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Lucas Representations of Positive Integers

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Abstract

Various kinds of representations of positive integers using nonconsecutive Lucas numbers are used to define arrays related to the Wythoff array. The columns of these arrays, or their order arrays, partition the positive integers. Limiting densities are found for numbers whose Lucas representations all have the same least term.

1 Introduction

Throughout, the set of positive integers is denoted by $\mathbb{N}$, and i, k, m, n, u represent elements of $\mathbb{N}$. The golden ratio, $(1+\sqrt{5})/2$, is denoted by τ. The sequence (L_n) of Lucas numbers is given by $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. In 1972, Carlitz, Scoville, and Hoggatt [2] proved the following uniqueness theorem for representations of positive integers as sums of nonconsecutive Lucas numbers.

Theorem 1. *Every n has a unique representation in exactly one of these two forms:*

$$n = L_{k_1} + \cdots + L_{k_u} + L_0 \ (\text{or } n = 2), \tag{1}$$

$$\text{where } k_i - k_{i+1} \geq 2 \text{ for } 1 \leq i \leq u-1, k_u \geq 3 \tag{2}$$

or

$$n = L_{k_1} + \cdots + L_{k_u} \tag{3}$$

$$\text{where } k_i - k_{i+1} \geq 2 \text{ for } 1 \leq i \leq u-1, k_u \geq 1. \tag{4}$$

Following the notation in [2], let B_0 be the sequence of numbers n given by (1) and (2), and let B_k be the sequence given by (3) and (4); e.g.,

$$\begin{aligned}B_0 &= (2, 6, 9, 13, 17, 20, 24, 27, 31, 35, 38, 42, 46, 49, \cdots)\\ B_1 &= (1, 5, 8, 12, 16, 19, 23, 26, 30, 34, 37, 41, 45, 48, \cdots)\\ B_2 &= (3, 10, 14, 21, 28, 32, 39, 43, 50, 57, 61, 68, 75, 79, \cdots).\end{aligned}$$

In general, B_k consists of all n such that the least term in (1), or (3), is L_k. Clearly the sequences B_k partition $\mathbb{N}$.

The representations in Theorem 1 are patterned after Zeckendorf representations, which we review as follows. The sequence (F_n) of Fibonacci numbers is given by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Zeckendorf representation of n is the unique sum

$$n = F_{k_1} + \cdots + F_{k_u} \tag{5}$$

$$\text{where } k_i - k_{i+1} \geq 2 \text{ for } i = 1 \cdots u - 1, k_u \geq 2. \tag{6}$$

For every n, the greedy algorithm can be used to find the successive terms in all three representations (1) and (2), (3) and (4), and (5) and (6).

Historically, the Zeckendorf representation dates back to Zeckendorf's work as early as 1939, but he did not submit the result for publication until April, 1972; remarkably, his reference section includes the 1972 papers [1] and [2]. Zeckendorf's theorems [8] are quoted here:

> THÉORÈME I.a. *Tout nombre naturel N peut être représenté par une somme de nombres de Fibonacci distincts non consécutifs.*
> THÉORÈME I.b. *Pour tout nombre naturel, cette somme est unique.*
> THÉORÈME II.a. *Tout nombre naturel N peut être représenté par une somme de nomberes de Lucas distincts non consécutifs.*
> THÉORÈME II.b. *La représenté des nombres naturels par une somme de nombres de Lucas non consécutifs est unique, sauf pour les nombres $L_{2v+1} + 1$.*

Aside from the Lucas representations in Theorem 1, another kind of Lucas representation is given by Luo [5], in which some but not all n have a unique representation; indeed those n having more than one representation have exactly two representations.

The main purpose of this article can now be stated: to partition $\mathbb{N}$, or some subset of $\mathbb{N}$, as the set of columns of certain arrays (as in Tables 2-5) obtained from various kinds of Lucas representations, and to consider corresponding order arrays, densities, and limiting densities. These results can be compared to similar results already well known for the Wythoff array.

