpha + \beta \pm \sqrt{(\alpha^2 - 4)\, b_n^2 + 2(\alpha + 2)\beta\, b_n + \beta^2}$

[three-step formula] $\qquad b_{n+1}b_n + b_{n+1}b_{n-1} + b_n b_{n-1} = (\alpha + 1)b_n^2 \qquad$ (only α)

[three-step formula] $\qquad b_n b_n - b_{n+1}b_{n-1} = \beta b_n \qquad\qquad\qquad$ (only β)

[four-step formula] $\qquad b_{n+2}b_{n-1} = (b_{n+1} - \beta)(b_n - \beta)$

[four-step formula] $\qquad b_{n+1}b_{n-1} + b_n b_{n-2} = \alpha b_n b_{n-1}$

The sequence constants have a geometric meaning: α codes the angle under which the circles forming the lens intersect (if they do), or, more generally, the product of the lens circles. The value of β reflects the size of the system. There are two basic properties determined by geometry: (a) the sum of the inverses is determined by the length of the lens, and (b) the limit of the ratio of consecutive terms is determined by the aforementioned lens angle:

$$\sum_n \frac{2}{b_n} = \frac{\sqrt{\alpha^2 - 4}}{-\beta} \quad \text{and} \quad \lim_{i\to\infty} \frac{b_{i+1}}{b_i} = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2} = \lambda,$$

The number λ, the *characteristic constant* of (b_i), allows one to express the lens sequence by a Binet-type formula

$$b_n = w\lambda^n + \bar{w}\bar{\lambda}^n + \gamma$$

where

$$w = \frac{a - 2b + c}{2(\alpha - 2)} + \frac{c - a}{2(\alpha^2 - 4)}\sqrt{\alpha^2 - 4}, \quad \gamma = \frac{-\beta}{\alpha - 2},$$

and where the bar denotes natural conjugation in the field $\mathbb{Q}(\sqrt{\alpha^2 - 4})$. The constant λ is an example of a (quadratic) Pisot number, an algebraic integer, the powers of which approximate natural numbers. In particular, $b_n \approx w\lambda^n + \gamma$. Lens sequences can be expressed also as combinations of Chebyshev polynomials.

The most mysterious property of a lens sequence is that its terms may be formed by taking products of pairs of consecutive terms of another sequence. This "underground" sequence has an alternating recurrence rule, different for odd and even terms. Namely, $b_n = f_{n-1}f_n$, where:

$$f_n = \begin{cases} kf_{n-1} - f_{n-2}, & \text{if } n \text{ is even;} \\ sf_{n-1} - f_{n-2}, & \text{if } n \text{ is odd.} \end{cases}$$

The constants of the sequence (b_i) may now be expressed as

$$\begin{cases} \alpha &= ks - 2; \\ \beta &= sf_0^2 + kf_1^2 - ks\, f_0 f_1. \end{cases}$$

It follows that the integer lens sequences may be determined by four arbitrary integers

$$^{s}(f_0, f_1)^k \,.$$

Choosing for f_0 the term with smallest absolute value allows one to treat the above quadruple as a *symbol* that labels the corresponding lens sequence.

The underground sequences automatically satisfy the following two recurrence formulas:

$$(i) \ f_{n+2} + f_{n-2} = \alpha f_n$$

$$(ii) \ f_n f_{n+1} - f_{n+2} f_{n-1} = \beta \quad \text{or} \quad \det \begin{bmatrix} f_{n-1} & f_n \\ f_{n+1} & f_{n+2} \end{bmatrix} = -\beta$$

More precisely, the set of the sequences that are underground sequences for lens sequences coincides with the intersection of two sets, each defined by one of the above equations.

An intriguing property holds — the eigenvectors of matrices assembled from the terms of a lens sequence are vectors with entries from the corresponding underground sequence:

$$\begin{bmatrix} b_{n+1} & b_{n+2} \\ b_{n-1} & b_n \end{bmatrix} \begin{bmatrix} f_{n+1} \\ -f_{n-1} \end{bmatrix} = \beta \begin{bmatrix} f_{n+1} \\ -f_{n-1} \end{bmatrix} \,.$$

But the geometric meaning of the underground sequences remains to be understood.

7   Acknowledgments

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