2 Wythoff array and Lucas-Wythoff arrays

In 1980, Morrison defined the Wythoff array $(w(n, k))$ by the formulas

$$w(n, 1) = \lfloor \lfloor n\tau \rfloor \tau \rfloor \text{ and } w(n, 2) = \lfloor \lfloor n\tau \rfloor \tau^2 \rfloor$$

together with the Fibonacci recurrence

$$w(n,k) = w(n,k-1) + w(n,k-2) \text{ for } k \geq 3.$$

The Wythoff array, which shows all the winning pairs for the Wythoff game, has been widely studied; see, for example, the Comments and Links at A035513 in [7] and "Wythoff visions" [3] In Theorem 4, we state and verify a formula for $w(n,k)$ that appears elsewhere (e.g., [4]) without proof. We begin with lemmas that account for the first two columns of the Wythoff array. The notation $\{x\}$ is used for the fractional part of a real number x, defined by $\{x\} = x - \lfloor x \rfloor$.

Lemma 2. *If $n \geq 1$, then $\lfloor \lfloor n\tau \rfloor \tau \rfloor = \lfloor n\tau \rfloor + n - 1$.*

Proof. Since the fractional part $\{n\tau\}$ of $n\tau$ is in $(0,1)$, we have

$$\lfloor n - \{n\tau\}(\tau - 1) \rfloor = n - 1.$$

The identity $\tau^2 = \tau + 1$ then gives

$$\begin{aligned} n - 1 &= \lfloor n\tau^2 - n\tau - \{n\tau\}(\tau - 1) \rfloor \\ &= \lfloor (n\tau - \{n\tau\})(\tau - 1) \rfloor \\ &= \lfloor \lfloor n\tau \rfloor (\tau - 1) \rfloor \\ &= \lfloor \lfloor n\tau \rfloor \tau \rfloor - \lfloor n\tau \rfloor. \end{aligned}$$

□

Lemma 3. $\lfloor \lfloor n\tau \rfloor \tau^2 \rfloor = 2\lfloor n\tau \rfloor + n - 1$.

Proof. Using Lemma 2, we have

$$\begin{aligned} \lfloor \lfloor n\tau \rfloor \tau^2 \rfloor &= \lfloor \lfloor n\tau \rfloor (\tau + 1) \rfloor \\ &= \lfloor \lfloor n\tau \rfloor \tau \rfloor + \lfloor n\tau \rfloor \\ &= (\lfloor n\tau \rfloor + n - 1) + \lfloor n\tau \rfloor. \end{aligned}$$

□

Theorem 4. *The Wythoff array is given by*

$$w(n,k) = \lfloor n\tau \rfloor F_{k+1} + (n-1)F_k \tag{7}$$

for $n \geq 1, k \geq 1$.

Proof. For every n, equation (7) holds for $k = 1$ and $k = 2$, by the two lemmas. Assume (7) for all n and arbitrary $k \geq 2$. Then

$$\begin{aligned} w(n,k+1) &= w(n,k) + w(n,k-1) \\ &= \lfloor n\tau \rfloor F_{k+1} + (n-1)F_k + \lfloor n\tau \rfloor F_k + (n-1)F_{k-1} \\ &= \lfloor n\tau \rfloor (F_{k+1} + F_k) + (n-1)(F_k + F_{k-1}) \\ &= \lfloor n\tau \rfloor F_{k+2} + (n-1)F_{k+1}; \end{aligned}$$

so that inductively, (7) holds for all k. □

1	2	3	5	8	13	21	34	55	89	...
4	7	11	18	29	47	76	123	199	322	
6	10	16	26	42	68	110	178	288	466	
9	15	24	39	63	102	165	267	432	699	
12	20	32	52	84	136	220	356	576	932	
14	23	37	60	97	157	254	411	665	1076	
⋮										

Table 1: Wythoff array

1	2	3	4	7	11	18	29	47	76	...
5	6	10	15	25	40	65	105	170	275	
8	9	14	22	36	58	94	152	246	398	
12	13	21	33	54	87	141	228	369	597	
16	17	28	44	72	116	188	304	492	796	
⋮										

Table 2: 1st Lucas-Wythoff array

Referring to Table 1, the Fibonacci numbers and Lucas numbers occupy rows 1 and 2, respectively, and every row satisfies the recurrence $r_n = r_{n-1} + r_{n-2}$, and every n occurs exactly once. Furthermore, for all k, column k consists of those numbers m having F_{k+1} as least term in the Zeckendorf representation ((5) and (6)), as proved in [4]. Indeed, the Zeckendorf array as defined in [4] is identical to the Wythoff array.

We now define the *1st Lucas-Wythoff array*, $(r(n,k))$, by columns: (column 1) $= B_1$, (column 2) $= B_0$, and, for $k \geq 2$, (column k) $= B_k$. See Table 2.

The inclusion of column 2 ensures that every n in $\mathbb{N}$ occurs (exactly once) in the array and that all rows and columns are strictly increasing. However, column 2 interrupts the Fibonacci row recurrence seen in the Zeckendorf array. Here, instead, we have $r(n,4) = r(n,1) + r(n,3)$ for all n, and $r(n,k) = r(n,k-1) + r(n,k-2)$ for $n \geq 1$ and $k \geq 5$. Deleting column 2 results in the *2nd Lucas-Wythoff array*, $(r^*(n,k))$, for which we have a formula much like (7), shown in Theorem 5. See Table 3.

Theorem 5. *The 2nd Lucas-Wythoff array is given by*

$$r^*(n,k) = \lfloor n\tau \rfloor L_k + (n-1)L_{k-1}, \tag{8}$$

for $n \geq 1, k \geq 1$.

Proof. Let $b(n,k)$ denote the nth term of the sequence B_k. First we prove that $b(n,k) = w(n,k-2) + w(n,k)$ for $k \geq 1$. Following [2], let

$$a(n) = \lfloor \tau n \rfloor, b(n) = \lfloor \tau^2 n \rfloor, \text{ and let}$$

1	3	4	7	11	18	29	47	76	123	...
5	10	15	25	40	65	105	170	275	445	
8	14	22	36	58	94	152	246	398	644	
12	21	33	54	87	141	228	369	597	966	
16	28	44	72	116	188	304	492	796	1288	
19	32	51	83	134	217	351	568	919	1487	
23	39	62	101	163	264	427	691	1118	1809	
⋮										

Table 3: 2nd Lucas-Wythoff array

$$A_{2t} = (ab^{t-1}a(n) : n \geq 1, t \geq 1)$$
$$A_{2t+1} = (b^{t-1}a(n) : n \geq 1, t \geq 1),$$

where concatenation of functions abbreviates composition. As proved in [1], we have $w(n,k) = A_k$. Likewise [2],

$$\begin{aligned} B_0 &= (a^2(n) + n : n \geq 1) \\ B_1 &= (a^2(n) + n - 1 : n \geq 1) \\ B_{2t} &= (b^{t-1}a(n) + b^t a(n) : n \geq 1, t \geq 1) \\ B_{2t+1} &= (ab^{t-1}a(n) + ab^t a(n) : n \geq 1, t \geq 1). \end{aligned}$$

Thus,

$$\begin{aligned} B_{2t} &= A_{2t-1} + A_{2t+1} = (w(n, 2t-1) + (w(n, 2t+1)) \\ B_{2t+1} &= A_{2t} + A_{2t+1} = (w(n, 2t) + w(n, 2t+1)), \end{aligned}$$

so that $b(n,k) = w(n,k-2) + w(n,k)$ for all n and $k \geq 3$. By Theorem 4,

$$\begin{aligned} b(n,k) &= \lfloor n\tau \rfloor F_{k-1} + (n-1)F_{k-2} + \lfloor n\tau \rfloor F_{k+1} + (n-1)F_k \\ &= \lfloor n\tau \rfloor (F_{k-1} + F_{k+1}) + (n-1)(F_{k-2} + F_k) \\ &= \lfloor n\tau \rfloor L_k + (n-1)L_{k-1}. \end{aligned}$$

□

The first rows of Table 3 provide some interesting examples involving the Wythoff array (W, as in Table 1):

Row 1 $\sim$ row 2 of W, as they have in common $(4, 7, \ldots)$
Row 2 $\sim$ row 16 of W, with common tail $(40, 65, \ldots)$
Row 3 $\sim$ row 9 of W, with common tail $(22, 36, \ldots)$
Row 4 $\sim$ row 13 of W, with common tail $(33, 54, \ldots)$.

Because *every* positive Fibonacci sequence is represented in W, every row of both Lucas-Wythoff arrays must, in the sense indicated by the examples, be tail-equivalent to a row of W. We leave open further investigation of this equivalence, which can be cast as follows: in each row of either Lucas-Wythoff array, where does a Wythoff pair first occur? (After the first pair, all subsequent pairs in a row are Wythoff pairs.)

3 Admissible representations

The requirement $k_u \geq 1$ in connection with the representation (4) shows that the number 2 is disallowed as a term. If 2 is allowed, then uniqueness is lost; e.g., 5 can be represented by both $4 + 1$ and $3 + 2$. Luo [5] proved that if 2 is allowed, then each n has at most two representations, so that any n having at least two representations must have exactly two. We shall identify them explicitly.

Definition 6. A representation

$$n = L_{k_1} + \cdots + L_{k_u} \tag{9}$$

is an admissible representation of n if

$$k_i - k_{i+1} \geq 2 \text{ for } 1 \leq i \leq u - 1, k_1 \geq 0. \tag{10}$$

Note that, unlike (4), in (10), the index k_1 can be 0. Clearly, both of the representations ((1) and (2)) and ((3) and (4)) are admissible.

Theorem 7. *If $n = L_{k_1} + \cdots + L_{k_u}$, where $u \geq 2$, $k_u = 1$, and k_{u-1} is odd, then n has exactly two admissible representations.*

Proof. Suppose that $k_1 = 1$ and k_2 is odd. We consider two cases.

Case 1: $u = 2$. Here,

$$n = 1 + L_{2i+1} \text{ for some } i \geq 1.$$

As a first induction step, if $i = 1$, then $n = 5 = 1 + 4 = 2 + 3$, two representations. Assume for arbitrary $i \geq 1$ that $n = 1 + L_{2i+1}$ has a second admissible representation, $n = s$, where 1 is not a term of s, and the greatest term of s is less than L_{2i+1}. Then

$$1 + L_{2i+3} = 1 + L_{2i+1} + L_{2i+2} \tag{11}$$

$$= s + L_{2i+2}. \tag{12}$$

This shows that $1 + L_{2i+3}$ has two admissible representations.

Case 2: $u \geq 3$. Here, suppose that

$$n' = 1 + L_{2i+1} + s' \tag{13}$$

where $i \geq 1$ and the Lucas representation ((3) and (4)) of s' has least term $\geq L_{2i+3}$. Then by (12), a second admissible representation of n' is $s + L_{2i+2}$.

For both cases, if n or n' has a third admissible representation, then one of the representations ((1) and (2)) or ((3) and (4)) is not unique, contrary to Theorem 1. Therefore, numbers of the forms in cases 1 and 2 have exactly two admissible representations. $\square$

Theorem 8. *(Converse of Theorem 7) If $n = L_{k_1} + \cdots + L_{k_u}$ has two admissible representations, then one of them has $u \geq 2$, $k_u = 1$, and k_{u-1} odd.*

Proof. Suppose that n has two admissible representations,

$$n = L_{k_1} + \cdots + L_{k_u} = L_{i_1} + \cdots + L_{i_v}.$$

By Theorem 1, either $k_u = 0$ or $i_v = 0$; assume the latter, so that

$$n = L_{i_1} + \cdots + L_{i_v - 1} + 2, \text{ where } i_v - 1 \geq 3; i.e., L_{i_{v-1}} \geq 4.$$

By the uniqueness of ((2) and (4)), we must have $i_v - 1 = 2$ because of (2), leading to two cases.

Case 1: $n = 3 + 2$, so that $n = 4 + 1$, as asserted.

Case 2: $n = w + 3 + 2$, where, by (3) and (4), the number w has a representation (3) with least term L_m for some $m \geq 4$. If $m \geq 5$, then

$$n = \cdots + L_m + 3 + 2 = \cdots + L_m + 4 + 1, \text{ as asserted.}$$

This leaves the possibility that $m = 4$, so that $n = w' + 7 + 3 + 2$, where, again by (3) and (4), the number w' has a representation (3) with least term L_m for some $m \geq 6$. This procedure can be continued, leading to the asserted form of representation in fewer than $2i_1$ steps. $\square$

Examples:

$$\begin{aligned}
12 &= 11 + 1 = 7 + 3 + 2 \\
16 &= 11 + 4 + 1 = 11 + 3 + 2 \\
19 &= 18 + 1 \text{ (unique)} \\
30 &= 29 + 1 = 18 + 7 + 3 + 2 \\
34 &= 29 + 4 + 1 = 29 + 3 + 2
\end{aligned}$$

Definition 9. The Luo-Lucas array, $(\ell(n, k))$, consists of the numbers that have exactly two admissible representations: column k of the array is the increasing sequence of numbers n whose representation (3) has least term $L_{2k+1} + 1$.

5	12	30	77	200	522	1365	3572	...
16	41	106	276	721	1886	4936	12921	
23	59	153	399	1043	2729	7143	18699	
34	88	229	598	1564	4093	10714	28048	
45	117	305	797	2085	5457	14285	37397	
52	135	352	920	2407	6300	16492	43175	
⋮								

Table 4: Luo-Lucas array

8	19	48	124	323	844	2208	...
26	66	171	446	1166	3051	7986	
37	95	247	645	1687	4415	11557	
55	142	370	967	2530	6622	17335	
73	189	493	1289	3373	8829	23113	
84	218	569	1488	3894	10193	26684	
⋮							

Table 5: Dual of Luo-Lucas array

Theorem 10. *The Luo-Lucas array is given by*

$$\ell(n,k) = 1 + \lfloor n\tau \rfloor L_{2k+1} + (n-1)L_{2k}, \ \ n \geq 1, k \geq 1. \tag{14}$$

Proof. Column k of $(\ell(n,k))$ has first term $m(1,k) = L_{2k+1} + 1$, and all the terms thereafter are, in order, of the form $m(1,k) + U$, where U ranges through the positive integers u having least term $L \geq L_{2k+3}$ in the Lucas-Wythoff representation of U. These numbers are, in the same order, the numbers in column $2k+1$ of the 2nd Lucas-Wythoff array. Therefore, by Theorem 5, equation (14) holds. □

Every number in the Luo-Lucas array is in column 1 of the Lucas-Wythoff array 3. The remaining numbers in column 1, excluding the initial 1, form a sequence $(d(n,k))$ given by

$$d(n,k) = (8, 19, 26, 37, 48, 55, 66, \ldots),$$

whose terms can be naturally arranged to form a dual, $(\ell^*(n,k))$ of the Luo-Lucas array, given by

$$\ell^*(n,k) = 1 + \lfloor n\tau \rfloor L_{2k+2} + (n-1)L_{2k+1}, n \geq 1, k \geq 1.$$

4 Order arrays

Following the definition at A333029, suppose that $(a(n,k))$, for $n \geq 1, k \geq 1$, is an array of distinct numbers. If each $a(n,k)$ is replaced by its position when all the numbers $a(n,k)$ are ordered by $<$, the resulting array is the *order array of* $(a(n,k))$.

In this section, we shall see that the order array of the 2nd Lucas-Wythoff array is the Wythoff array. A proof depends on several lemmas. The first three involve properties of the Fibonacci numbers that follow readily from the fact that the fractions F_{k+1}/F_k are the convergents to τ. Proofs of these three are omitted.

Lemma 11. *If $k \geq 3$ is odd, then $0 < \tau F_{k-2} - F_{k-1} + 1$.*

Lemma 12. *If $k \geq 3$ is odd, then $F_{k-2} - 1 < \tau(F_{k-3} + 1)$.*

Lemma 13. *If $k \geq 3$ is even, then $-1 \leq \tau F_{k-3} - F_{k-2} < 1$.*

Let $\beta(n,k)$ be the number of numbers i in B_0 such that $i \leq r^*(n,k)$. Several lemmas will be used to prove the following equation, to be used in proving Theorem 19:

$$\beta(n,k) = \lfloor n\tau \rfloor F_{k-1} + (n-1)F_{k-2}, \tag{15}$$

for $n \geq 1$ and $k \geq 1$, where $F_0 = 0$ and $F_{-1} = 1$.

Lemma 14. *If $n \geq 1$ and $k \geq 3$, then $\beta(n,k) = \lfloor \frac{\lfloor n\tau \rfloor L_k + (n-1)L_{k-1} + 1}{\tau + 2} \rfloor$.*

Proof. The numbers in $B_0 = (B(m))$, for $m \geq 1$, are given by

$$\begin{aligned} B(m) &= \lfloor \tau \lfloor \tau m \rfloor \rfloor + m \\ &= \lfloor m\tau \rfloor + m - 1 + m \text{ by Lemma 2}, \end{aligned}$$

so that we seek the number of numbers m, hence the greatest such m, satisfying

$$\lfloor m\tau \rfloor + 2m - 1 \leq \lfloor n\tau \rfloor L_k + (n-1)L_{k-1} + 1 + \{m\tau\}.$$

This inequality can be recast as

$$m(\tau + 2) \leq \lfloor n\tau \rfloor L_k + (n-1)L_{k-1} + 1 - \{m\tau\},$$

and dividing by $\tau + 2$ finishes the proof. □

Lemma 15. *The equation (15) holds for $k = 1$ and all n.*

Proof.

$$\begin{gathered} 0 < 1 - \{n\tau\} < \tau + 2 \\ (n-1)(\tau+2) < n\tau + \{n\tau\} + 2n - 1 < n(\tau + 2) \\ n - 1 < \frac{n\tau + \{n\tau\} + 2n - 1}{\tau + 2} < n, \end{gathered}$$

so that (15), the right-hand side of which is $n-1$ when $k = 1$, holds. □

Lemma 16. *The equation (15) holds for $k = 2$ and all n.*

Proof. First, $\tau\lfloor n\tau\rfloor < \tau^2 n = (\tau+1)n$, and also

$$\begin{aligned} n = n\tau^2 - n\tau &< (\tau-1)(n\tau + \{n\tau\}) + \tau + 2, \text{ so that} \\ \tau\lfloor n\tau\rfloor < \lfloor n\tau\rfloor + n &< \tau\lfloor n\tau\rfloor + \tau + 2. \end{aligned}$$

Adding $2\lfloor n\tau\rfloor$ and dividing by $\tau+2$ give

$$\lfloor n\tau\rfloor < \frac{3\lfloor n\tau\rfloor + n}{\tau+2} < \lfloor n\tau\rfloor + 1,$$

so that (15), the right-hand side of which is $\lfloor n\tau\rfloor$ when $k = 2$, holds. □

Lemma 17. *If $k \geq 3$, then $\beta(n,k) = \lfloor \frac{(n-1+3\lfloor n\tau\rfloor)F_{k-1}+(2n-2+\lfloor n\tau\rfloor)F_{k-1}+1}{\tau+2} \rfloor$.*

Proof. This results readily by substituting $L_k = 3F_{k-1} + F_{k-2}$ and $L_{k-1} = 2F_{k-2} + F_{k-1}$ into the formula in Lemma 14. □

Lemma 18. *The equation (15), for $k \geq 3$ and all n, is equivalent to*

$$0 < \tau F_{k-2} - F_{k-1} + 1 + (F_k - \tau F_{k-1})\{n\tau\} < \tau + 2. \tag{16}$$

Proof. Equation (15) and Lemma 17 give the following inequalities equivalent to (15):

$$\begin{aligned} \lfloor n\tau\rfloor F_{k-1} + (n-1)F_{k-2} &< \frac{(n-1+3\lfloor n\tau\rfloor)F_{k-1} + (2n-2+\lfloor n\tau\rfloor)F_{k-1} + 1}{\tau+2} \\ &< \lfloor n\tau\rfloor F_{k-1} + (n-1)F_{k-2} + 1 \end{aligned}$$

Multiplying by $\tau+2$ and then expanding the products and canceling like terms leave

$$\begin{aligned} \tau\lfloor n\tau\rfloor F_{k-1} + \tau(n-1)F_{k-2} &< (\lfloor n\tau\rfloor + n - 1)F_{k-1} + \lfloor n\tau\rfloor F_{k-2} + 1 \\ &< \tau\lfloor n\tau\rfloor F_{k-1} + \tau(n-1)F_{k-2} + \tau + 2, \end{aligned}$$

which is successively equivalent to each of these:

$$\begin{aligned} &0 < ((1-\tau)\{n\tau\} - 1)F_{k-1} + (\tau + \{n\tau\})F_{k-2} + 1 < \tau + 2 \\ &0 < \tau F_{k-2} - F_{k-1} + 1 + (F_{k-2} + (1-\tau)F_{k-1})\{n\tau\} < \tau + 2 \\ &0 < \tau F_{k-2} - F_{k-1} + 1 + (F_k - \tau F_{k-1})\{n\tau\} < \tau + 2. \end{aligned}$$

□

Theorem 19. *The order array of the 2nd Lucas-Wythoff array (Table 3 is the Wythoff array.*

Proof. By Lemma 18, we must prove that (16) holds for $k \geq 3$ and arbitrary n. (For $k = 1$ and $k = 2$, the proof is already established by Lemmas 15 and 16.)

Case 1: k odd. Here, $F_k - \tau F_{k-1} > 0$, so that by Lemma 11,

$$0 < \tau F_{k-2} - F_{k-1} + 1 < \tau F_{k-2} - F_{k-1} + 1 + (F_k - \tau F_{k-1})\{n\tau\}.$$

Also,

$$\tau F_{k-2} - F_{k-1} + 1 + (F_k - \tau F_{k-1})\{n\tau\} < \tau F_{k-2} - F_{k-1} + 1 + (F_k - \tau F_{k-1}),$$

and this last expression simplifies to $F_{k-2} - \tau F_{k-3} + 1$, which by Lemma 12 is $< \tau + 2$.

Case 2: k even. Here, $F_k - \tau F_{k-1} < 0$, so that by Lemma 13,

$$\begin{aligned} 0 &< F_{k-2} - \tau F_{k-3} + 1 < \tau F_{k-2} - F_{k-1} + 1 + F_k - \tau F_{k-1} \\ &< \tau F_{k-2} - F_{k-1} + 1 + (F_k - \tau F_{k-1})\{n\tau\}. \end{aligned}$$

Also,

$$\tau F_{k-2} - F_{k-1} + 1 + (F_k - \tau F_{k-1})\{n\tau\} < \tau F_{k-2} - F_{k-1} + 1 + (F_k - \tau F_{k-1}).$$

Abbreviating this last expression as E, the desired inequality $E < \tau + 2$ is equivalent to

$$F_k - F_{k-1} - 1 < \tau(F_{k-1} - F_{k-2}),$$

which is equivalent to $F_{k-2} - 1 \leq \tau F_{k-3}$, which holds by Lemma 13. $\square$

The reader may wish to prove the following proposition.

Theorem 20. *The Wythoff difference array, A080164, is the order array of both the Luo-Lucas array (Table 4) and its dual (Table 5).*

5 Densities and limiting densities

Suppose that $s = (s_k)$ for $k \geq 1$, is a sequence in $\mathbb{N}$. Define

$$c(s, m) = \text{number of numbers in } s \text{ that are } \leq m,$$

and define the *density of s in* $[1, m]$, by

$$D(s, m) = \frac{m}{c(s, m)}. \tag{17}$$

In order to estimate densities of column sequences of the Wythoff array, $(w(n, k))$, we start with a lemma:

Lemma 21. *For every n and k, $w(n+1,k) - w(n,k) \in \{F_{k+2}, F_{k+3}\}$.*

Proof. By Theorem 4,

$$\begin{aligned} w(n+1,k) - w(n,k) &= \lfloor (n+1)\tau \rfloor F_{k+1} + nF_k - (\lfloor n\tau \rfloor F_{k+1} + (n-1)F_k) \\ &= (\tau + \{(n+1)\tau\} - \{n\tau\})F_{k+1} + F_k \\ &= \delta F_{k+1} + F_k, \text{ where } \delta \in \{1,2\}. \end{aligned}$$

□

Example 22. For fixed n and all k, let $s_k = w(n,k)$. The density

$$D(w(n,k), m),$$

which is the proportion of numbers in column k (that is, numbers whose Zeckendorf representation has F_{k+1} as least term) of the Wythoff array that are $\leq m$, is estimated as follows. Let n be the number satisfying

$$w(n,k) \leq m < w(n+1,k).$$

Then by Lemma 21,

$$\frac{n}{w(n,k) + F_{k+3}} \leq \frac{n}{w(n+1,k)} \leq D(w(n,k), m) = \frac{n}{m} < \frac{n}{w(n,k)}.$$

Applying Theorem 4 and dividing by n lead to a limiting density:

$$\lim_{n\to\infty} D(w(n,k)) = \frac{1}{\tau F_{k+1} + F_k}.$$

Since the columns of $(w(n,k))$ partition $\mathbb{N}$, we have

$$\sum_{k=1}^{\infty} \frac{1}{\tau F_{k+1} + F_k} = 1.$$

Example 23. We turn now to the second column, $B_0 = (r(2,k)) =$ A188378, of the 1st Lucas-Wythoff array:

$$D(r(2,k), n) = \frac{n}{\lfloor \tau \lfloor n\tau \rfloor \rfloor + n},$$

so that the limiting density is

$$\frac{1}{\tau^2 + 1} = \frac{2}{5 + \sqrt{5}} \approx 27.64\%.$$

Next, consider the 2nd Lucas-Wythoff array, $(r^*(n,k))$.

Lemma 24. *For every n and k, $r^*(n+1,k) - r^*(n,k) \in \{L_{k+1}, L_{k+2}\}$.*

Proof. By Theorem 5,

$$\begin{aligned} r^*(n+1,k) - r^*(n,k) &= \lfloor (n+1)\tau \rfloor L_k + nL_{k-1} - (\lfloor n\tau \rfloor L_k + (n-1)L_{k-1}) \\ &= (\tau + \{n\tau\})L_k - L_{k-1} \\ &= \delta L_k + L_{k-1}, \text{ where } \delta \in \{1,2\}. \end{aligned}$$

□

Example 25. For fixed n and all k, let $s_k = r^*(n,k)$. The density

$$D(r^*(n,k), m),$$

which is the proportion of numbers in column k (that is, numbers whose Lucas representation has L_k as least term) of the 2nd Lucas-Wythoff array that are $\leq m$, is estimated as follows. Let n be the number satisfying

$$r^*(n,k) \leq m < r^*(n+1,k).$$

Then by Lemma 24,

$$\frac{n}{r^*(n,k) + L_{k+2}} \leq \frac{n}{r^*(n+1,k)} \leq D(r^*(n,k), m) = \frac{n}{m} < \frac{n}{r^*(n,k)}.$$

Applying Theorem 4 and dividing by n lead to a limiting density:

$$\lim_{n\to\infty} D(r^*(n,k)) = \frac{1}{\tau L_k + L_{k-1}}.$$

Since the columns of $(r^*(n,k))$ partition $\mathbb{N} - B_0$, we have, as a corollary of Example 23,

$$\sum_{k=1}^{\infty} \frac{1}{\tau L_k + L_{k-1}} = 1 - \frac{1}{\tau^2 + 1} \approx 72.36\%.$$

Example 26. Recall the Luo-Lucas array, $(\ell(n,k))$. Following the proof of Lemma 24, it is easy to find that

$$\ell(n+1,k) - \ell(n,k) \in \{L_{2k+2}, L_{2k+3}\}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{\tau L_{2k+1} + L_{2k}} = \frac{1}{3\tau + 1} = \frac{3\sqrt{5} - 5}{10} \approx 17.08\%,$$

which agrees with the limit obtained in a different manner by Luo [5].

Example 27. Finally, recall the dual of the Luo-Lucas array, $(\ell^*(n,k))$. As in Example 26,

$$\ell^*(n+1,k) - \ell^*(n,k) \in \{L_{2k+2}, L_{2k+3}\}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{\tau L_{2k+2} + L_{2k+1}} = 1 - \frac{2}{\sqrt{5}} \approx 10.55\%.$$

6 Concluding conjectures

In addition to the open question posed at the end of Section 2, we recall conjectures from A214979 and A214981.

1. Let $I(n) = \{1, 2, \ldots, n\}$. Let $U(n)$ be the number of terms in the unique greedy Lucas representations of the numbers in $I(n)$, and let $V(n)$ be the number of terms in the Zeckendorf representations of $I(n)$. Then $V(n) \geq U(n)$ for all n, and $V(n) = U(n)$ for infinitely many n.

2. Let $S = (1, 2, 3, 4, 5, 7, 8, 11, 13, 18, \ldots)$ be the sequence, in increasing order, of all Fibonacci and Lucas numbers. Let $W(n)$ be the number of terms in the greedy S-representations of the numbers in $I(n)$ (as in A214981). Then the limit of $V(n)/W(n)$ exists and is the interval $(1.2, 1.4)$.

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